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Spectral aspects of symmetric matrix signings☆

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ABSTRACT

The spectra of signed matrices have played a fundamental role in social sciences, graph theory, and control theory. In this work, we investigate the computational problems of *finding* symmetric signings of matrices with natural spectral properties. Our results are the following:

1. We characterize matrices that have an invertible signing: a symmetric matrix has an invertible symmetric signing *if and only if* the support graph of the matrix contains a perfect 2-matching. Further, we present an efficient algorithm to search for an invertible symmetric signing.
2. We use the above-mentioned characterization to give an algorithm to find a minimum increase in the support of a given symmetric matrix so that it has an invertible symmetric signing.
3. We show NP-completeness of the following problems: verifying whether a given matrix has a symmetric *off-diagonal* signing that is singular/has bounded eigenvalues. However, we also illustrate that the complexity could differ substantially for input matrices that are adjacency matrices of graphs.

We use combinatorial techniques in addition to classic results from matching theory.

1. Introduction

The spectra of several graph-related matrices such as the adjacency and the Laplacian matrices have become fundamental objects of study in computer science. In this work, we undertake a systematic and

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comprehensive investigation of the spectrum and the invertibility of *symmetric signings* of matrices. We study natural spectral properties of symmetric signed matrices and address the computational problems of finding and verifying the existence of symmetric signings to achieve these spectral properties.

For a real-valued symmetric $n \times n$ matrix M and a $\{\pm 1\}$-valued $n \times n$ matrix s—which we refer to as a *signing*—we define the *signed matrix* $M(s)$ to be the matrix obtained by taking entry-wise products of M and s. Signed adjacency matrices (respectively, Laplacians) correspond to signed matrices $M(s)$ where M is the adjacency matrix (respectively, Laplacian) of a graph. We say that s is a *symmetric* signing if s is a symmetric matrix and an *off-diagonal* signing if all the diagonal entries of s are $+1$. In this work we are interested in the following computational problems:

BoundedEvalueSigning: Given a real symmetric matrix M and a real number λ, verify if there exists a symmetric signing s such that the largest eigenvalue $\lambda_{\max}(M(s))$ is at most λ.
IncludeSigning: Given a real symmetric matrix M and a real number λ, verify if there exists a symmetric signing s such that $M(s)$ has λ as an eigenvalue.
AvoidSigning: Given a real symmetric matrix M and a real number λ, verify if there exists a symmetric signing s such that $M(s)$ does not have λ as an eigenvalue.

We focus our attention specifically on the variants of the above problem where λ is assumed to be 0:

NsdSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is negative semi-definite.
SingularSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is singular.
InvertibleSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is invertible (that is, non-singular).

It will also be useful to further consider the same specialization but where we further restrict the problem to only allow *off-diagonal* symmetric signings.

NsdODSigning: Given a real symmetric matrix M, verify if there exists a symmetric off-diagonal signing s such that $M(s)$ is negative semi-definite.
SingularODSigning: Given a real symmetric matrix M, verify if there exists a symmetric off-diagonal signing s such that $M(s)$ is singular.
InvertibleODSigning: Given a real symmetric matrix M, verify if there exists a symmetric off-diagonal signing s such that $M(s)$ is invertible (that is, non-singular).

We note that NsdODSigning would be interesting only if the input matrix has non-positive diagonal entries. However, if the input matrix has non-positive diagonal entries, then the problem exactly corresponds to NsdSigning. Furthermore, if the input to BoundedEvalSigning is a *graph-related* matrix (for example, the adjacency matrix), then we can reduce it to an instance of NsdSigning: indeed, solving BoundedEvalSigning on inputs of the form (M, λ) where M is graph-related, reduces to solving NsdSigning on $M - \lambda I$.

1.1. Motivations

Spectra of signed matrices and expanders. Let G be a connected d-regular graph on n vertices and let $d = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1}$ be the eigenvalues of its adjacency matrix. Then G is a Ramanujan expander if $\max_{|\lambda_i|<d} |\lambda_i| \leq 2\sqrt{d-1}$. Efficient construction of Ramanujan expanders of arbitrary degrees remains an important open problem[4]. A combinatorial approach to this problem, initiated by Friedman [2], is to

[4] While efficient construction of bipartite Ramanujan *multi-graphs* of all degrees is known [1], it still remains open to efficiently construct bipartite Ramanujan *simple* graphs of all degrees.

obtain larger Ramanujan graphs from smaller ones while preserving the degree. A 2-*lift* H of G is obtained as follows: introduce two copies of each vertex u of G, say u_1 and u_2, as the vertices of H and for each edge $\{u, v\}$ in G, introduce either $\{u_1, v_2\}, \{u_2, v_1\}$ or $\{u_1, v_1\}, \{u_2, v_2\}$ as edges of H. There is a bijection between 2-lifts and symmetric signed adjacency matrices of G. Furthermore, the eigenvalues of the adjacency matrix of a 2-lift H are given by the union of the eigenvalues of the adjacency matrix of the base graph G (also called the "old" eigenvalues) and the signed adjacency matrix of G that corresponds to the 2-lift (the "new" eigenvalues).

Marcus, Spielman, and Srivastava [3] showed that every d-regular bipartite graph has a 2-lift whose new eigenvalues are bounded in absolute value by $2\sqrt{d-1}$. However, this result [3] is not constructive and their work raises the question of whether there is an efficient algorithm to find a symmetric signing that minimizes the largest eigenvalue. This motivates investigating BoundedEvalueSigning which is the decision variant of the computational problem. More precisely, it motivates investigating BoundedEvalueSigning when the input matrix is an adjacency matrix.

It is also natural to investigate the complexity of several related problems. As we will see in the next section, BoundedEvalueSigning is NP-hard for arbitrary symmetric matrices. Our reduction showing that BoundedEvalueSigning is NP-hard suggests a close relationship with SingularODSigning which we also show to be NP-hard. Hoping to make progress on BoundedEvalueSigning for adjacency matrices, we investigate SingularODSigning for adjacency matrices—this is equivalent to SingularSigning when the graph has no self-loops. IncludeSigning is a natural generalization of SingularSigning. Moreover, InvertibleODSigning, InvertibleSigning and AvoidSigning arise naturally as the complements of SingularODSigning, SingularSigning and IncludeSigning.

Solvability index of a signed matrix. The notion of *balance* of a symmetric signed matrix has been studied extensively in social sciences [4–7]. A signed adjacency matrix is *balanced* if there is a partition of the vertex set such that all edges within each part are positive, and all edges in between two parts are negative (one of the parts could be empty). A number of works [7–12] have explored the problem of minimally modifying signed graphs (or signed adjacency matrices) to convert it into a balanced graph.

In this work, we introduce a related problem regarding symmetric signed matrices: Given a symmetric matrix M, what is the smallest number of off-diagonal zero entries of M whose replacement by non-zeros gives a symmetric matrix M' that has an invertible symmetric signing? We define this quantity to be the *solvability index*[5]. Knowing this number might be helpful in studying systems of linear equations in signed matrices that might be ill-defined, and thus do not have a (unique) solution and in minimally modifying such matrices so that the resulting linear system becomes (uniquely) solvable. We use classic graph-theoretic techniques to show that solvability index is indeed computable efficiently.

1.2. Our results

Intriguingly, the complexity of BoundedEvalueSigning has not been studied in the literature even though it is widely believed to be a difficult problem in the graph sparsification community. We shed light on this problem by showing that it is NP-complete.

Theorem 1. NsdSigning *and* SingularODSigning *are NP-complete.*

Theorem 1 also implies that BoundedEvalueSigning is NP-complete. The hard instances generated by our proof of Theorem 1 are real symmetric matrices with non-zero diagonal entries and hence, it does not

[5] Our definition of *solvability index* is similar to the notion of *frustration index* [13,14]. The *frustration index* of a matrix M is the minimum number of non-zero off-diagonal entries of M whose deletion results in a *balanced signed graph.* Computing the frustration index of a signed graph is NP-hard [15].

resolve the computational complexity of the problem of finding a signing of a given graph-related matrix that minimizes its largest eigenvalue.

In contrast to SingularODSigning, we show that InvertibleSigning admits an efficient algorithm. In fact, we show that there exists an efficient algorithm to solve the search variant of this problem, which we denote by SearchInvertibleSigning: here the goal is to *find* a symmetric signing (not necessarily off-diagonal) of a given matrix M such that $M(s)$ is invertible.

Theorem 2. *There exists a polynomial-time algorithm to solve* SearchInvertibleSigning*.*

Our proof of Theorem 2 leads to a structural characterization for the existence of invertible signings through the existence of *perfect* 2*-matchings* in the support graph of the matrix. We believe that this structural characterization could be of independent interest and hence, discuss it in detail in Section 1.2.1.

We next investigate the number of invertible (symmetric but not necessarily off-diagonal) signings of the adjacency matrix of graphs. One of the motivations behind studying this quantitative version arises in the design of randomized algorithms: if the number is sufficiently large, then it could reduce the amount of randomness in a popular algebraic algorithm for verifying the existence of a perfect matching in a bipartite graph due to Lovász [16].

It is well-known that flipping the signs on the edges of a *cut* preserves the spectrum of the signed adjacency matrix. Thus, the existence of one invertible signed adjacency matrix for a (connected) graph G on n vertices also implies the existence of 2^{n-1} invertible signed adjacency matrices. In comparison, the lower bound obtained in our next result is much larger (if the number of edges in the graph is much larger than the number of vertices). We emphasize that our lower bound holds for arbitrary simple graphs (and not just bipartite graphs).

