Thin Partitions: Isoperimetric Inequalities and a Sampling Algorithm for Star Shaped Bodies

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Abstract

Star-shaped bodies are an important nonconvex generalization of convex bodies (e.g., linear programming with violations). Here we present an efficient algorithm for sampling a given star-shaped body. The complexity of the algorithm grows polynomially in the dimension and inverse polynomially in the fraction of the volume taken up by the kernel of the star-shaped body. The analysis is based on a new isoperimetric inequality. Our main technical contribution is a tool for proving such inequalities when the domain is not convex. As a consequence, we obtain a polynomial algorithm for computing the volume of such a set as well. In contrast, linear optimization over star-shaped sets is NP-hard.

1 Introduction

Convexity has been a cornerstone of fundamental polynomial-time algorithms for continuous as well as discrete problems [GLS88]. The basic problems of optimization, integration and sampling in $\mathbb{R}^n$ can be solved efficiently (to arbitrary approximation) for convex bodies given only by oracles. More precisely,

- Optimization. Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$, a convex body K specified by a membership oracle and a point in K, and $\epsilon > 0$, find a point $x^* \in K$ s.t. $f(x^*) \leq \min_K f(x)$. This can be done using either the Ellipsoid algorithm [YN76, GLS88], Vaidya's algorithm [Vai96] or the random walk approach [BV04, LV06a]. For important special cases such as linear programming, there are several alternative approaches.

- Integration. Given an integrable logconcave function $f : \mathbb{R}^n \to \mathbb{R}_+$ as an oracle, and $\epsilon > 0$, find A s.t. $(1-\epsilon)\int f < A < (1+\epsilon)\int f$. This is done using a Monte Carlo algorithm based on sampling logconcave densities [DFK91, AK91, LV06b, LV06a].

- Sampling. Any logconcave density can be sampled efficiently [LV06a]. The sampling algorithm is based on a suitable random walk.

For the above problems and related applications, both the algorithms and their analyses rely heavily on the assumption of convexity or its natural extension, logconcavity. For example, for optimization, all the known algorithms use the fact that a local optimum is a global optimum. Similarly, a key step in the analysis of sampling algorithms is the derivation of isoperimetric inequalities, which are currently known for logconcave functions. Even the proofs of these inequalities (more on this presently) are based on techniques that fundamentally assume convexity. The main motivation of this paper is the following: *for what nonconvex bodies/distributions, can the above basic problems be solved efficiently?*

In this paper, we consider a well-studied generalization of convex bodies called *star-shaped* bodies. Star-shaped sets come up naturally in many fields, including computational geometry [PS85], integral geometry, mixed integer programming, etc. [Cox73]. A star-shaped set has at least one point such that every line through the point has a convex intersection with the set. Alternatively, star-shaped sets can be viewed as the union of convex sets, with all the convex sets having a nonempty intersection. The subset of points that can "see" the full set is called the kernel of the star-shaped set.

A compelling example of a star-shaped set is the "k-out-of-m-inequalities" set, i.e., the set of points that satisfy at least k out of a given set of m linear inequalities, with the assumption that there is a feasible solution to all m. In this case the kernel is the intersection of all m inequalities. Another interesting special case is that of "k-out-of-m-polytopes", where we have m polytopes with a nonempty intersection and feasible points are required to lie in at least k of the m polytopes. These and other special cases have been studied and applied extensively in operations research [RP94, Mat94, Cha05]. Not surprisingly, linear optimization over even these special cases is NP-hard [LSN07].

This might suggest that the problems of sampling and integration are also intractable over star-shaped bodies. Indeed convex optimization is reducible to sam-

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pling. Our main result (Theorem 1.3) is that, to the contrary, *star-shaped bodies can be sampled efficiently*, with the complexity growing as a polynomial in $n, 1/\epsilon, \ln D$ and $1/\eta$, where n is the dimension, ϵ is an error parameter denoting distance to the true uniform distribution, D is the diameter of the body and η is the fraction of the volume taken up by the kernel; we assume that we are given membership oracles for S as well as for its kernel K and a point x_0 so that the unit ball around x_0 is contained in S. (For the particular cases considered above, these oracles are readily available). The sampling algorithm leads to an efficient algorithm for computing the volume of such a set as well. We note here that linear optimization remains NP-hard even when the kernel takes up most of the volume.

A reader familiar with sampling algorithms for convex bodies will recall that such an analysis crucially uses isoperimetric inequalities. Here we prove isoperimetric inequalities for star-shaped sets (Theorems 1.1, 1.2). The key technical contribution of this paper is the proof of these inequalities and a new tool we develop for this purpose, which is also of independent interest. We refer to this tool as a *thin decomposition* of a set. The other crucial ingredients for efficient sampling (local conductance, coupling, etc...) extend naturally from the convex case to the star-shaped one. Therefore building on this new isoperimetry, we are able to show that the *ball walk* provides an efficient sampler for star-shaped bodies.

In the rest of this section, we give some context for thin partitions.

The common ingredient of most proofs of isoperimetric inequalities for convex bodies is the localization lemma, introduced by Lovász and Simonovits [LS93]. The approach is based on proof by contradiction. If a certain target inequality is false in $\mathbb{R}^n$, then there exists an essentially one-dimensional object over which it is still false. The proof is then completed by proving a one-dimensional inequality. This approach has been quite successful for convex bodies and logconcave functions and for proving many other inequalities in convex geometry. These, in turn, have played an essential role in the analysis of algorithms for convex bodies.

However, this approach does not seem to work for nonconvex sets, since the resulting one-dimensional versions could be nonconvex or nonlogconcave (e.g., for star-shaped bodies, convexity holds along lines that intersect the kernel but is not required along lines that do not intersect the kernel). To overcome this, we use partitions of $\mathbb{R}^n$ induced by hyperplanes where each part is "long" in at most one direction. The overall proof strategy in applying the partition is proof by induction: we combine inequalities on all the parts to derive an inequality for the full set. The advantage of this (as opposed to proof by contradiction) is that a suitably strong inequality does not need to hold for *every* part; it suffices to hold for most parts.

1.1 Preliminaries

Let $S \subseteq \mathbb{R}^n$ be a compact body. Define the kernel of S as $K_S := \{x : x \in S : \forall y \in S \; [x, y] \subseteq S\}$. We say S is *star-shaped* if K_S is nonempty and let $\eta(S) = \mathsf{vol}(K_S)/\mathsf{vol}(S)$.

We denote the n-dimensional ball of radius r centered around a point x as $\mathbb{B}_n(x, r)$. The ball walk with step size δ in a set S is the following Markov process: At a point x in S, we pick a uniform random point in $\mathbb{B}_n(x, \delta)$ and move to the chosen point if it is in S and otherwise stay put. Let π_S denote the uniform measure on S and let σ_m denote the measure after m ball walk steps. For two probability distributions σ, τ, the *total variation* distance is

$$d_{tv}(\sigma, \tau) = sup_A(\sigma(A) - \tau(A))$$

1.2 Results

We begin with two isoperimetric inequalities for star-shaped bodies, one parametrized using the diameter and the other using the second moment.

THEOREM 1.1. *Let $S \subseteq R^n$ be a star-shaped body with diameter D and $\eta(S) > 0$. Then for any measurable partition (S_1, S_3, S_2) of S, we have that*

$$\mathsf{vol}(S_3) \geq \frac{\eta(S)}{4D} d(S_1, S_2) \min \{\mathsf{vol}(S_1), \mathsf{vol}(S_2)\}$$

where $d(S_1, S_2)$ is the minimum distance between a point in S_1 and a point in S_2.

The above theorem is nearly the best possible as shown by a construction in Theorem 3.1.