Theorem 3. *Let G be a simple graph with m edges that contains a perfect* 2*-matching, and let M be the adjacency matrix of G. Let $\gamma(M)$ denote the number of symmetric signings of M that are invertible and let $\mu(G)$ denote the number of perfect* 2*-matchings in G. Then,*

$$\gamma(M) \cdot \mu(G) \geq 2^m.$$

As a consequence of Theorem 3, we obtain that the fraction of invertible signed adjacency matrices of a graph G with n vertices containing a perfect 2-matching is at least $2^{-O(n \log n)}$ (since $\mu(G)$, the number of perfect 2-matchings in a n-vertex graph, is at most $n!$). An upper bound of $2^{-\Omega(n)}$ on the fraction is demonstrated by the graph that is a disjoint union of 4-cycles. While our bound from Theorem 3 does not reduce the amount of randomness needed in the algebraic algorithm for verifying the existence of a perfect matching in a bipartite graph, we believe that it could be of interest from the perspective of combinatorics.

Our next result provides some evidence that one might be able to design efficient algorithms to solve the NP-complete problems appearing in Theorem 1 for graph-related matrices. In particular, we show that SingularSigning and its search variant admit efficient algorithms when the input matrix corresponds to the adjacency matrix of a *bipartite* graph. We note that since bipartite graphs do not have self-loops, SingularSigning and SingularODSigning are equivalent on such input.

Theorem 4. *There exists a polynomial-time algorithm to verify if the adjacency matrix A_G of a given bipartite graph G has a symmetric signing s such that $A_G(s)$ is singular; and if so, find such a signing.*

Finally, we recall the *solvability index* of a real symmetric matrix M that we defined earlier: it is the smallest number of off-diagonal zero entries that need to be converted to non-zeros so that the resulting symmetric matrix has an invertible symmetric signing. We emphasize that the support-increase operation that we consider preserves symmetry, that is, if we replace the zero entry $A[i, j]$ by α, then the zero entry $A[j, i]$ is also replaced by α. We give an efficient algorithm to find the solvability index of a given symmetric matrix M.

Theorem 5. *There exists a polynomial-time algorithm to find the solvability index of a given real symmetric matrix.*

1.2.1. Structural characterization for invertible signings

Theorem 2, in particular, implies that InvertibleSigning is solvable efficiently. In fact, our proof-technique gives an efficient characterization for the existence of an invertible signing. This characterization also leads to an alternative algorithm to solve InvertibleSigning. We believe that this characterization might be of independent interest and hence describe it here.

The *support graph* of a real symmetric $n \times n$ matrix M is an undirected graph G where the vertex set of G is $[n] := \{1, \ldots, n\}$, and the edge set of G is $\{\{u, v\} \mid M[u, v] \neq 0\}$. We note that G could have self-loops depending on the diagonal entries of M. A *perfect 2-matching* in a graph G with edge set E is an assignment $x : E \to \{0, 1, 2\}$ of values to the edges such that $\sum_{e \in \delta(v)} x_e = 2$ holds for every vertex v in G (where $\delta(v)$ denotes the set of edges incident to v). Equivalently, a perfect 2-matching in a graph G is a vertex-disjoint union of edges and cycles (self-loops are cycles) in G such that each vertex is incident to at least one edge. We show the following characterization:

Theorem 6. *Let M be a symmetric $n \times n$ matrix and let G be the support graph of M. The following are equivalent:*

(i) There exists a symmetric signing s such that the signed matrix $M(s)$ is invertible.
(ii) The support graph G contains a perfect 2-matching.

Remark 1. The structural characterization of Theorem 6 leads to a polynomial-time algorithm to solve InvertibleSigning—it suffices to verify if the support graph of the input matrix contains a perfect 2-matching which can be done in polynomial-time.

Remark 2. For Theorem 6, we present two proofs: the first is a constructive proof via a generalization (see Theorem 11 in Section 3) and the second is a non-constructive proof via Combinatorial Nullstellensatz which also leads to the proof of Theorem 3 (see Section 4).

1.3. Related work

Skew symmetric matrix of indeterminates. A square skew-symmetric matrix of indeterminates with zeros on the diagonal is known as the *Tutte matrix* of its support graph. A well-known result by Tutte shows that the determinant polynomial of the Tutte matrix is non-zero if and only if the corresponding support graph has a perfect matching. Our result in Theorem 6 can be interpreted as a variant of Tutte's result to square *symmetric* matrices of indeterminates with zeros on the diagonal.

Cunningham and Geelen [17] extended Tutte's work along a different direction by giving a characterization of invertible submatrices of the Tutte matrix using *path-matchings*. Given a graph G with vertex set V and vertex subsets $R, L \subseteq V$, a (R, L)-*path-matching* in G is a collection of vertex-disjoint paths from R to L and edges in $G[V \setminus (R \cup L)]$. A *perfect* (R, L)-path-matching is a (R, L)-path-matching in which every vertex in G is incident to some edges of the vertex-disjoint paths. They showed that the determinant polynomial of a square submatrix of the Tutte matrix of G with column set R and row set L is non-zero if and only if there exists a perfect (R, L)-path-matching in G. The notion of *cycle-covers* that we introduce in Section 3 and our result in Theorem 11 can be interpreted as variants of Cunningham and Geelen's result to square *symmetric* matrices of indeterminates with zeros on the diagonal.

Our results in Theorems 6 and 11 go further than Cunningham and Geelen's result by not only giving similar characterizations for the determinant to be a non-zero polynomial but also by giving polynomial-time algorithms to find a point in $\{\pm 1\}^E$ at which the polynomial is non-zero.

Minimum rank problems. A line of work seemingly related to ours is the minimum rank problem (e.g., see [18,19]): given a graph G, the goal is to compute the minimum rank of the weighted adjacency matrix of a graph obtained by giving non-zero real-valued weights to the edges of G. We emphasize that the allowed weights in the minimum rank problem are arbitrary and are not simply signs of the given adjacency matrix as in the case of our work. A signed variant of the minimum rank problem has also been addressed in the literature: given a sign pattern matrix S, the goal is to compute the minimum rank of a matrix with real-valued entries whose sign pattern is identical to S. Once again, we emphasize the distinction between the signed variant of the minimum rank problem and the problems studied in our work: in the signed variant of the minimum rank problem, the sign pattern is the input and the goal is to find a matrix with real-valued entries matching the input sign pattern and achieving minimum rank. In contrast, the problems studied in our work have real-valued entries as inputs and the goal is to find a symmetric sign pattern of the entries to achieve the specified spectral properties.

A year after posting our work on arXiv [20], Akbari, Ghafari, Kazemian, and Nahvi [21] also posted an article addressing InvertibleSigning[6]. They show the same characterization as Theorem 6 with a proof identical to the non-constructive proof appearing Section 4. We emphasize that in addition to showing the structural characterization in Theorem 6, this work resolves the search problem in Theorem 2, and moreover shows a much more general structural characterization in Theorem 11 with a constructive proof.

1.4. Organization

In Section 1.5, we review definitions and notations. In Section 2, we show that NsdSigning and SingularODSigning are NP-complete (Theorem 1). In Section 3, we describe an efficient algorithm to find an invertible signing (Theorem 2). In Section 4, we consider the problem of counting the number of invertible signings (Theorem 3). In Section 5, we give an efficient algorithm to find a singular signing of adjacency matrices of bipartite graphs (Theorem 4). In Section 6, we give an efficient algorithm to find the solvability index of symmetric matrices (Theorem 5). Finally, in Section 7, we conclude with some open questions.

1.5. Preliminaries

Unless otherwise specified, all matrices are symmetric and take values over the reals. Since all of our results are for symmetric signings, we will just use the term *signing* to refer to a symmetric signing in the rest of this work. We denote the entry-wise product of two $n \times n$ matrices M and s as $M(s)$ (even when s is not necessarily a signing).

Let S_n be the set of permutations of n elements, M be a real symmetric $n \times n$ matrix, and s be a symmetric $n \times n$ signing. Then, the *permutation expansion* of the determinant of a signed matrix $M(s)$ is given by

$$\det M(s) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} M(s)[i, \sigma(i)].$$

[6] Our arXiv post dated Nov, 2016: https://arxiv.org/abs/1611.03624; the post by Akbari, Ghafari, Kazemian, and Nahvi dated Aug, 2017: https://arxiv.org/abs/1708.07118.

A permutation σ in S_n has a unique cycle decomposition and hence corresponds to a vertex-disjoint union of directed cycles on n vertices. Removing the orientation gives an undirected graph which is a vertex disjoint union of cycles, self-loops, and matching edges.

2. Hardness of eigenvalue problems

In this section, we focus on our hardness results and prove Theorem 1. We use the notion of *Schur complement*. The following lemma summarizes the definition and the relevant properties of the Schur complement.

Lemma 7 (*Schur Complement [22]*)**.** *Suppose $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times q}$, $C \in \mathbb{R}^{q\times q}$ are matrices such that A is invertible and the matrix*

$$D := \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

is a symmetric matrix. Then the Schur complement *of A in matrix D is defined to be*

$$D_A := C - B^T A^{-1} B.$$

We have the following properties:

(i) Suppose A is positive definite. Then, the matrix D is positive semi-definite if and only if the Schur complement of A in D, namely D_A, is positive semi-definite.
(ii) $\det(D) = \det(A) \cdot \det(D_A)$.

To show that NsdSigning is NP-complete, it is convenient to first show that the following problem is NP-complete:

PsdSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is positive semi-definite.

We note that a positive semi-definite matrix cannot have negative entries on its diagonal. Hence, the above problem is equivalent to the following off-diagonal variant:

PsdODSigning: Given a real symmetric matrix M, verify if there exists a symmetric off-diagonal signing s such that $M(s)$ is positive semi-definite.

In order to show the NP-completeness result, we reduce from the partition problem, which is a well-known NP-complete problem [23]. We recall the problem below:

Partition: Given an n-dimensional vector b of non-negative integers, determine if there is a ± 1-signing vector z such that the inner product $\langle b, z\rangle$ equals zero.

Lemma 8. PsdODSigning *and* SingularODSigning *are NP-complete.*

Proof. Both problems are in NP since if there is a symmetric off-diagonal signing of the given matrix that is positive semi-definite/singular, then this signing gives the witness. In particular, we can verify if a given symmetric off-diagonal signed matrix is positive semi-definite/singular in polynomial time by computing its spectrum [24].

We show NP-hardness of both problems by reducing from Partition. Let the n-dimensional vector $b := (b_1, \dots, b_n)^T$ be the input to the Partition problem, where each b_i is a non-negative integer. We construct a matrix M as an instance of either problem as follows: Consider the following $(n+2)\times(n+2)$-matrix

$$M := \begin{bmatrix} I_n & b & \mathbf{1}_n \\ b^T & \langle b, b\rangle & 0 \\ \mathbf{1}_n^T & 0 & n \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n-dimensional column vector of all ones.

Now consider a symmetric off-diagonal signing s of M. The symmetric off-diagonal signed matrix is of the following form:

$$M' := M(s) = \begin{bmatrix} I_n & \hat{b} & z \\ \hat{b}^T & \langle b, b\rangle & 0 \\ z^T & 0 & n \end{bmatrix},$$

where the n-dimensional vector z takes values in $\{\pm 1\}^n$ and $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_n)^T$, where $\hat{b}_i$ takes value in $\{\pm b_i\}$ for every i. Let

$$\begin{aligned} A &:= I_n, \\ B &:= \begin{bmatrix} \hat{b} & z \end{bmatrix}, \text{ and} \\ C &:= \begin{bmatrix} \langle b, b\rangle & 0 \\ 0 & n \end{bmatrix}. \end{aligned}$$

Since $A = I_n$ is invertible, the Schur complement of A in M' is well-defined and is given by

$$M'_A = \begin{bmatrix} \langle b, b\rangle & 0 \\ 0 & n \end{bmatrix} - \begin{bmatrix} \hat{b}^T \\ z^T \end{bmatrix} I_n^{-1} \begin{bmatrix} \hat{b} & z \end{bmatrix} = \begin{bmatrix} \langle b, b\rangle & 0 \\ 0 & n \end{bmatrix} - \begin{bmatrix} \langle \hat{b}, \hat{b}\rangle & \langle \hat{b}, z\rangle \\ \langle \hat{b}, z\rangle & \langle z, z\rangle \end{bmatrix} = \begin{bmatrix} 0 & -\langle \hat{b}, z\rangle \\ -\langle \hat{b}, z\rangle & 0 \end{bmatrix},$$

where the last equation follows because we have $\langle \hat{b}, \hat{b}\rangle = \langle b, b\rangle$ and $\langle z, z\rangle = n$.

We note that $A = I_n$ is positive definite. Therefore, by property (i) of Lemma 7, the matrix M' is positive semi-definite if and only if M'_A is positive semi-definite. Likewise, using property (ii) of Lemma 7, we have that

$$\det M' = \det(I_n) \cdot \det(M'_A) = \det(I_n) \cdot \det\left(\begin{bmatrix} 0 & -\langle \hat{b}, z\rangle \\ -\langle \hat{b}, z\rangle & 0 \end{bmatrix}\right) = -\langle \hat{b}, z\rangle^2.$$

Therefore, $\det M' = 0$ if and only if $\langle \hat{b}, z\rangle = 0$. Thus, M' is positive semi-definite/singular if and only if $\langle \hat{b}, z\rangle = 0$. However, $\langle \hat{b}, z\rangle = 0$ if and only if there exists $\hat{z} \in \{\pm 1\}$ such that $\langle b, \hat{z}\rangle = 0$. ◀

We observe that a real symmetric $n \times n$ matrix is positive semi-definite if and only if $-M$ is negative semi-definite. Lemma 8 and this observation imply Theorem 1.

3. Finding invertible signings

In this section, we focus on invertible signings and prove Theorem 2. We prove a much more general statement in comparison to the one given in Theorem 6, which we believe could be of independent interest. We start by introducing the background needed to state the general version.

Symmetric signings of asymmetric sub-matrices. Let M be a symmetric $n \times n$ matrix. For $X, Y \subseteq [n]$ being a subset of row and column indices of the same cardinality, let $M[X, Y]$ denote the submatrix of M obtained by picking the rows in X and the columns in Y. We note that $M[X, Y]$ is a square matrix, but it may *not* be symmetric even though M is symmetric. We are interested in finding a symmetric $n \times n$ signing s so that the square submatrix $M(s)[X, Y]$ is invertible. We emphasize that for a symmetric signing s, the (possibly asymmetric) matrix $M(s)[X, Y]$ is symmetric on $X \cap Y$, that is, the $[i, j]$'th and the $[j, i]$'th entries of the matrix $M(s)[X, Y]$ are the same for every $i, j \in X \cap Y$.

Perfect 2-matchings in subgraphs. Let G be a simple undirected graph, possibly containing self-loops. Let X, Y be vertex subsets of G. We consider the subgraph $G[X \cup Y]$ induced by $X \cup Y$. An (X, Y)-*cycle-cover* is a collection of edges of the subgraph $G[X \cup Y]$ that induce a vertex-disjoint union of paths and cycles (cycles could be loop edges) in $G[X \cup Y]$ such that (1) every cycle is a subgraph of $G[X \cap Y]$, (2) every vertex

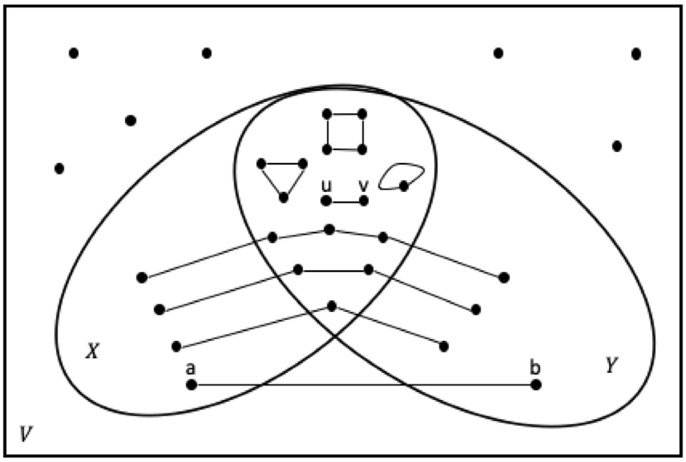

Fig. 1. An (X,Y)-cycle-cover F. Furthermore, by our definitions below, the edge $\{a,b\}$ is in Paths(F) while the edge $\{u,v\}$ is in Matchings(F).

of $X \cup Y$ is incident to at least one edge, and (3) every path either has one end-vertex in $X \setminus Y$, the other end-vertex in $Y \setminus X$, and all intermediate vertices in $X \cap Y$, or has both end-vertices in $X \cap Y$ with only one edge (see Fig. 1 for an example). We note that (X,X)-cycle-covers correspond to perfect 2-matchings in $G[X]$ and hence, (V,V)-cycle-covers correspond to perfect 2-matchings in G. It follows that in (X,X)-cycle-covers all paths are only a single edge in G. Furthermore, the existence of an (X,Y)-cycle-cover is possible only if $|X| = |Y|$. The following lemma states that the existence of an (X,Y)-cycle-cover in a given graph can be verified efficiently.

Lemma 9. *There exists a polynomial-time algorithm that decides if there is an (X,Y)-cycle-cover in a given graph G for given vertex subsets X, Y of G.*

Proof. We recall that determining if a bipartite graph has a perfect matching can be done in polynomial time. Hence, it will suffice to show that deciding if an (X,Y)-cycle-cover exists in G can be reduced in polynomial time to deciding if a perfect matching exists in a bipartite graph. Let $L := \{v_l \mid v \in X\}$ and $R := \{v_r \mid v \in Y\}$. Let H be the bipartite graph with vertex set $L \cup R$ and edge set $\{\{v_l, u_r\} \mid \{v,u\} \in (X \times Y) \cap E\}$. We will show that there is an (X,Y)-cycle-cover in G if and only if there is a perfect matching in H.

Let F be an (X,Y)-cycle-cover in G. Consider the edge set $M := \{\{v_l, u_r\} \mid \{v,u\} \in (X \times Y) \cap F\}$. We note that for each $v \in X \setminus Y$ the degree of v_l in M is one. Likewise, for each $v \in Y \setminus X$ the degree of v_r in M is one. Finally, if $v \in X \cap Y$ then the degrees of v_l and v_r in M is one or two. It follows that M is a perfect 2-matching in H and hence there must be a perfect matching in H.