THEOREM 1.2. *Let $S \subseteq R^n$ be a star-shaped body with $\eta(S) > 0$ and $M_S = \mathsf{E}_S[\|X - \mu_S\|^2]$ where μ_S is the centroid of S. Then for any measurable partition (S_1, S_3, S_2) of S, we have that either*

$$\mathsf{vol}(S_3) \geq \frac{\eta(S)}{4} \mathsf{vol}(S)$$

or

$$\mathsf{vol}(S_3) \geq \frac{\eta(S)^{\frac{3}{2}}}{16\sqrt{M_S}} d(S_1, S_2) \min \{\mathsf{vol}(S_1), \mathsf{vol}(S_2)\}$$

where $d(S_1, S_2)$ is the minimum distance between a point in S_1 and a point in S_2.

Next, we turn to the complexity of sampling. We assume that we have an oracle for the star-shaped body S, a lower bound on η and an M-warm start σ_O for the random walk, i.e. an initial distribution on S such that $\forall A \subseteq S$, $\sigma_0(A) \leq M\pi_S(A)$.

THEOREM 1.3. *Let $S \subseteq \mathbb{R}^n$ be a star-shaped body of diameter D with kernel K_S satisfying $\mathbb{B}_n(0,1) \subseteq K_S$. Let π_S be uniform distribution over S and $\epsilon > 0$. Given a random point x_0 from a distribution σ_0 such that σ_0 is an M-warm start for π_S, then there exists an absolute constant C such that, after*

$$m > \frac{Cn^2D^2M^2}{\eta(S)^2\epsilon^2} \log \frac{2M}{\epsilon}$$

steps of the ball walk with radius $\frac{\epsilon}{8M\sqrt{n}}$, we have $d_{TV}(\sigma_m, \pi_S) < \epsilon$.

THEOREM 1.4. *Let $S \subseteq \mathbb{R}^n$ be a star-shaped body of diameter D with kernel K_S. Suppose we are given membership oracles for K_S and S and a point x_0 such that $\mathbb{B}_n(x_0,1) \subseteq K_S$. Then, for any $\epsilon > 0$, a nearly random point X from S can be produced using amortized $O^*(n^3/(\eta(S)^4\epsilon^2))$ oracle calls with the guarantee that the distribution σ of X satisfies $d_{TV}(\sigma, \pi_S) < \epsilon$.*

We note that up to the polynomial in η, this matches the best-known bounds for sampling convex bodies.

2 Thin Decompositions via Bisection

DEFINITION 2.1. *Let $S \subseteq \mathbb{R}^n$. We define S to be a compact body if S is compact, has non-empty interior, and satisfies $\mathrm{cl}(S^\circ) = S$, where $\mathrm{cl}(S^\circ)$ denotes the closure of the interior of S.*

Let $S \subseteq \mathbb{R}^n$ be a compact body. A decomposition of S is a finite collection $\mathcal{P} = \{P_1, \ldots, P_k\}$ of compact bodies such that

1. $S = \cup_{i=1}^k P_i$
2. $P_i \cap P_j = \partial P_i \cap \partial P_j$, $1 \le i < j \le k$

Furthermore, we define a decomposition $\mathcal{P}$ to be ϵ-thin *if each $P \in \mathcal{P}$ is contained in a cylinder of radius at most ϵ.*

For completeness, we state without proof the following simple lemma.

LEMMA 2.1. *Let $S \subseteq \mathbb{R}^n$ be a compact body.*

1. *Let N be a decomposition of S, and let $S' \subseteq S$ be a compact body. Then $N' = \{\mathrm{cl}((P \cap S)^\circ) : P \in N, P \cap S^\circ \neq \emptyset\}$ is a decomposition of S'.*
2. *Let N be a decomposition of S, and let N' be a decomposition of an element $P \in N$. Then $N \cup N' \setminus P$ is a decomposition of S.*

The following simple lemma from [LS93] will be used repeatedly.

LEMMA 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be integrable, $n \ge 2$. Then for any point $z \in \mathbb{R}^n$, and any 2-dimensional linear subspace S of $\mathbb{R}^n$, there exists a hyperplane $H = \{x : a^T x = a^T z\}$, with $a \in S$, inducing halfspaces H^+, H^-, such that it equipartitions f, i.e.,*

$$\int_{H^+} f(x)\, dx = \int_{H^-} f(x)\, dx.$$

THEOREM 2.1. *For any integrable function $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{supp}(f) \subseteq S$, S a compact body, and $\int f dx = 0$, and any $\epsilon > 0$, there exists an ϵ-thin decomposition $\mathcal{P}$ of S such that each part $P \in \mathcal{P}$ is obtained by successive half space cuts from S and satisfies $\int_P f dx = 0$.*

Proof. Pick D such that $S \subseteq \mathbb{B}_n(0, D)$. Since S is compact we know that $D < \infty$. We start with the initial decomposition $\mathcal{P}_0 = \{S\}$ of S. From this decomposition, we will inductively build decompositions $\mathcal{P}_1, \ldots, \mathcal{P}_{n-1}$ with the following properties. For each i, $0 \le i \le n-1$, we have that for all $P \in \mathcal{P}_i$:

1. P is obtained from S via successive half space cuts.
2. $\int_P f dx = 0$
3. $\exists S \subseteq \mathbb{R}^n$, an i-dimensional linear subspace of $\mathbb{R}^n$ such that the orthogonal projection of P into S is contained inside of cuboid of side length at most $\delta = \frac{2\epsilon}{\sqrt{n}}$.

Assuming the above properties, one can easily see that each part in $\mathcal{P}_{n-1}$ is contained inside a cylinder of radius $\sqrt{n}\frac{\delta}{2} = \epsilon$, and hence $\mathcal{P}_{n-1}$ is an ϵ-thin decomposition of S compatible with f as needed. Hence, we only need to show how to perform the induction step.

Take $P \in \mathcal{P}_i$, $0 \le i \le n-2$. By assumption, there exists an i-dimensional linear subspace T such that $\pi_T(P)$, the orthogonal projection of P into T, is contained inside a cuboid of size length at most δ. Since $i \le n-2$, we may pick a 2 dimensional subspace $\widehat{T}$ orthogonal to T.

Let $Q = \mathrm{conv.hull}(P)$ and let $\Pi_{\widehat{T}}$ denote the orthogonal projection map from $\mathbb{R}^n$ onto $\widehat{T}$. Since $P \subseteq S \subseteq \mathbb{B}_n(0, D)$ and $\mathbb{B}_n(0, D)$ is convex, we know that $Q \subseteq \mathbb{B}_n(0, D)$. Therefore $\Pi_{\widehat{T}}(Q) \subseteq \mathbb{B}_n(0, D) \cap S \Rightarrow \mathsf{vol}_2(Q_{\widehat{T}}) \le \pi D^2$. Let $N = \{Q\}$. We perform the following iterative cutting procedure on N. Take an element $E \in N$. If $\mathsf{vol}_2(\Pi_{\widehat{T}}(E)) < \delta^2/2$ stop. Otherwise, letting μ denote the centroid of $\Pi_{\widehat{T}}(E)$, we have by Lemma 2.2 that there exists $H = \{x : a^t x = a^t \mu\}$, where $a \in S$, such that $\int_{E \cap H^-} f dx = \int_{E \cap H^+} f dx = 0$. Let $E_1 = E \cap H^-$, $E_2 = E \cap H^+$. Now set $N \leftarrow N \cup \{E_1, E_2\} \setminus E$. Since we are cutting through the centroid of $\Pi_S(E)$, and $\Pi_{\widehat{T}}(E)$ is convex, by Grunbaum's theorem we know that $\mathsf{vol}_2(\Pi_{\widehat{T}}(E_1)), \mathsf{vol}_2(\Pi_{\widehat{T}}(E_2)) \leq (1-\frac{1}{e})\mathsf{vol}_2(\Pi_{\widehat{T}}(E))$. Therefore, after a number of iterations depending only on D, we will have that every element $E \in N$ has $\mathsf{vol}_2(\Pi_{\widehat{T}}(E)) < \frac{\delta^2}{2}$.