Next, let M be a perfect matching in H. Consider the edge set $F := \{\{v,u\} \mid \{v_l, u_r\} \in M\}$. We note that F contains no edge between vertices in $Y \setminus X$. Likewise, F contains no edge between vertices in $X \setminus Y$. Moreover, for each $v \in X \setminus Y \cup Y \setminus X$ the degree of v is exactly one. Finally, for each $v \in X \cap Y$ the degree of v is one or two. It follows that F is an (X,Y)-cycle-cover in G. ◀

Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix, $X, Y \subseteq [n]$ with $|X| = |Y|$ and s be a symmetric $n \times n$ signing. Recall that we are interested in finding a symmetric $n \times n$ signing s so that the square submatrix $M(s)[X,Y]$ is invertible. We derive a convenient expression for $\det(M(s)[X,Y])$ that is based on (X,Y)-cycle-covers. For an (X,Y)-cycle-cover F, let Cycles(F), Paths(F), and Matchings(F) denote the set of cycles in F, paths in F with end-vertices in $X \setminus Y$ and $Y \setminus X$, and paths in F that are contained in $G[X \cap Y]$, respectively.

Moreover, let $\mathrm{Loops}(F)$ and $\mathrm{NTCs}(F)$ denote the set of self-loops and non-trivial-cycles in F. We emphasize that $\mathrm{Cycles}(F) = \mathrm{Loops}(F) \cup \mathrm{NTCs}(F)$. We also note that $\mathrm{Cycles}(F)$, $\mathrm{Paths}(F)$, and $\mathrm{Matchings}(F)$ are all vertex-disjoint from one another and if $X = Y$ then $\mathrm{Paths}(F) = \emptyset$. We define

$$\begin{aligned} M(s)_{\mathrm{Cycles}(F)} &\coloneqq \prod_{C \in \mathrm{Cycles}(F)} \prod_{\{u,v\} \in C} M(s)[u,v], \\ M(s)_{\mathrm{Paths}(F)} &\coloneqq \prod_{P \in \mathrm{Paths}(F)} \prod_{\{u,v\} \in P} M(s)[u,v], \text{ and} \\ M(s)_{\mathrm{Matchings}(F)} &\coloneqq \prod_{\{u,v\} \in \mathrm{Matchings}(F)} M(s)[u,v]^2. \end{aligned}$$

With this notation, we have the following claim that the determinant of $M(s)[X,Y]$ is a $\{\pm 1\}$-linear combination of terms corresponding to (X,Y)-cycle-covers in G.

Lemma 10 (*(X,Y)-cycle-cover Expansion*)**.** *Let $M \in \mathbb{R}^{n \times n}$ be a symmetric $n \times n$ matrix, $X, Y \subseteq [n]$ with $|X| = |Y|$, and s be a symmetric $n \times n$ matrix. Let G be the support graph of M and $\mathcal{F}$ be the set of all (X,Y)-cycle-covers in G. Then, there exists $\lambda_F \in \{\pm 1\}$ for all $F \in \mathcal{F}$ such that*

$$\det(M(s)[X,Y]) = \sum_{F \in \mathcal{F}} \lambda_F \cdot 2^{|NTCs(F)|} \cdot M(s)_{Cycles(F)} \cdot M(s)_{Paths(F)} \cdot M(s)_{Matchings(F)}.$$

Moreover, if there are $F_1, F_2 \in \mathcal{F}$ such that $Cycles(F_1) = Cycles(F_2)$ and $Paths(F_1) = Paths(F_2)$ then $\lambda_{F_1} = \lambda_{F_2}$.

Proof. For simplicity, we denote $M' = M[X,Y]$. Let $k \coloneqq |X|$ and let S_k denote the set of permutations on k elements. Then, by the permutation expansion of the determinant, we have

$$\det(M'(s)) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot \prod_{i=1}^{k} s[i,\sigma(i)] \cdot M'[i,\sigma(i)].$$

We recall that $\mathrm{sgn}(\sigma) \in \{\pm 1\}$. Moreover, if $\sigma_1, \sigma_2 \in S_k$ such that σ_1 and σ_2 have the same cycle structure then $\mathrm{sgn}(\sigma_1) = \mathrm{sgn}(\sigma_2)$. Now, we note that there is a one-to-one correspondence between S_k and bijections from X to Y. So, we may view $\sigma \in S_k$ as a bijection $\sigma' : X \to Y$. Now, consider the graph $H_{\sigma'}$ on vertex set $X \cup Y$ and edge set $F_{\sigma'} \coloneqq \{\{u,v\} \mid \sigma'(u) = v\}$. Since σ' is a bijection, it follows that $F_{\sigma'}$ is an (X,Y)-cycle-cover in the complete graph on vertex set $X \cup Y$. Moreover, since each non-trivial-cycle in an (X,Y)-cycle-cover can take one of two orientations in any corresponding permutation, there are $2^{|\mathrm{NTCs}(F)|}$

distinct permutations which map to each (X, Y)-cycle-cover F. Hence,

$$\prod_{i=1}^{k} s[i,\sigma(i)] \cdot M'[i,\sigma(i)] = \prod_{u\in X} s[u,\sigma'(u)] \cdot M[u,\sigma'(u)]$$
$$= M(s)_{\mathrm{Cycles}(F_{\sigma'})} \cdot M(s)_{\mathrm{Paths}(F_{\sigma'})} \cdot M(s)_{\mathrm{Matchings}(F_{\sigma'})}.$$

The above-term is non-zero only if $F_{\sigma'}$ is an (X, Y)-cycle-cover in the support graph of G. Furthermore, if $F_1, F_2 \in \mathcal{F}$ such that $\mathrm{Cycles}(F_1) = \mathrm{Cycles}(F_2)$ and $\mathrm{Paths}(F_1) = \mathrm{Paths}(F_2)$ then $\lambda_{F_1} = \lambda_{F_2}$ since the corresponding permutations would have the same cycle structure. ◀

To prove Theorems 6 and 2, we show the following theorem which gives a generalized structural characterization: it characterizes the existence of invertible symmetric signings for (potentially asymmetric) submatrices of symmetric matrices.

Theorem 11. *Let M be a real symmetric $n\times n$ matrix with support graph G and $X, Y \subseteq [n]$ with $|X| = |Y|$. The following are equivalent:*

(i) There exists an (X, Y)-cycle-cover in G.
(ii) There exists a symmetric signing s such that $M(s)[X, Y]$ is invertible.

Moreover, there exists a polynomial-time algorithm that takes a real symmetric $n\times n$ matrix M and $X, Y \subseteq [n]$ as input and verifies if there exists a symmetric signing s such that $M(s)[X, Y]$ is invertible and if so, find such a signing.

Notation. Let M be a real symmetric $n\times n$ matrix with support graph G. Let A and B be vertex subsets of G. We define $E[A, B]$ to be the set of edges with one end-vertex in A and the other end-vertex in B. We use $E[A]$ to denote $E[A, A]$. Let e be an edge in G corresponding to the non-zero entry $M[u, v]$ $(= M[v, u])$. We define $M^{\overline{e}}$ as the matrix obtained by setting $M[u, v]$ and $M[v, u]$ to 0. For a signing s and row and column indices $u, v \in [n]$, we can obtain another signing s' such that $s'[u, v] := -s[u, v]$, $s'[v, u] := -s[v, u]$ and $s'[i, j] := s[i, j]$ for every entry $(i, j) \in [n] \times [n] \setminus \{(u, v), (v, u)\}$. We call this operation as s' *obtained from* s *by flipping on* $\{u, v\}$.

Proof of Theorem 11. We first present a constructive proof of the characterization. We will then use the proof to design the algorithm.

Lemma 10 immediately shows that (ii) implies (i): If we have a symmetric signing s such that $M(s)[X, Y]$ is invertible, then at least one of the terms in the (X, Y)-cycle-cover expansion of $\det(M(s)[X, Y])$ is non-zero. Hence, there exists an (X, Y)-cycle-cover in G.

We show that (i) implies (ii). Suppose not. Among the counterexamples, consider the ones with $|X|$ minimum and among these, pick a matrix M with minimum number of non-zero entries. Since we chose a counterexample, we have that

(A) there exists an (X, Y)-cycle-cover in G, but
(B) there is no symmetric signing s such that $M(s)[X, Y]$ is invertible.

We will arrive at a contradiction by showing that a signing s satisfying (ii) exists. We begin with the following claim about the counterexample.

Claim 12. *$E[X \setminus Y, Y] = \emptyset$ and $E[Y \setminus X, X] = \emptyset$.*

Proof. Suppose there exists an edge $e \in E[X \setminus Y, Y]$. Let $e := \{u, v\}$ with $u \in X \setminus Y$ and $v \in Y$. Then there exists $\alpha \in \{\pm 1\}$ such that the determinant of $M(s)[X, Y]$ can be expressed as a linear function of $s[u, v]$:

$$\det(M(s)[X, Y]) = \alpha \cdot s[u, v] \cdot M[u, v] \cdot \det(M(s)[X - u, Y - v]) + \det(M^{\overline{e}}(s)[X, Y]). \tag{1}$$

Case 1. Suppose there exists an (X, Y)-cycle-cover F containing e. We observe that $F - e$ is an $(X - u, Y - v)$-cycle-cover in G. Since we have a smallest counterexample, it follows that there exists a symmetric signing s such that $\det(M(s)[X - u, Y - v]) \neq 0$. Since $\det(M(s)[X, Y])$ is a linear function of $s[u, v]$, it follows that $\det(M(s)[X, Y]) \neq 0$ or $\det(M(s')[X, Y]) \neq 0$ where s' is a signing obtained from s by flipping on $\{u, v\}$. Hence, we have a contradiction to assumption B about the counterexample.

Case 2. Suppose that every (X, Y)-cycle-cover in G does not contain e. Then there is no $(X - u, Y - v)$-cycle-cover in G. Since (ii) implies (i), it follows that $\det(M(s)[X - u, Y - v]) = 0$ for every symmetric signing s. Let F be an (X, Y)-cycle-cover in G (as promised to exist by A). Then F is an (X, Y)-cycle-cover in $G - e$. Since we have a smallest counterexample, it follows that there exists a symmetric signing s such that $\det(M^{\overline{e}}(s)[X, Y]) \neq 0$. By (1), we observe that $\det(M(s)[X, Y]) \neq 0$. Thus, the symmetric signing s is a contradiction to assumption B about the counterexample.

Hence, $E[X \setminus Y, Y] = \emptyset$. Similarly $E[Y \setminus X, X] = \emptyset$. ◀

Now, if $X \setminus Y \neq \emptyset$ and there is no edge $e \in E[X \setminus Y, Y]$, then there is no (X, Y)-cycle-cover in G, a contradiction to assumption A about the counterexample. Hence, $X \setminus Y = \emptyset$. Similarly, $Y \setminus X = \emptyset$. Thus, we have $X = Y$ in the counterexample. We next show that the counterexample cannot have any self-loop edges.

Claim 13. *There are no self-loop edges in $E[X]$.*

Proof. Suppose there exists a self-loop edge in $E[X]$. Let $e = \{u, u\}$ for some $u \in X$. Then, there exists $\alpha \in \{\pm 1\}$ such that $\det(M(s)[X, Y])$ is a linear function of $s[u, u]$:

$$\det(M(s)[X, X]) = \alpha \cdot s[u, u] \cdot M[u, u] \cdot \det(M(s)[X - u, X - u]) + \det(M^{\overline{e}}(s)[X, X]). \tag{2}$$

We arrive at a contradiction by proceeding similar to the proof of the previous claim. We avoid restating the proof in the interests of brevity. ◀

By Claim 13, the counterexample has no self-loop edges in $E[X]$. Our next claim strengthens this further by showing that the counterexample has no (X, Y)-cycle-cover with cycle edges.

Claim 14. *Every (X, X)-cycle-cover in G has no cycles.*

Proof. Suppose there exists an (X, X)-cycle-cover F in G with a cycle C induced by F. Let $e = \{u, v\}$ be an edge in the cycle. By Claim 13, we know that $u \neq v$. We observe that $\det(M(s)[X, X])$ is a quadratic function of $s[u, v]$, i.e., there exists $\alpha \in \{\pm 1\}$ such that the determinant of $M(s)[X, X]$ can be expressed as

$$\begin{aligned} \det(M(s)[X, X]) = {} & -s[u, v]^2 \cdot M[u, v]^2 \cdot \det(M(s)[X - \{u, v\}, X - \{u, v\}]) \\ & + 2\alpha \cdot s[u, v] \cdot M[u, v] \cdot \det(M^{\overline{e}}(s)[X - u, X - v]) \\ & + \det(M^{\overline{e}}(s)[X, X]). \end{aligned} \tag{3}$$

Furthermore, $F-e$ is an $(X-u,X-v)$-cycle-cover in G. Since we have a smallest counterexample, it follows that there exists a symmetric signing s such that $\det(M^{\overline{e}}(s)[X-u,X-v])\neq 0$. We now define the quadratic function

$$\begin{aligned}f(x) &:= -x^2\cdot M[u,v]^2\cdot\det(M(s)[X-\{u,v\},X-\{u,v\}])\\ &\quad+2\alpha x\cdot M[u,v]\cdot\det(M^{\overline{e}}(s)[X-u,X-v])+\det(M^{\overline{e}}(s)[X,X]),\end{aligned}$$

and consider its roots. Since $\det(M^{\overline{e}}(s)[X-u,X-v])\neq 0$, the sum of the roots of this quadratic equation is non-zero. Since the real roots of a quadratic function are symmetric about the critical point of the parabola defined by the function (i.e., symmetric about $\arg\min f(x)$), there exists $x\in\{\pm 1\}$ that is *not* a root of $f(x)$. Hence, either $\det(M(s)[X,Y])\neq 0$ or $\det(M(s')[X,Y])\neq 0$ where s' is a signing obtained from s by flipping on $\{u,v\}$. Thus, either s or s' contradict assumption B about the counterexample. ◀

By Claims 12 and 13, the counterexample has $X=Y$ with no loop edges in $E[X]$. Furthermore, by Claim 14, every (X,X)-cycle-cover in G has no cycles. By definition of (X,X)-cycle-covers, it follows that each (X,X)-cycle-cover in G corresponds to a perfect matching in $G[X]$. Let N be an (X,X)-cycle-cover in G.

Claim 15. *N is the unique (X,X)-cycle-cover in G.*

Proof. Let e be an arbitrary edge in N. Suppose there exists an (X,X)-cycle-cover N' in $G-e$. Then, Claims 13 and 14 imply that N' is also a perfect matching in $G[X]$. We consider $N'':=N\cup N'$. Since N and N' are perfect matchings in $G[X]$, the set of edges N'' induces a vertex-disjoint union of edges and cycles of even length in $G[X]$. Hence, N'' is an (X,X)-cycle-cover in G. Furthermore, since $e\in N\setminus N'$, it follows that N'' contains at least one cycle. This contradicts Claim 14. Thus, every edge $e\in N$ belongs to every (X,X)-cycle-cover in G. Consequently, N is the unique (X,X)-cycle-cover in G. ◀

Since N is the unique (X,X)-cycle-cover in G, by Lemma 10, we have that

$$\det(M(s)[X,X])=(-1)^{|N|}\prod_{\{u,v\}\in N}M(s)[u,v]^2$$

which is non-zero for every signing s. Thus, there exists a symmetric signing s such that $\det(M(s)[X,X])\neq 0$, a contradiction to assumption B about the counterexample. This completes the proof of the characterization. We note that the above proof of the characterization is constructive and immediately leads to the algorithm FindSigning(M,X,Y) in Fig. 2.

We now describe an efficient implementation of the non-trivial steps in FindSigning. In Step 1, the algorithm performs a brute-force search. We note that the search needs to be conducted only for the entries $s[u,v]$ where $u,v\in X\cup Y$ since $\det(M(s)[X,Y])$ is independent of the remaining entries of the signing s. Since $|X\cup Y|\leq 2$, the search can be conducted in constant time by picking an arbitrary sign for the remaining entries.

Lemma 9 implies that Steps 2 and 3.2 can be implemented to run in polynomial time. We recall that any cycle edge in an (X,X)-cycle-cover must be a cycle edge in some perfect 2-matching in $G[X]$. Claim 16 shows that Step 4.1 can be implemented to run in polynomial time. Finally, the recursive algorithm terminates in polynomial time since each recursive call reduces either $|X\cup Y|$ or the number of non-zero entries in M. ◀

Claim 16. *There is a polynomial-time algorithm that given a graph, finds an edge that belongs to a cycle in some perfect 2-matching of the graph or decides that no such edge exists.*

FindSigning(M, X, Y):

Input: $M \in \mathbb{R}^{n\times n}$ with support graph G, and $X, Y \subseteq [n]$ satisfying $|X| = |Y|$.

Output: A symmetric signing $s \in \{\pm 1\}^{n\times n}$ such that $M(s)[X, Y]$ is invertible if such a signing exists.

1. If $|X| = |Y| \leq 1$:
   1.1. If $M[X, Y] > 0$, then return $\mathbf{1}$ *(the all-positive signing)*.
   1.2. Else if $M[X, Y] < 0$, then return $-\mathbf{1}$ *(the all-negative signing)*.
   1.3. Else return “No Invertible Signing”.
2. If there exists no (X, Y)-cycle-cover then return “No Invertible Signing”.
3. If $E[X \setminus Y, Y] \cup E[Y \setminus X, X] \neq \emptyset$:
   3.1. Pick $e := \{u, v\} \in E[X \setminus Y, Y]$ such that $u \in X \setminus Y$ and $v \in Y$ or pick $e := \{u, v\} \in E[Y \setminus X, X]$ such that $u \in Y \setminus X$ and $v \in X$.
   3.2. If there is no $(X - u, Y - v)$-cycle-cover in G:
      3.2.1 $s \leftarrow$ FindSigning($M^{\overline{e}}, X, Y$).
   3.3. Else: *(when there is an $(X - u, Y - v)$-cycle-cover in G)*
      3.3.1 $s \leftarrow$ FindSigning($M, X - u, Y - v$).
   3.4. If $M(s)[X, Y]$ is invertible then return s.
   3.5. Else return s' obtained from s by flipping on $\{u, v\}$.
4. Else: *(when sets X and Y are identical)*
   4.1. If there exists an (X, Y)-cycle-cover in G with a cycle edge $\{u, v\}$:
      4.1.1. $s \leftarrow$ FindSigning($M, X - u, Y - v$).
      4.1.2. If $M(s)[X, Y]$ is invertible then return s.
      4.1.3. Else return s' obtained from s by flipping on $\{u, v\}$.
   4.2. Else: *(when all (X, Y)-cycle-covers are perfect matchings in $G[X]$)*
      4.2.1 Return $\mathbf{1}$ *(the all-positive signing)*.

Fig. 2. The algorithm FindSigning(M, X, Y).

FindCycleEdge(G):

Input: A graph G with vertex set V.

Output: An edge e that is a cycle edge in some perfect 2-matching in G if one exists.

1. If there exists no perfect 2-matching in G then return “No edge”.
2. Let F be a perfect 2-matching in G.
3. If F contains a cycle C then return any edge in C.
4. For $e \in F$:
   4.1. Let N_e be a perfect 2-matching in $G - e$ if one exists.
   4.2. If N_e exists and has a cycle C then return any edge in C.
5. If Step 4 finds N_e for some $e \in F$, then return e.
6. Else return “No edge”.

Fig. 3. The algorithm FindCycleEdge(G).