LEMMA 2.3. *Let $E \in N$. There exists $v \in \widehat{T}$, $\|v\| = 1$, such that* $\text{width}_v(E) = \sup_{x\in E} v^T x - \inf_{x \in E} v^T x \leq \delta$.

Proof. Assume not, then note that the diameter of $\Pi_{\widehat{T}}(E)$ is at least δ. Let $[u, v]$ be a diameter inducing chord in $\Pi_{\widehat{T}}(E)$. Let w, z be points on opposite sides of $[u, v]$ such that their distances from the line $l(u, v)$ are maximum. Then the sum of the distances from w, z to $l(u, v)$ is at least δ and therefore the area of the quadrilateral induced by these four points is at least $\delta^2/2$, a contradiction. Hence there exists a direction v such that $\text{width}_v(E) \leq \delta$.

Note then that the orthogonal projection of E into the subspace spanned by v and T is contained inside a cuboid of size length at most δ as needed.

Hence N is now a decomposition of $Q = \text{conv.hull}(P)$, such that each element of $E \in N$ has $i+1$ orthogonal δ-thin directions. To transform N into a decomposition of P, we let $N' = \{\text{cl}((E \cap P)^\circ) : E \in N, E \cap P^\circ \neq \emptyset\}$. By adding N' to $\mathcal{P}_{i+1}$, we complete the induction step as needed to prove the theorem.

3 Application to Nonconvex Isoperimetry

The benefit of Theorem 2.1 is that it will allow us to derive isoperimetric inequalities for high-dimensional sets without requiring convexity along every line. We show an application to star-shaped bodies. To gain some intuition, it is useful to understand what the obstructions to isoperimetry in the star-shaped setting are, as well as to understand why star-shaped bodies have good isoperimetry at all. The following Theorem illustrates what a "canonical" bottleneck looks like in the star-shaped setting.

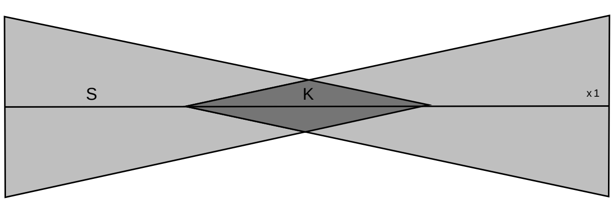

Figure 1: Star-Shaped Gluing of 2 Truncated Cones

THEOREM 3.1. *Let $H_n = \{x : x \in \mathbb{R}^n, x_1 = 0\}$. There exists an absolute constant $C > 0$, such that for all $\eta > 0$, there exists a sequence of symmetric star-shaped bodies $S_n \in \mathbb{R}^n$ centered at 0 such that for all $n \geq N_\eta$, we have that $\eta(S_n) = \Omega(\eta)$ and*

$$\mathsf{vol}_{n-1}(H_n \cap S_n) \leq C\left(\frac{\eta \ln(\frac{1}{\eta})}{(1-\eta)\text{Diam}(S_n)}\right)\mathsf{vol}_n(S_n)$$

Proof. [Proof of Theorem 3.1 (Isoperimetry: Upper Bound)] Our strategy here will be to reduce the above statement to one about one dimensional distributions on the real line. For each η, we will construct a candidate sequence S_n of star-shaped bodies which are rotationally symmetric about the x_1 axis. Then by analyzing the cross-sectional distributions of S_n and K_{S_n} along the x_1-axis, we will explicitly construct one dimensional asymptotic densities f_η, f_η^K to which the cross-sectional distributions of S_n and K_{S_n} respectively converge. We will then establish the required isoperimetry and kernel volume constraints for the sequence S_n and K_{S_n} by direct computation on f_η, f_η^K.

The geometry of our constructions is simple. As shown in Figure 1 previously, we will take two n-dimensional rotational cones with variance 1 along the x_1 axis, truncate them at their ends removing exactly an η fraction of their volume, and glue them together at the truncation sites. Choose l_n such that $\left(1 - \frac{l_n}{\sqrt{n(n+2)}}\right)^n = \eta$. Since $l_n \to \ln\left(\frac{1}{\eta}\right)$ we may choose N_η such that for $n \geq N_\eta$, $2l_n \leq \sqrt{n(n+2)}$. Now let

$$S_n = \left\{x : \sqrt{\sum_{i=2}^n x_i^2} \leq 1 - \frac{l_n + x}{\sqrt{n(n+2)}}, x_1 \in [-l_n, 0]\right\}$$
$$\bigcup \left\{x : \sqrt{\sum_{i=2}^n x_i^2} \leq 1 - \frac{l_n - x}{\sqrt{n(n+2)}}, x_1 \in [0, l_n]\right\}$$

From here one can easily verify that the kernel of S_n is

$$K_{S_n} = \left\{x : \sqrt{\sum_{i=2}^n x_i^2} \leq 1 - \frac{l_n - x}{\sqrt{n(n+2)}}, x_1 \in [-l_n, 0]\right\}$$
$$\bigcup \left\{x : \sqrt{\sum_{i=2}^n x_i^2} \leq 1 - \frac{l_n + x}{\sqrt{n(n+2)}}, x_1 \in [0, l_n]\right\}$$

Next, a simple computation reveals that the cross-sectional distribution of S_n is

$$f_n(x) = \begin{cases} \frac{\sqrt{n(n+2)}}{2(1-\eta)n}(1 - \frac{l_n + x}{\sqrt{n(n+2)}})^{n-1} : & x \in [-l_n, 0] \\ \frac{\sqrt{n(n+2)}}{2(1-\eta)n}(1 - \frac{l_n - x}{\sqrt{n(n+2)}})^{n-1} : & x \in [0, l_n] \\ 0 : & \text{otherwise} \end{cases}$$

Another computation, shows us that the cross-sectional density of K_{S_n} relative to S_n (we normalize by the volume of S_n) is

$$f_n^K(x) = \begin{cases} \frac{\sqrt{n(n+2)}}{2(1-\eta)n}(1 - \frac{l_n - x}{\sqrt{n(n+2)}})^{n-1}: & x \in [-l_n, 0] \\ \frac{\sqrt{n(n+2)}}{2(1-\eta)n}(1 - \frac{l_n + x}{\sqrt{n(n+2)}})^{n-1}: & x \in [0, l_n] \\ 0: & \text{otherwise} \end{cases}$$

From here, one can easily verify that the sequence f_n converges pointwise to the density function $f_\eta : \mathbb{R} \to \mathbb{R}^+$ where

$$f_\eta(x) = \begin{cases} \frac{\eta}{2(1-\eta)} e^{-x}: & x \in [-\ln(\frac{1}{\eta}), 0] \\ \frac{\eta}{2(1-\eta)} e^{x}: & x \in [0, \ln(\frac{1}{\eta})] \\ 0: & \text{otherwise} \end{cases}$$

Similarly the sequence f_n^K converges pointwise to f_η^K where

$$f_\eta^K(x) = \begin{cases} \frac{\eta}{2(1-\eta)} e^{x}: & x \in [-\ln(\frac{1}{\eta}), 0] \\ \frac{\eta}{2(1-\eta)} e^{-x}: & x \in [0, \ln(\frac{1}{\eta})] \\ 0: & \text{otherwise} \end{cases}$$