Proof. To prove the claim we consider the algorithm FindCycleEdge(G) in Fig. 3. If at any point we find a perfect 2-matching with a cycle then we return an edge from it. Hence, it only remains to show the correctness of Steps 5 and 6. Let F be a perfect 2-matching with no cycle edge. Suppose there exists a perfect 2-matching N_e for some edge e with no cycle edge. Then N_e and F are both perfect matchings in G. It follows that $N_e \cup F$ will be a perfect 2-matching where e is in a cycle and hence Step 5 is correct to return e. Now suppose that for all e there is no perfect 2-matching N_e. It follows that G has one unique perfect 2-matching F that is a perfect matching and hence Step 6 correctly returns that no cycle edge exists.

Using the algorithm from Lemma 9 we can perform Steps 1, 2, and 4.1 in polynomial time. Thus, FindCycleEdge(G) runs in polynomial time. ◀

4. Counting invertible signings

In this section we turn our attention to counting the number of invertible signings and prove Theorem 3. To this end, we identify a linear multivariate polynomial f such that it is the identically zero polynomial if and only if an associated graph has no perfect 2-matching.

Definition 17. Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and let s be a symmetric $n \times n$ matrix of variables. Let G be the support graph of M, let $\mathcal{F}$ be the set of all perfect 2-matchings in G, and let $\lambda_F \in \{\pm 1\}$ for all $F \in \mathcal{F}$ be the signs guaranteed by Lemma 10. We define f_G as the polynomial

$$f_G(s) := \sum_{F\in\mathcal{F}} \lambda_F \cdot 2^{|\mathrm{NTCs}(F)|} \cdot M(\mathbf{1})_{\mathrm{Matchings}(F)} \cdot M(s)_{\mathrm{Cycles}(F)}$$

where $\mathbf{1}$ is the all-positive signing.

Remark 1. We note that $f_G(s)$ is a linear function of $s[u,v]$ for every u, v (when all other entries are kept constant) and every term $\lambda_F \cdot 2^{|\mathrm{NTCs}(F)|} \cdot M(\mathbf{1})_{\mathrm{Matchings}(F)} \cdot M(s)_{\mathrm{Cycles}(F)}$ is a non-zero multiple of the monomial

$$\prod_{C\in\mathrm{Cycles}(F)} \prod_{\{u,v\}\in C} s[u,v].$$

Remark 2. We also note that $f_G(s) = \det(M(s))$ for all symmetric $n \times n$ signing s since $s[u,v]^2 = 1$ for all $u, v \in [n]$.

Lemma 18. *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and s be a symmetric $n \times n$ matrix of variables. Let G be the support graph of M and $\mathcal{F}$ be the set of all perfect 2-matchings in G. With this notation, the polynomial $f_G(s)$ is the zero polynomial if and only if $\mathcal{F} = \emptyset$.*

Proof. If $\mathcal{F} = \emptyset$ then $f_G(s)$ is the zero polynomial by definition of $f_G(s)$. We now show that if $\mathcal{F} \neq \emptyset$ then there exists a monomial in $f_G(s)$ that has non-zero coefficient. Let F be a perfect 2-matching in G. We will now show that the monomial

$$g(s) := \prod_{C\in\mathrm{Cycles}(F)} \prod_{\{u,v\}\in C} s[u,v]$$

has non-zero coefficient in $f_G(s)$. It is sufficient to prove that there exists at least one term in $f_G(s)$ that is a non-zero multiple of $g(s)$ and every term in $f_G(s)$ that is a non-zero multiple of $g(s)$ has the same sign. Since F is a perfect 2-matching in G we know from the definition of $f_G(s)$ that $\lambda_F \cdot 2^{|\mathrm{NTCs}(F)|} \cdot M(\mathbf{1})_{\mathrm{Matchings}(F)} \cdot M(s)_{\mathrm{Cycles}(F)}$ is a term in $f_G(s)$ and is a non-zero multiple of $g(s)$. Now suppose that there exists another $F' \in \mathcal{F}$ such that $\lambda_{F'} \cdot 2^{|\mathrm{NTCs}(F')|} \cdot M(\mathbf{1})_{\mathrm{Matchings}(F')} \cdot M(s)_{\mathrm{Cycles}(F')}$ is a non-zero multiple of $g(s)$. We note that

$$\{\{u,v\} \mid \{u,v\} \in C, C \in \mathrm{Cycles}(F)\} = \{\{u,v\} \mid \{u,v\} \in C, C \in \mathrm{Cycles}(F')\}.$$

That is, every edge in a cycle of F is also in a cycle of F'. Hence, $\mathrm{Cycles}(F) = \mathrm{Cycles}(F')$. We recall that $\mathrm{Paths}(F) = \mathrm{Paths}(F') = \emptyset$ since F and F' are both perfect 2-matchings in G. Hence, by Lemma 10 we know that $\lambda_F = \lambda_{F'}$ and thus every term in $f_G(s)$ that is a non-zero multiple of $g(s)$ has the same sign as the term $\lambda_F \cdot 2^{|\mathrm{NTCs}(F)|} \cdot M(\mathbf{1})_{\mathrm{Matchings}(F)} \cdot M(s)_{\mathrm{Cycles}(F)}$. Hence, $f_G(s)$ is not the zero polynomial. ◀

Before we begin the proof of Theorem 3 we demonstrate the power of Lemma 18 by providing a second nonconstructive proof of Theorem 6 using the following celebrated result of Alon.

Theorem 19 (*Combinatorial Nullstellensatz [25]*). *Let $\mathbb{F}$ be an arbitrary field and f be a multivariate polynomial over $\mathbb{F}$ with variables $x_1, \ldots, x_n$. Let $t_1, \ldots, t_n$ be non-negative integers such that the degree of f equals $\sum_{i=1}^{n} t_i$, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is non-zero. Given subsets $S_1, \ldots, S_n$ of $\mathbb{F}$ with $|S_i| > t_i$ for each i, there is a tuple $(s_1, \ldots, s_n)$ satisfying $f(s_1, \ldots, s_n) \neq 0$, where each s_i is selected from S_i.*

Alternative Proof of Theorem 6. Suppose there exists a perfect 2-matching in G. We will apply Combinatorial Nullstellensatz (Theorem 19) by taking the field $\mathbb{F}$ to be the reals. Let s be a $n \times n$ symmetric matrix of variables. Then by Lemma 18 the linear multivariate polynomial $f(s)$ is not identically zero. That is, the polynomial $f_G(s)$ has a term

$$\prod_{u=1}^{n} \prod_{v=u}^{n} s[u,v]^{t_{\{u,v\}}}$$

with non-zero coefficient such that $t_{\{u,v\}} \in \{0,1\}$ for all $u, v \in [n]$ where $v \leq u$ and

$$\sum_{u=1}^{n} \sum_{v=u}^{n} t_{\{u,v\}}$$

is the degree of $f(s)$. Now consider $S_{\{u,v\}} := \{\pm 1\}$ for every $u, v \in [n]$ with $v \leq u$. We immediately have that $|S_{\{u,v\}}| > t_{\{u,v\}}$ for all $u, v \in [n]$ where $v \leq u$. Hence, by Combinatorial Nullstellensatz (Theorem 19), there must exist an $n \times n$ matrix z where every entry of z takes a value from $\{\pm 1\}$ with $z[u,v] = z[v,u]$ for all $u, v \in [n]$, and such that $f_G(z) \neq 0$. That is, there must exist a symmetric $n \times n$ signing z such that $\det(M(z)) = f_G(z) \neq 0$.

Now suppose there exists a symmetric $n \times n$ signing s' such that $M(s')$ is invertible. Then it follows that $f_G(s') = \det(M(s')) \neq 0$. Hence, $f_G(s)$ is not the zero polynomial and by Lemma 18 we know that there must exist a perfect 2-matching in G. ◀

To show Theorem 3 we use the following variant of Combinatorial Nullsetllensatz. We say that a monomial m in a multivariate polynomial f is *maximal* among the monomials in f with non-zero coefficients if for any other monomial m' in f with non-zero coefficient, there exists a variable such that its degree in m is greater than or equal to its degree in m'.

Lemma 20 (*Coefficient Formula [26,27]*). *Let $\mathbb{F}$ be an arbitrary field and f be a multivariate polynomial over $\mathbb{F}$ with variables $x_1, \ldots, x_n$. Let $t_1, \ldots, t_n$ be non-negative integers such that the degree vector $(t_1, \ldots, t_n)$ of variables $(x_1, \ldots, x_n)$ is maximal among the monomials in f with non-zero coefficients. For every $i \in [n]$, let S_i be a subset of $\mathbb{F}$ with $|S_i| = t_i + 1$ and let $\phi_i(x_i) := \prod_{s_i \in S_i}(x_i - s_i)$. Then the coefficient of $x_1^{t_1} \cdots x_n^{t_n}$ is equal to*

$$\sum_{s_1 \in S_1} \cdots \sum_{s_n \in S_n} \frac{f(s_1, \ldots, s_n)}{\phi_1'(s_1) \cdots \phi_n'(s_n)},$$

where $\phi_i'(x_i)$ is the derivative of $\phi_i(x_i)$ with respect to x_i.

We now have all the ingredients necessary to prove Theorem 3.

Proof of Theorem 3. Let

$$\Gamma_1(M) := \left\{ s \in \{\pm 1\}^{E(G)} \,\middle|\, M(s) \text{ is invertible} \right\}, \text{ and}$$
$$\Gamma_0(M) := \left\{ s \in \{\pm 1\}^{E(G)} \,\middle|\, M(s) \text{ is singular} \right\}.$$

With this notation, we have that $\gamma(M) = |\Gamma_1(M)|$.