Notice that $f_\eta^K \leq f_\eta$ and that f_η^K is log-concave. We get that

$$\begin{aligned} \int_{\mathbb{R}} f_\eta^K(x)dx &= 2\frac{\eta}{2(1-\eta)} \int_0^{\ln(\frac{1}{\eta})} e^{-x} dx \\ &= 2\frac{\eta}{2(1-\eta)}(1-\eta) = \eta. \end{aligned}$$

The above computation shows that the volume fraction of the asymptotic kernel is indeed η as required. Clearly the length of the support of f_η is $2\ln(\frac{1}{\eta})$. Therefore we see that

$$f_\eta(0) = \frac{\eta}{2(1-\eta)} = \frac{\eta \ln(\frac{1}{\eta})}{(1-\eta)|\text{supp}(f_\eta)|}$$

where $|\text{supp}(f_\eta)|$ denotes the length of the support of f_η. Since $\frac{\mathsf{vol}(S_n \cap H_n)}{\mathsf{vol}(S_n)} = f_n(0) \to f_\eta(0)$ as $n \to \infty$, the above computation verifies the claim of the theorem passing through the asymptotics. The one thing left to justify is that $\text{Diam}(S_n) \to |\text{supp}(f_\eta)|$. As it is, this is not the case, but this can easily be achieved by scaling S_n orthogonally to the x_1 axis by a factor of $\frac{1}{n^2}$. By doing this, we are collapsing the sequence S_n onto the x_1 axis, without changing the cross sectional distribution along the axis, and hence asymptotically $\text{Diam}(S_n)$ will converge to $|\text{supp}(f)_\eta|$ as needed.

The above theorem gives us an upper bound on the isoperimetric coefficient of general star shaped sets. We note that the implicit isoperimetric cut above is $(H_n^- \cap S_n, H_n^+ \cap S_n)$, H_n^+ and H_n^- being the halfspaces induced by H_n, and where $\mathsf{vol}_n(H_n^+ \cap S_n) = \mathsf{vol}_n(H_n^+ \cap S_n) = \frac{1}{2}\mathsf{vol}_n(S_n)$ by symmetry of S_n. In the convex setting the isoperimetric coefficient is always $\Omega(1/\text{Diam}(S))$, and hence the above theorem shows us the rate at which isoperimetry must degrade from the convex setting as $\eta(S)$ decreases. In particular, from Figure 1, we observe how contrary to the convex setting we can get a V-shaped break in logconcavity of the cross-sectional volume distribution of a star-shaped body. On the other hand, as we will see later via Lemma 5.1, the severity of these breaks is strictly controlled by the kernel of S. For reference, in Lemma 5.1 we show that the cross-sectional distributions of a star-shaped body satisfy a form of restricted logconcavity with respect to the kernel. The rest of this section will be devoted to proving isoperimetric inequalities for star-shaped sets. In particular, in Theorem 1.1 we show isoperimetry for star-shaped bodies in terms of the diameter and η which in light of Theorem 3.1 is optimal within a factor of $O\left(\frac{\ln(\frac{1}{\eta})}{1-\eta}\right)$.

Lemma 3.3 forms the technical core of the isoperimetry proofs for star-shaped sets. Informally, we prove that for any thin enough hyperplane cut decomposition of a star-shaped set S, the parts of the decomposition that intersect the kernel of S are "almost" convex. This will in essence allow us to apply the isoperimetric inequalities developed for convex sets to the "almost" convex pieces from which we will extract the isoperimetric properties of S.

First we state and prove a few technical lemmas which will be useful.

Lemma 3.1. *Let $K_1, K_2 \subseteq \mathbb{R}^n$ be compact bodies and let π_1, π_2 denote the uniform measures on K_1, K_2 respectively. Then*

$$\mathsf{vol}(K_1 \triangle K_2) \leq \epsilon \min\{\mathsf{vol}(K_1), \mathsf{vol}(K_2)\} \Rightarrow d_{tv}(\pi_1, \pi_2) \leq \epsilon$$

Proof. Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}^+$ denote the associated densities with respect to π_1, π_2. We recollect that

$$d_{tv}(\pi_1, \pi_2) = \frac{1}{2}\int_{\mathbb{R}^n} |f_1(x) - f_2(x)|dx.$$

Expanding the above formula, the result follows by a direct computation.

Lemma 3.2. *For $S \subseteq \mathbb{R}^n$, K_S is a convex set. Furthermore, if S is compact then K_S is compact.*

Proof. If $K_S = \emptyset$ we are done. Therefore assume $K_S \neq \emptyset$, and pick $x, y \in K_S$. Now take $z \in [x, y]$. We need to show that $\forall p \in S$, $[z, p] \subseteq S$. Assume not, then there exists $p \in S$, $q \notin S$, such that $q \in [z,p]$. Since $x, y \in S$ we have that $[x,p], [y,p] \subseteq S$. Furthermore we see that q is in the interior of the triangle defined by x, y, p. Let $l(x,q)$ denote the line through x, q. Since q is in the interior of conv.hull$\{x,y,p\}$ we must have that $l(x,q)$ intersects the segment $[y,p]$ in some point r. But now note that $r \in S$, since $r \in [y,p]$, and by construction $[x,r] \not\subseteq S$, a contradiction. This proves the statement.

For the furthermore, we assume that S is compact. To show that K_S is compact, we need only show that K_S is closed. If x is a limit point of K_S, we have a sequence $\{x_i\}_{i=1}^{\infty} \subseteq K_S$ converging to x. Now take any point $p \in S$. We see that $[x_i, p] \subseteq S$ for all $i \geq 1$, and we note that the sequence of line segments $[x_i, p]$ converge to $[x,p]$ as $i \to \infty$. By compactness of S, we have that the limit segment $[x,p]$ is indeed contained in S. Since this holds for all p, we see that $x \in K_S$ as needed.

LEMMA 3.3. *Let S be a star-shaped body with $\eta(S) > 0$, and let (S_1, S_3, S_2) denote a measurable partition of S where $\mathsf{vol}(S_1), \mathsf{vol}(S_2) > 0$. Then for every $\epsilon > 0$, there exists a decomposition $\mathcal{P}$ of S such that*

1. $\forall P \in \mathcal{P}$, $\mathsf{vol}(S_1 \cap P)\mathsf{vol}(S_2) - \mathsf{vol}(S_1)\mathsf{vol}(S_2 \cap P) = 0$.
2. $\exists N \subseteq \mathcal{P}$ *such that*
   (a) $\sum_{P \in N} \frac{\mathsf{vol}(P)}{\mathsf{vol}(S)} \geq (1-\epsilon)\eta(S)$
   (b) $\forall P \in N$, *P is ϵ-convex, i.e., there exists $P' \subseteq \mathbb{R}^n$ a convex body, such that* $\mathsf{vol}(P \triangle P') \leq \epsilon \min\{\mathsf{vol}(P), \mathsf{vol}(P')\}$.

Proof. [Proof of Lemma 3.3 (Near Convex Decomposition)] First we will show that we can find subset $K_S^r \subseteq K_S$ that takes up most and the kernel and that lies deep inside it, i.e. that $K_S^r + \mathbb{B}_n(0,r) \subseteq K_S$. Formally, let $K_S^r = \{x : \mathbb{B}_n(x,r) \subseteq K_S\}$ where $r > 0$. Let K_S° denote the interior of K_S. We note that $K_S^\circ = \cup_{i=1}^{\infty} K_S^{\frac{1}{i}}$. By the continuity of measure, there exists a positive integer j, such that for $\epsilon_0 = \frac{1}{j}$, $\mathsf{vol}(K_S^{\epsilon_0}) \geq (1-\epsilon)\mathsf{vol}(K_S^\circ)$. Since K_S is convex, we know that $\mathsf{vol}(K_S^\circ) = \mathsf{vol}(K_S)$ and hence

$$\frac{\mathsf{vol}(K_S^{\epsilon_0})}{\mathsf{vol}(S)} \geq \frac{(1-\epsilon)\mathsf{vol}(K_S)}{\mathsf{vol}(S)} = (1-\epsilon)\eta(S)$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be

$$f(x) = \mathsf{vol}(S_2)1_{S_1}(x) - \mathsf{vol}(S_1)1_{S_2}(x)$$

where $1_{S_1}, 1_{S_2}$ are the indicator functions for S_1 and S_2 respectively. We note that $\int_S f = \mathsf{vol}(S_2)\mathsf{vol}(S_1) - \mathsf{vol}(S_1)\mathsf{vol}(S_2) = 0$. By Theorem 2.1, for every $\epsilon_1 > 0$, there exists an ϵ_1-thin decomposition $\mathcal{P}_{\epsilon_1}$ of S such that each part $P \in \mathcal{P}$ is obtained by successive half space cuts from S and $\int_P f dx = 0$. We note that the condition $\int_P f dx = 0$ immediately implies condition (1) for $\mathcal{P}_{\epsilon_1}$. For the time being we will assume that $\epsilon_1 < \frac{1}{2}\epsilon_0$ and determine its exact value later.

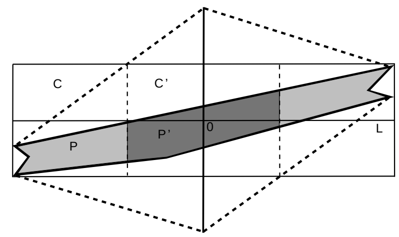

Figure 2: P is ϵ-convex

Let $N = \{P : P \in \mathcal{P}_{\epsilon_1}, P \cap K_S^{\epsilon_0} \neq \emptyset\}$. Since $\mathcal{P}_{\epsilon_1}$ is a decomposition of S, we note that $\cup_{P \in N} P \supseteq K_S^{\epsilon_0}$ and hence

$$\frac{\sum_{P\in N}\mathsf{vol}(P)}{\mathsf{vol}(S)} = \frac{\mathsf{vol}(\cup_{P\in N}P)}{\mathsf{vol}(S)} \geq \frac{\mathsf{vol}(K_S^{\epsilon_0})}{\mathsf{vol}(S)} \geq (1-\epsilon)\eta(S)$$

We will now show that for an appropriately chosen ϵ_1 every $P \in N$ is ϵ-convex. Our strategy is as follows. We analyze a minimal cylinder C of radius ϵ_1 containing P, which exists by our assumption on $\mathcal{P}_{\epsilon_1}$. We will use the fact that P contains a point deep inside the kernel to show that a subcylinder C' of C is fully contained inside S. Lastly we will show that $P' = C' \cap P$ is a convex body whose volume is at least a $(1-\epsilon)$ fraction of the volume of P.

Take $P \in N$. Let C be the cylinder of radius at most ϵ_1 and let $L \subseteq C$ denote the axis of C. Without loss of generality, we may assume that L is a subset of the x_1 axis, i.e.

$$C = \left\{ x : x \in \mathbb{R}^n, a \leq x_1 \leq b, \sum_{i=2}^{n} x_i^2 \leq \epsilon_1^2 \right\}.$$

By assumption, we have that $P \cap K_S^{\epsilon_0} \neq \emptyset$, so pick $c \in P \cap K_S^{\epsilon_0}$. Since $\epsilon_1 < \frac{1}{2}\epsilon_0$, there exists $d \in L$ such that $\|c - d\| \leq \epsilon_1 < \frac{1}{2}\epsilon_0$. Hence $\mathbb{B}_n(c, \epsilon_0) \subseteq K_S \Rightarrow \mathbb{B}_n(d, \frac{\epsilon_0}{2}) \subseteq K_S$. Let $\delta = \frac{1}{2}\epsilon_0$. Without loss of generality, we may assume that $d = 0$. Furthermore, by choosing a, b minimal subject to containing P, we may assume there exist points $v, w \in P$ such that $v_1 = a$ and $w_1 = b$. By possibly rotating C, we may assume that $v = (a, r, 0, \ldots, 0)$ where $0 \leq r \leq \epsilon_1$. By assumption on d, we know that $t = (0, -\delta, 0, \ldots, 0) \in K_S$. Therefore the line segment $[v,t] \subseteq S$. By a simple computation, we see that $[v,t]$ intersects the x_1 axis at $v' = (\frac{\delta}{r+\delta}a, 0, \ldots, 0)$. Since $0 \in K_S$, we also have that $[v',0] \in S \Rightarrow v^* = (\frac{\delta}{\epsilon_1+\delta}a, 0, \ldots, 0) \in S$. By symmetric reasoning with respect to w, we have that $w^* = (\frac{\delta}{\epsilon_1+\delta}b, 0, \ldots, 0) \in S$.

Now, consider the subcylinder

$$C' = \left\{ x : \frac{\delta - \epsilon_1}{\delta + \epsilon_1} a \le x_1 \le \frac{\delta - \epsilon_1}{\delta + \epsilon_1} b, \sum_{i=2}^n x_i^2 \le \epsilon_1^2 \right\}.$$

LEMMA 3.4. *$C' \subseteq S$.*

Proof. Take $x \in C'$. By symmetry we may assume that $x = (e, f, 0, \ldots, 0)$ where $0 \le e \le \frac{\delta-\epsilon_1}{\delta+\epsilon_2}b$ and $0 \le f \le \epsilon_1$. Now examine the line $l(x, w^*)$. A simple computation reveals that $l(x, w^*)$ intersects the x_2 axis at the point $x^* = (0, \frac{\delta b}{\delta b - (\delta+\epsilon_1)e} f, 0, \ldots)$. Now we note that

$$\begin{aligned}\frac{\delta b}{\delta b - (\delta + \epsilon_1)e} f &\le \frac{\delta b}{\delta b - (\delta + \epsilon_1)e} \epsilon_1 \\ &\le \frac{\delta b}{\delta b - (\delta + \epsilon_1)\frac{\delta-\epsilon_1}{\delta+\epsilon_1} b} \epsilon_1 \\ &= \frac{\delta b}{\epsilon_1 b} \epsilon_1 = \delta.\end{aligned}$$

Therefore by assumption on d we know that $x^* \in K_S$. Since $x \in [x^*, w^*]$, we have that $x \in S$ as needed.

Now define $P' := P \cap C'$.

LEMMA 3.5. *P' is convex.*

Proof. To see this note that P is obtained from S via halfspace cuts, i.e. $P = S \bigcap_{i=1}^m H_i$ where each H_i denotes a halfspace. Now we see that

$$P' = P \cap C' = S \cap C' \bigcap_{i=1}^m H_i = C' \bigcap_{i=1}^m H_i$$

since $C' \subseteq S$. Since the intersection of convex sets is convex, we have that P' is convex as needed.

Now note that $P \triangle P' = P \setminus P' = P \setminus C'$. We will now show that for an appropriate choice of ϵ_1, depending only on δ and ϵ, we have that $P \triangle P' \le \epsilon \mathsf{vol}(P')$ which will prove that P is indeed ϵ-convex. In fact, letting $P_+ = P \cap \{x : x_1 \ge 0\}, P'_+ = P \cap \{x : x_1 \ge 0\}$, we will prove that

$$\mathsf{vol}(P_+ \triangle P'_+) \le \epsilon \mathsf{vol}(P'_+)$$

By symmetry, the same inequality will follow for the $x_1 \le 0$ side, and by summing up the two inequalities the result follows.