Let $\mathcal{F}$ be the collection of perfect 2-matchings in G. By assumption, we know that $\mathcal{F}$ is non-empty. We recall that $f_G(s) = \det M(s)$ for every signing s.

Among the perfect 2-matchings in G with maximum number of cycles, let F^* be one with maximum number of cycle edges. Let k denote the number of cycle edges in F^* with cycle edges labeled as $1, \ldots, k$ and let ℓ denote the number of matching edges in F^* with matching edges labeled as $k+1, \ldots, k+\ell$. Thus, the number of vertices in G is $k+2\ell$ and the edges with labels $k+\ell+1, \ldots, m$ are not in F^*. We define the polynomial

$$P(x) := f_G(x) \cdot \left(\prod_{i=k+1}^{m} x_i \right).$$

We will apply the coefficient formula (Lemma 20) on P, with $t_i = 1$ and $S_i = \{\pm 1\}$ for all i. Since F^* is a perfect 2-matching with maximum number of cycle edges, the degree vector of the monomial $M(x)_{\text{Cycles}(F^*)} \cdot \left(\prod_{i=k+1}^{m} x_i\right)$ is maximal with the absolute value of the coefficient of this monomial being at least $2^{|\text{NTCs}(F^*)|}$. We emphasize that this maximal term might not be the one with maximum degree in P. We observe that $\phi_i'(s_i) \in \{\pm 2\}$ for every $s_i \in S_i$. By Lemma 20, we have the inequality

$$\begin{aligned}
2^{|\text{NTCs}(F^*)|} &\le \left| \frac{\sum_{s \in \{\pm 1\}^{E(G)}} P(s)}{2^m} \right| \\
&\le \frac{\sum_{s \in \Gamma_1(M)} |P(s)| + \sum_{s \in \Gamma_0(M)} |P(s)|}{2^m} = \frac{\sum_{s \in \Gamma_1(M)} |P(s)|}{2^m} \\
&\le \frac{\gamma(M) \cdot \max_s \left| f_G(s) \cdot \prod_{i=k+1}^{m} s_i \right|}{2^m} \\
&= \frac{\gamma(M) \cdot \max_s |\det M(s)|}{2^m} \\
&\le \frac{\gamma(M) \cdot 2^{|\text{NTCs}(F^*)|} \cdot \mu(G)}{2^m},
\end{aligned}$$

where the last inequality follows from the fact that F^* is chosen to be a perfect 2-matching with maximum number of cycles, so each perfect 2-matching contributes at most $2^{|\text{NTCs}(F^*)|}$ to the term $\max_s |\det M(s)|$. Thus, we have

$$\gamma(M)\mu(G) \ge 2^m. \quad \blacktriangleleft$$

5. Finding singular signings of bipartite graphs

In this section, we characterize bipartite graphs whose signed adjacency matrix is invertible for all signings. We use this characterization to prove Theorem 4. We need two structural results (Lemma 21 and Theorem 22) which are extensions of results due to Little [28] for our characterization. We include their proofs for the sake of completeness.

Lemma 21 (*Little [28]*)**.** *Let G be a graph with adjacency matrix A_G. Then $\det(A_G(s))$ is even for all signings s if and only if G has an even number of perfect matchings.*

Proof. We recall that the permutation expansion of the determinant of $A_G(s)$ is

$$\det(A_G(s)) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \cdot \prod_{i=1}^{n} A_G(s)[i, \sigma(i)]$$

for all signings s.

To prove the claim, it is sufficient to show that each perfect 2-matching in G that is not a perfect matching must contribute an even number of distinct nonzero terms of the same sign to the determinant of $A_G(s)$ for all signings s.

Let M be a perfect 2-matching in G that is not a perfect matching. Let C be the set of vertex disjoint cycles in M. Then M corresponds to $2^{|C|}$ distinct permutations of n and thus, leads to $2^{|C|}$ distinct terms in the permutation expansion of the determinant of $A_G(s)$ for all signings s. Since all such permutations have the same cycle structure it follows that all corresponding terms have the same sign. ◀

Theorem 22 (*Little [28]*). *Let G be a graph. Then G has an even number of perfect matchings if and only if there is a set $S \subseteq V(G)$ such that every vertex in G has even number of neighbors in S. Moreover, if G has an even number of perfect matchings, then such a set S can be found in polynomial time.*

Proof. Suppose G has an even number of perfect matchings. Then by Lemma 21, we have that $\det(A(G))$ is even. Thus, the matrix $A(G)$ is singular under modular 2 arithmetic. Consequently, there must exist a subset of columns of $A(G)$ that sum to the zero vector modulo 2. Let Q be such a set of columns and $S \subseteq V(G)$ be the set of vertices that correspond to the columns. We note that using Gaussian elimination on $A(G)$ under modular 2 arithmetic we can find such a Q in polynomial time and thus can find S in polynomial time. Then $\sum_{q\in Q} A(G)[r,q] \equiv 0 \mod 2$ for all $r \in [n]$. Thus, every vertex $v \in V(G)$ must have an even number of neighbors in S.

Now suppose there exists $S \subseteq V(G)$ such that $|N_G(v) \cap V(G)|$ is even for all $v \in V(G)$ where $N_G(v)$ denotes the *non-inclusive neighborhood* of v in G. Let Q be the set of columns of $A(G)$ that correspond to the vertices of S. Then $\sum_{q\in Q} A(G)[r,q] \equiv 0 \mod 2$ for all $r \in [n]$. Thus, $A(G)$ is singular under modular 2 arithmetic and $\det(A(G)) \equiv 0 \mod 2$. Therefore, by Lemma 21, the number of perfect matchings must be even. ◀

We now have the ingredients to characterize bipartite graphs whose signed adjacency matrix is invertible for all signings.

Lemma 23. *Let G be a bipartite graph and let A_G be the adjacency matrix of G. Then $\det(A_G(s)) \neq 0$ for all signings s if and only if G has an odd number of perfect matchings.*

Proof. Suppose G has an odd number of perfect matchings. By Lemma 21, we have that $\det(A_G(s)) \neq 0$ for all signings s.

Now suppose that G has an even number of perfect matchings. By Theorem 22, there exists a set $S \subseteq V(G)$ such that $|N_G(v) \cap S|$ is even for all $v \in V(G)$, where $N_G(v)$ is the non-inclusive neighborhood of v in G. We observe that the subgraph $G[S]$ induced by S is bipartite with every vertex having even degree. Thus, any closed walk on $G[S]$ has even number of edges and every connected component in $G[S]$ has an Eulerian tour with even number of edges. Let C be a connected component of $G[S]$ with m edges and let $T := (e_1, e_2, \dots, e_m)$ be an ordering of the edges that represents an Eulerian tour of C. Then we sign edge e_i to be positive if i is even and negative otherwise. Every vertex $v \in V(G) \setminus S$ has even number of edges between v and vertices in S. We partition the edges incident to v into two arbitrary parts of equal size and sign all the edges in one part to be positive and the rest of the edges in the other part to be negative. Let $\hat{s}$ denote the resulting signing.

Under the signing $\hat{s}$ every vertex v of G has an equal number of positive and negative edges to vertices in S. Thus, the sum of the column vectors corresponding to the vertices in S will be zero and hence $\det(A_G(\hat{s})) = 0$. ◀

We note that the proof of Lemma 23 is constructive since we can find a set S for which every vertex has even number of neighbors in S in polynomial time by Theorem 22. Thus, Theorem 4 follows from Theorem 22 and Lemma 23.

6. Minimum support increase to obtain an invertible signing

In this section, we study the problem of computing the solvability index of real symmetric matrices, thus proving Theorem 5. We recall the following definition: For a real symmetric matrix M, the *solvability index* of M is the smallest number of off-diagonal zero entries that need to be converted to non-zeros so that the resulting *symmetric* matrix has an invertible signing. We remind the reader that the support-increase operation preserves symmetry.

By our characterization in Theorem 6, computing the solvability index of a matrix reduces to the following edge addition problem:

> EdgeAdd: Given a graph G (possibly with self-loops), find $\min |F|$ where F ranges over all sets of non-edges of G with no loops such that $G + F$ has a perfect 2-matching.

In the above, $G + F$ denotes the graph obtained by adding the edges in F to G. In the rest of the section, we will show that EdgeAdd can be solved efficiently, which will imply Theorem 5.

Theorem 24. *There is a polynomial-time algorithm to solve* EdgeAdd.

We need some terminology from matching theory. Let G be a graph on vertex set V and edge set E. For a subset S of vertices, denote the *induced subgraph* of G on S as $G[S]$ and the *non-inclusive neighborhood* of S in G by $N_G(S)$. We recall that a *matching* M in G is a subset of edges where each vertex is incident to at most one edge in M. Let $\nu(G)$ denote the cardinality of a *maximum matching* in G and let

$$\nu_f(G) := \max \left\{ \sum_{e\in E} x_e \;\middle|\; \sum_{e\in\delta(v)} x_e \le 1, \text{ and } x_e \ge 0 \text{ for all } e \in E \right\}$$

denote the value of a *maximum fractional matching* in G. For a matching M, we define a vertex u to be *M-exposed* if none of the edges of M are incident to u, and a vertex v to be an *M-neighbor* of u if edge $\{u, v\}$ is in M. A vertex u in V is said to be *inessential* if there exists a maximum cardinality matching M in G such that u is M-exposed, and is said to be *essential* otherwise. A graph H is *factor-critical* if there exists a perfect matching in $H - v$ for every vertex v in H. The following result is an immediate consequence of the odd-ear decomposition characterization of Lovász [29].