Let $S(t) = \{x : x \in P_+, x_1 = t\}$, and $s(t) = \mathsf{vol}_{n-1}(S(t))$. Now let $b' = \frac{\delta-\epsilon_1}{\delta+\epsilon_1}b$ and let $t^* = \operatorname{argmax}_{b' \le t \le b} s(t)$. We have that

$$\begin{aligned}\mathsf{vol}_n(P_+ \setminus C') &= \int_{b'}^b s(t)dt \le \int_{b'}^b s(t^*)dt \\ &= (b - b')s(t^*) = \left(\frac{2\epsilon_1}{\delta + \epsilon_1}\right) bs(t^*).\end{aligned}$$

Now by construction the section $S(0) \subseteq K_S$. We claim that $S(0) \subseteq K_{P_+}$. Take $x \in S(0)$ and $y \in P_+$. Since $x \in K_S$, we have that $[x, y] \subseteq S$. Now $P_+ = S \bigcap_{i=1}^m H_i$. Clearly $x, y \in P \Rightarrow x, y \in H_i$, for $1 \le i \le m$. Furthermore, since each H_i is convex, we have that $[x, y] \subseteq H_i$. Therefore $[x, y] \subseteq P_+$ as needed. Choose $\alpha \in [0, 1]$ such that $(1 - \alpha)0 + \alpha t^* = b'$. Since $S(0) \subseteq K_{P_+}$, we see that

$$(1 - \alpha)S(0) + \alpha S(t^*) \subseteq S(b')$$

Therefore by the Brunn-Minkowski inequality, we have that

$$\begin{aligned}s(b') &\ge ((1 - \alpha)\mathsf{vol}_{n-1}(S(0))^{\frac{1}{n-1}} \\ &\qquad + \alpha \mathsf{vol}_{n-1}(S(t^*))^{\frac{1}{n-1}})^{n-1} \\ &\ge \alpha^{n-1}\mathsf{vol}_{n-1}(S(t^*)) \ge \left(\frac{\delta - \epsilon_1}{\delta + \epsilon_1}\right)^{n-1} s(t^*)\end{aligned}$$

Since P'_+ is convex we note that conv.hull $\{0, S(b')\} \subseteq P'_+$ and hence

$$\begin{aligned}\mathsf{vol}_n(P'_+) &\ge \mathsf{vol}_n\left(\text{conv.hull}\,\{0, S(b')\}\right) \\ &= \frac{1}{n}b's(b') \ge \frac{1}{n}\left(\frac{\delta - \epsilon_1}{\delta + \epsilon_1}\right)^n bs(t^*).\end{aligned}$$

Now by choosing ϵ_1 small enough such that

$$\left(\frac{2\epsilon_1}{\delta + \epsilon_1}\right) \le \epsilon \frac{1}{n}\left(\frac{\delta - \epsilon_1}{\delta + \epsilon_1}\right)^n,$$

we get that $\mathsf{vol}(P_+ \triangle P'_+) \le \epsilon \mathsf{vol}_n(P'_+)$ as needed.

Using the above lemma, we now prove Theorem 1.1.

Proof. [Proof of Theorem 1.1 (Diameter isoperimetry)] Let (S_1, S_3, S_2) be a measurable partition of S. Without loss of generality we may assume that $\mathsf{vol}(S_1) \le \mathsf{vol}(S_2)$. Let $\alpha = \frac{\mathsf{vol}(S_1)}{\mathsf{vol}(S_2)}$, where we see that $\alpha \le 1$. Our goal is now to show that $\mathsf{vol}(S_3)$ is "large" with respect to $\mathsf{vol}(S_1)$. Note that

$$\mathsf{vol}(S) \ge \mathsf{vol}(S_1) + \mathsf{vol}(S_2) = \frac{\alpha + 1}{\alpha}\mathsf{vol}(S_1)$$

implies that

$$\frac{\alpha}{\alpha + 1}\mathsf{vol}(S) \ge \mathsf{vol}(S_1)$$

Let $\mathcal{P}_\epsilon$ be the decomposition of S with respect to (S_1, S_3, S_2) as defined in Lemma 3.3 with parameter ϵ. Let N denote the set of ϵ-convex needles. Let $N_+ = \left\{P : P \in N, \mathsf{vol}(S_1 \cap P) \geq \frac{1}{2}\frac{\alpha}{\alpha+1}\mathsf{vol}(P)\right\}$ and $N_- = N \setminus N_+$. Since $N = N_+ \cup N_-$ and by assumption on N

$$\sum_{P \in N} \frac{\mathsf{vol}(P)}{\mathsf{vol}(S)} \geq (1-\epsilon)\eta(S)$$

we must have that either

$$(a) \sum_{P \in N_-} \frac{\mathsf{vol}(P)}{\mathsf{vol}(S)} \geq \frac{1}{2}(1-\epsilon)\eta(S) \quad \text{or}$$

$$(b) \sum_{P \in N_+} \frac{\mathsf{vol}(P)}{\mathsf{vol}(S)} \geq \frac{1}{2}(1-\epsilon)\eta(S).$$

Assume first that (a) is true. We will show that S_1 and S_2 take up a small fraction of most partition parts and that consequently S_3 must take up a large fraction of S. Take $P \in N^-$, and let $S_1^P = S_1 \cap P, S_2^P = S_2 \cap P, S_3^P = S_3 \cap P$. By assumption on $\mathcal{P}_\epsilon$ we know that $\mathsf{vol}(S_1^P) = \alpha\mathsf{vol}(S_2^P)$. Therefore we have that

$$\mathsf{vol}(S_1^P) + \mathsf{vol}(S_2^P) = \frac{\alpha+1}{\alpha}\mathsf{vol}(S_1^P) \leq \frac{1}{2}\mathsf{vol}(P)$$

by assumption on N^-. Since $\mathsf{vol}(S_1^P) + \mathsf{vol}(S_2^P) + \mathsf{vol}(S_3^P) = \mathsf{vol}(P)$, we must have that $\mathsf{vol}(S_3^P) \geq \frac{1}{2}\mathsf{vol}(P)$. Therefore, we have that

$$\begin{aligned} \frac{\mathsf{vol}(S_3)}{\mathsf{vol}(S)} &\geq \sum_{P \in N^-} \frac{\mathsf{vol}(S_3^P)}{\mathsf{vol}(S)} \\ &\geq \sum_{P \in N^-} \frac{1}{2}\frac{\mathsf{vol}(P)}{\mathsf{vol}(S)} \geq \frac{1}{4}(1-\epsilon)\eta(S). \end{aligned}$$

Since $\frac{d(S_1,S_2)}{D} \leq 1$, this proves the theorem for case (a).

Now assume that (a) is not true. Then we must have that (b) is true to satisfy our assumption on N. Now take $P \in N^+$. Our strategy here will be to derive isoperimetry for P using the fact that P is ϵ-convex. By approximating the measure of P by that of its convex approximation, we will derive an isoperimetric inequality for P with an additive error depending on ϵ. Since in this case S_1 and S_2 take up a lower bounded fraction of P, we will be able to transform the additive error into multiplicative error by making ϵ sufficiently small. The statement will follow as a result.

So let Q be a convex body such that $\mathsf{vol}(P \triangle Q) \leq \epsilon \min\{\mathsf{vol}(P), \mathsf{vol}(Q)\}$. We may assume that $Q \subseteq \text{conv.hull}(P)$, since otherwise $Q \cap \text{conv.hull}(P)$ is a convex body and strictly closer to P. Next, since $\text{Diam}(P) = \text{Diam}(\text{conv.hull}(P))$ we have that $\text{Diam}(Q) \leq \text{Diam}(P) \leq \text{Diam}(S) = D$.

Let π_P, π_Q denote the uniform measures on P, Q respectively. Let $S_1^Q = S_1^P \cap Q, S_2^Q = S_2^P \cap Q$ and $S_3^Q = Q \setminus (S_1^Q \cup S_2^Q)$. Since (S_1^P, S_3^P, S_2^P) partition P, we note that $S_3^Q = (S_3^P \cap Q) \cup (Q \setminus P)$. Then we have that $d(S_1^Q, S_2^Q) \geq d(S_1^P, S_2^P) \geq d(S_1, S_2)$. By lemma 3.1 we know that $d_{tv}(\pi_Q, \pi_P) \leq \epsilon$, hence we see that

$$\pi_Q(S_i^Q) = \pi_Q(S_i^P) \geq \pi_P(S_i^P) - \epsilon \geq \pi_P(S_i^P) - 3\epsilon : i = 1, 2,$$

and

$$\begin{aligned} \pi_Q(S_3^Q) &\leq \pi_P(S_3^Q) + \epsilon = \pi_P(S_3^Q \cap P) + \epsilon \\ &\leq \pi_Q(S_3^Q \cap P) + 2\epsilon = \pi_Q(S_3^P) + 2\epsilon \\ &\leq \pi_P(S_3^P) + 3\epsilon \end{aligned}$$

Since Q is convex, using the isoperimetric inequality proved in [LS93] we have that

$$\begin{aligned} \pi_Q(S_3^Q) &\geq \frac{d(S_1^Q, S_2^Q)}{\text{Diam}(Q)} \min\left\{\pi_Q(S_1^Q), \pi_Q(S_2^Q)\right\} \\ &\geq \frac{d(S_1, S_2)}{D} \min\left\{\pi_Q(S_1^Q), \pi_Q(S_2^Q)\right\} \end{aligned}$$

Now bringing the above inequalities together, we get that

$$\pi_P(S_3^P) + 3\epsilon \geq \frac{d(S_1, S_2)}{D} \min\left\{\pi_P(S_1^P) - 3\epsilon, \pi_P(S_2^P) - 3\epsilon\right\}$$

$$\Rightarrow$$

$$\pi_P(S_3^P) \geq \frac{d(S_1, S_2)}{D} \min\left\{\pi_P(S_1^P) - 3\epsilon, \pi_P(S_2^P) - 3\epsilon\right\} - 3\epsilon$$

$$\Rightarrow$$

$$\pi_P(S_3^P) \geq \frac{d(S_1, S_2)}{D}\left(\pi_P(S_1^P) - 3\epsilon\right) - 3\epsilon$$

since $\pi_P(S_1^P) \leq \pi_P(S_2^P)$. Now choose

$$\epsilon \leq \min\left\{\frac{\epsilon_0}{12}\frac{d(S_1, S_2)}{D}\frac{\alpha}{\alpha+1}, \frac{\epsilon_0}{12}\frac{\alpha}{\alpha+1}\right\}$$

where $\epsilon_0 > 0$. Since $P \in N^+$ we have that

$$\mathsf{vol}(S_1^P) \geq \frac{1}{2}\frac{\alpha}{\alpha+1}\mathsf{vol}(P) \Rightarrow \pi_P(S_1^P) \geq \frac{1}{2}\frac{\alpha}{\alpha+1}$$

Hence $3\epsilon \leq \frac{\epsilon_0}{4}\frac{\alpha}{\alpha+1} \leq \frac{\epsilon_0}{2}\pi_P(S_1^P)$. A simple computation now gives us that

$$\pi_P(S_3^P) \geq (1-\epsilon_0)\frac{1}{2}\frac{d(S_1, S_2)}{D}\frac{\alpha}{\alpha+1}$$

Now $\mathsf{vol}(S_3^P) = \pi_P(S_3^P)\mathsf{vol}(P)$, so we see that

$$\begin{aligned}\mathsf{vol}(S_3) &\geq \sum_{P\in N^+} \mathsf{vol}(S_3^P)\\ &\geq (1-\epsilon_0)\frac{1}{2}\frac{d(S_1,S_2)}{D}\frac{\alpha}{\alpha+1}\sum_{P\in N^+}\mathsf{vol}(P)\\ &\geq (1-\epsilon_0)\frac{d(S_1,S_2)}{D}\frac{\alpha}{\alpha+1}(1-\epsilon)\frac{\eta(S)}{2}\mathsf{vol}(S)\\ &\geq (1-\epsilon_0)(1-\epsilon)\frac{\eta(S)}{4D}d(S_1,S_2)\mathsf{vol}(S_1)\end{aligned}$$

Finally, letting $\epsilon_0 \to 0$ yields the result.

The proof of Theorem 1.2 follows a similar proof strategy as Theorem 1.1. We need the following lemma about second moments.

LEMMA 3.6. *Let $f_1,\ldots,f_m : \mathbb{R}^n \to \mathbb{R}^+$ be densities with associated random variables $X_1,\ldots,X_m$ and centroids $\mu_1,\ldots,\mu_m$ respectively. Let $g = \sum_{i=1}^m p_i f_i$ be a mixture of the f_is with associated random variable Y and centroid μ. Then we have that*

$$\mathsf{E}[\|Y-\mu\|^2] = \sum_{i=1}^m p_i\left(\mathsf{E}[\|X_i-\mu_i\|^2] + \|\mu_i-\mu\|^2\right)$$

Proof. Since g is a mixture, we note that

$$\mathsf{E}[\|Y-\mu\|^2] = \sum_{i=1}^m p_i\mathsf{E}[\|X_i-\mu\|^2] \tag{1}$$

Now $\|X_i-\mu\|^2 = \langle X_i-\mu, X_i-\mu\rangle$. Now we have that

$$\mathsf{E}[\langle X_i-\mu, X_i-\mu\rangle] = \mathsf{E}[\langle X_i-\mu_i+(\mu_i-\mu), X_i-\mu_i+(\mu_i-\mu)\rangle]$$

Expanding the above we get

$$\mathsf{E}[\|X_i-\mu_i\|^2] + 2\mathsf{E}[\langle X_i-\mu_i, \mu_i-\mu\rangle] + \mathsf{E}[\|\mu_i-\mu\|^2] = \mathsf{E}[\|X_i-\mu_i\|^2] + \|\mu_i-\mu\|^2$$

where the equality holds since $\mathsf{E}[X_i-\mu_i]=0$, and μ_i,μ are constant vectors. Plugging the above into (1) yields the result.