Lemma 25 (*Lovász [29]*)**.** *If G is a factor-critical graph, then G has a perfect 2-matching.*

The *Gallai–Edmonds decomposition* [30–32] of a graph G is a partition of the vertex set of G into three sets (B, C, D), where B is the set of inessential vertices, $C := N_G(B)$, and $D := V \setminus (B \cup C)$. Let B_1 denote the set of isolated vertices in $G[B]$ and $B_{\ge 3} := B \setminus B_1$. For notational convenience, we will denote the Gallai–Edmonds decomposition as $(B = (B_1, B_{\ge 3}), C, D)$. The Gallai–Edmonds decomposition of a graph is unique and can be found efficiently [32]. The following theorem summarizes the properties of the Gallai–Edmonds decomposition that we will be using (properties (i) and (ii) are well-known and can be found in Schrijver [33] while property (iii) follows from results due to Balas [34] and Pulleyblank [35]—see Bock, Chandrasekaran, Könemann, Peis, and Sanità [36] for a proof of property (iii)):

Theorem 26. *Let $(B = (B_1, B_{\ge 3}), C, D)$ be the Gallai–Edmonds decomposition of a graph G. We have the following properties:*

EdgeAdd(G):
Input: A graph G with no isolated vertices and no self-loops.
Output: A set F of non-edges of G such that $G+F$ contains a perfect 2-matching.

1. Find the Gallai-Edmonds decomposition $(B=(B_1,B_{\geq 3}),C,D)$ of G.
2. Find a maximum matching M that matches the largest number of B_1 vertices.
3. Let $S := \{u \in B_1 \mid u \text{ is } M\text{-exposed}\}$.
4. If $|S|$ is even:
   Pick an arbitrary pairing of the vertices in S.
5. If $|S|$ is odd:
   Consider a vertex s in S, pick a vertex t in $N_G(s)$ and let u be the M-neighbor of t.
   Pair up u with s and pick an arbitrary pairing of the vertices in $S \setminus \{s\}$.
6. Return the set of pairs F.

Fig. 4. The algorithm EdgeAdd(G).

(i) Each connected component in $G[B]$ is factor-critical.
(ii) Every maximum matching M in G contains a perfect matching in $G[D]$ and matches each vertex in C to distinct components in $G[B]$.
(iii) Let M be a maximum matching that matches the largest number of B_1 vertices. Then there are $2(\nu_f(G)-\nu(G))$ M-exposed vertices in $B_{\geq 3}$.

We observe that G contains a perfect 2-matching if and only if $\nu_f(G)=|V|/2$. Therefore, adding edges to get a perfect 2-matching in G is equivalent to adding edges to increase the maximum fractional matching value to $|V|/2$.

Proof of Theorem 24. We will assume that G has no isolated vertices and no self-loops in the rest of the proof. We make this assumption here in order to illustrate the main idea underlying the algorithm. This assumption can be relaxed by a case analysis in the algorithm as well as the proof of correctness.

We use the algorithm EdgeAdd(G) given in Fig. 4. We briefly describe an efficient implementation for Step 2, since it is easy to see that other steps can be implemented efficiently. In order to find a maximum matching that matches the largest number of B_1 vertices (as mentioned in property (iii) of Theorem 26), we first find the Gallai–Edmonds decomposition and a maximum matching M. Then, we repeatedly augment M by searching for M-alternating paths (of even-length) from M-exposed B_1 vertices. This approach can be implemented to run in polynomial time. Alternatively, Step 2 can also be implemented by solving a maximum weight matching with suitably chosen weights.

We now argue the correctness of the algorithm. We first show that if $|S|$ is odd, then there is a choice of vertices t and u as described in the algorithm EdgeAdd(G). This is because G has no isolated vertices and hence there exists a vertex t in $N_G(s)$. Moreover, by Theorem 26, since s is in B_1, it follows that t is in C and thus t is matched by M to a vertex u in B. Now, Claim 27 proves feasibility and bounds the size of the returned solution F while Claim 28 proves the optimality. ◀

Claim 27. *The algorithm* EdgeAdd(G) *returns a set F of non-edges of G such that (1) $G+F$ contains a perfect 2-matching, and (2) $|F|=\lceil |V|/2-\nu_f(G)\rceil$.*

Proof. By property (ii) of Theorem 26, the set F is a set of non-edges of G. We will construct a perfect 2-matching in $G+F$. By property (i) of Theorem 26, every component in $G[B_{\geq 3}]$ is factor-critical. By Lemma 25, every component K in $G[B_{\geq 3}]$ contains a perfect 2-matching x^K. Let N_K denote the support of x^K. Let $\mathcal{K}$ denote the components in $G[B_{\geq 3}]$ that contain an M-exposed vertex. We have two cases:

Case 1: Suppose $|S|$ is even. Let N denote the set of edges of M that do not match any vertices in $\bigcup_{K\in\mathcal{K}} V(K)$. Now, the set of edges induced by $\left(\bigcup_{K\in\mathcal{K}} N_K\right) \cup N \cup F$ has a perfect 2-matching. A perfect 2-matching x in $G+F$ can be obtained by assigning $x(e) := x^K(e)$ for edges e in $\bigcup_{K\in\mathcal{K}} N_K$, $x(e) := 2$ for edges e in $N \cup F$, and $x(e) := 0$ for the remaining edges in $G+F$.

Case 2: Suppose $|S|$ is odd. Let N denote the set of edges of $M \setminus \{\{t,u\}\}$ that do not match any vertices in $\bigcup_{K\in\mathcal{K}} V(K)$. Now, $\left(\bigcup_{K\in\mathcal{K}} N_K\right) \cup N \cup (F \setminus \{s,u\}) \cup \{\{t,u\},\{s,t\},\{s,u\}\}$ has a perfect 2-matching. We note that the edges $\{t,u\},\{s,t\}$ were already present in the graph owing to the choice of c and u while the edge $\{s,u\}$ was added as an edge from F. A perfect 2-matching x in $G+F$ can be obtained by assigning $x(e) := x^K(e)$ for edges $e \in \bigcup_{K\in\mathcal{K}} N_K$, $x(e) := 1$ for edges e in $\{\{t,u\},\{s,t\},\{s,u\}\}$, $x(e) := 2$ for edges e in $N \cup (F \setminus \{s,u\})$, and $x(e) := 0$ for the remaining edges in $G+F$.

Next we find the size of the set F returned by the algorithm. We observe that $|F| = \lceil |S|/2 \rceil$. It remains to bound $|S|$. For this, we count the number of vertices in the graph using the matched and exposed vertices. We have that $|V| = 2|M| + |S| + \{\text{number of } M\text{-exposed vertices in } B_{\geq 3}\}$. By property (iii) of Theorem 26 and the choice of the matching M, we have $|V| = 2|M| + |S| + 2(\nu_f(G) - \nu(G))$. Since M is a maximum cardinality matching, we know that $|M| = \nu(G)$ and hence, $|S| = |V| - 2\nu_f(G)$. ◀

Our next claim shows a lower bound on the optimal solution that matches the upper bound and hence proves the optimality of the returned solution.

Claim 28. *Let F' be a set of non-edges of G. Suppose $G + F'$ has a perfect 2-matching. Then $|F'| \geq \lceil |V|/2 - \nu_f(G) \rceil$.*

Proof. We first note that the addition of a non-edge can increase the value of the maximum fractional matching by at most one. That is, for every graph H and every non-edge e of H, we have $\nu_f(H+e) - \nu_f(H) \leq 1$ (this can be shown by considering the dual problem, namely the minimum fractional vertex cover). Now, consider an arbitrary ordering of the edges in the solution F' and let F'_i denote the set of first i edges according to this order and let $F'_0 = \emptyset$. Then,

$$\nu_f(G+F') - \nu_f(G) = \sum_{i=1}^{|F'|} \left(\nu_f(G+F_i) - \nu_f(G+F_{i-1})\right) \leq |F'|.$$

Thus, we have $|F'| \geq \nu_f(G+F') - \nu_f(G)$. We observe that if $G+F'$ has a perfect 2-matching, then $\nu_f(G+F') = |V|/2$. Hence, $|F'| \geq |V|/2 - \nu_f(G)$. Finally, we observe that $|F'|$ has to be an integer and hence, $|F'| \geq \lceil |V|/2 - \nu_f(G) \rceil$. ◀

7. Conclusion

In this work we investigated several problems related to finding signings of symmetric matrices with natural spectral properties. We showed that NsdSigning and SingularODSigning are NP-complete. In contrast, we showed that InvertibleSigning admits an efficient algorithm. Moreover, we proved that SingularSigning admits an efficient algorithm when the input is the adjacency matrix of a bipartite graph. We complement our algorithmic and hardness results with a non-trivial lower bound on the number of invertible symmetric signings of a matrix whose support graph contains a perfect 2-matching. Finally, we gave a polynomial-time algorithm to find the solvability index of a symmetric matrix.

Our work raises several interesting open questions related to signing symmetric matrices. We state some of them here: Is it true that SingularSigning and InvertibleODSigning have the same complexity as SingularODSigning and InvertibleSigning respectively? Does there exist an efficiently verifiable characterization for the existence of singular signings for all simple graphs without self-loops (i.e., extend Theorem 4 to all simple graphs without self-loops)? For those signing problems which are solvable efficiently, is it also true that their search variants are efficiently solvable? We believe that answering these questions would be helpful in understanding the complexity of BoundedEvalueSigning for graph-related matrices. In addition to exact algorithms, designing approximation algorithms for BoundedEvalueSigning is also of interest.

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