Proof. [Proof of Theorem 1.2 (Second moment isoperimetry)] Let (S_1,S_3,S_2) be the measurable partition of S. We may assume $\mathsf{vol}(S_1) \leq \mathsf{vol}(S_2)$ and so $\alpha = \frac{\mathsf{vol}(S_1}{\mathsf{vol}(S_2)} \leq 1$. Let $\mathcal{P}_\epsilon$, N, N^+, N^- be defined as in the proof of Theorem 1.1. Again as in Theorem 1.1 we have the cases (a) and (b). If case (a) occurs, then by the proof of Theorem 1.1 we have that

$$\mathsf{vol}(S_3) \geq \frac{1}{4}(1-\epsilon)\eta(S)\mathsf{vol}(S)$$

as needed. So we may assume assume that we are in case (b), i.e that

$$\sum_{P\in N_+}\frac{\mathsf{vol}(P)}{\mathsf{vol}(S)} \geq \frac{1}{2}(1-\epsilon)\eta(S)$$

Now for each $P \in \mathcal{P}_\epsilon$, let π_P denote the uniform measure on P, μ_P denote the centroid of P, and let $M_P = \mathsf{E}_P[\|X-\mu_P\|^2]$. Now we note that π_S, the uniform measure on S, is a mixture of the π_Ps, i.e.

$$\pi_S = \sum_{P\in\mathcal{P}_\epsilon}\frac{\mathsf{vol}(P)}{\mathsf{vol}(S)}\pi_P$$

Therefore by Lemma 3.6 we have that

$$\begin{aligned}M_S &= \sum_{P\in\mathcal{P}_\epsilon}\frac{\mathsf{vol}(P)}{\mathsf{vol}(S)}\left(M_P + \|\mu_P-\mu_S\|^2\right)\\ &\geq \sum_{P\in N^+}\frac{\mathsf{vol}(P)}{\mathsf{vol}(S)}M_P.\end{aligned}$$

Let $V = \sum_{P\in N^+}\mathsf{vol}(P)$. By assumption $V \geq \frac{1}{2}(1-\epsilon)\eta(S)\mathsf{vol}(S)$, and hence

$$\begin{aligned}\sum_{P\in N^+}\frac{\mathsf{vol}(P)}{V}M_P &\leq \sum_{P\in N^+}\frac{2}{(1-\epsilon)\eta(S)}\frac{\mathsf{vol}(P)}{\mathsf{vol}(S)}\\ &\leq \frac{2}{(1-\epsilon)\eta(S)}M_S.\end{aligned}$$

Let $N^* = \left\{P : P\in N^+, M_P \leq \frac{4}{(1-\epsilon)\eta(S)}M_S\right\}$. Since $\sum_{P\in N^+}\frac{\mathsf{vol}(P)}{V}M_P$ is an average of positive numbers by Markov's inequality we must have that

$$\sum_{P\in N^*}\mathsf{vol}(P) \geq \frac{1}{2}V \geq \frac{1}{4}(1-\epsilon)\eta(S)\mathsf{vol}(S)$$

Now take $P \in N^*$. By assumption on $N^* \subseteq N$, there exists Q a convex body such that $\mathsf{vol}(P \triangle Q) \leq \epsilon\min\{\mathsf{vol}(P),\mathsf{vol}(Q)\}$. In particular, by the construction of Lemma 3.3 we may assume that $Q \subseteq P$. Let $\bar{Q} = P \setminus Q$, and let $\pi_{\bar{Q}}, \pi_Q$ denote the uniform measures on $\bar{Q}, Q$ respectively. We now see that

$$\pi_P = \frac{\mathsf{vol}(\bar{Q})}{\mathsf{vol}(P)}\pi_{\bar{Q}} + \frac{\mathsf{vol}(Q)}{\mathsf{vol}(P)}\pi_Q$$

As done previously above from Lemma 3.6 we readily see that

$$\begin{aligned}M_P &\geq \frac{\mathsf{vol}(Q)}{\mathsf{vol}(P)}M_Q\\ \Rightarrow \frac{\mathsf{vol}(P)}{\mathsf{vol}(Q)}M_P &\geq M_Q\\ \Rightarrow (1+\epsilon)M_P &\geq M_Q.\end{aligned}$$

As in the proof of Theorem 1.1, let $S_1^Q = S_1^P \cap Q$, $S_2^Q = S_2^P \cap Q$, and $S_3^Q = Q \setminus (S_1^Q \cup S_2^Q)$. Since Q is a convex set, using the isoperimetric inequality proved in [KLS95] we get that

$$\begin{aligned}\pi_Q(S_3^Q) &\geq \frac{d(S_1^Q, S_2^Q)}{2\sqrt{M_Q}} \min\left\{\pi_Q(S_1^Q), \pi_Q(S_2^Q)\right\} \\ &\geq \left(\frac{d(S_1, S_2)}{2\sqrt{(1+\epsilon)M_P}}\right) \min\left\{\pi_Q(S_1^Q), \pi_Q(S_2^Q)\right\} \\ &\geq \left(\frac{d(S_1, S_2)}{\sqrt{\frac{8(1+\epsilon)M_S}{(1-\epsilon)\eta(S)}}}\right) \min\left\{\pi_Q(S_1^Q), \pi_Q(S_2^Q)\right\}\end{aligned}$$

Using the same analysis as in Theorem 1.1, the above inequality gives us that

$$\pi_P(S_3^P) \geq \left(\frac{d(S_1, S_2)}{\sqrt{\frac{8(1+\epsilon)M_S}{(1-\epsilon)\eta(S)}}}\right) (\pi_P(S_1^P) - 3\epsilon)$$

Now choose

$$\epsilon \leq \left(\frac{\epsilon_0}{12}\right)\left(\frac{\alpha}{\alpha+1}\right) \min\left\{\frac{d(S_1, S_2)}{\sqrt{\frac{8(1+\epsilon)M_S}{(1-\epsilon)\eta(S)}}}, 1\right\}$$

for any $\epsilon_0 > 0$. By the same analysis as in Theorem 1.1, we get that

$$\pi_P(S_3^P) \geq \left(\frac{d(S_1, S_2)}{\sqrt{\frac{8(1+\epsilon)M_S}{(1-\epsilon)\eta(S)}}}\right)\left(\frac{(1-\epsilon_0)\alpha}{\alpha+1}\right)$$

Using the fact that $\sum_{P \in N^*} \mathsf{vol}(P) \geq \frac{1}{4}(1-\epsilon)\eta(S)\mathsf{vol}(S)$ we get that

$$\begin{aligned}\mathsf{vol}(S_3) &\geq \sum_{P \in N^*} \mathsf{vol}(S_3^P) \\ &\geq \left(\frac{d(S_1, S_2)}{\sqrt{\frac{8(1+\epsilon)M_S}{(1-\epsilon)\eta(S)}}}\right) \frac{(1-\epsilon_0)\alpha}{\alpha+1} \sum_{P \in N^*} \mathsf{vol}(P) \\ &\geq \left(\frac{d(S_1, S_2)}{\sqrt{\frac{8(1+\epsilon)M_S}{(1-\epsilon)\eta(S)}}}\right) \frac{(1-\epsilon)(1-\epsilon_0)\eta(S)\alpha}{4(\alpha+1)} \mathsf{vol}(S) \\ &\geq \left(\frac{d(S_1, S_2)}{\sqrt{\frac{8(1+\epsilon)M_S}{(1-\epsilon)\eta(S)}}}\right) \frac{(1-\epsilon_0)(1-\epsilon)\eta(S)}{4} \mathsf{vol}(S_1) \\ &\geq \frac{(1-\epsilon_0)(1-\epsilon)^{\frac{3}{2}}}{(1+\epsilon)^{\frac{1}{2}}} \left(\frac{\eta(S)^{\frac{3}{2}} d(S_1, S_2)}{16\sqrt{M_S}}\right) \mathsf{vol}(S_1)\end{aligned}$$

Finally, letting $\epsilon_0 \to 0$ yields the result.

4 Conductance and mixing time

4.1 Local Conductance. Ball walk on star-shaped bodies could potentially get stuck in points with very low local conductance. Here we prove that most of the points in a star-shaped body have good local conductance. First, we extend a lemma from [KLS97] from convex bodies to star-shaped bodies which leads to the proof of good local conductance. The proof is essentially identical to the case of convex bodies.

LEMMA 4.1. *Let $v_n = \mathsf{vol}_n(\mathbb{B}_n(0,1))$. Let L be a measurable subset of the surface of a star-shaped set S in $\mathbb{R}^n$ and let*

$$S_L := \{(x, y) : x \in S, y \notin S, ||x-y|| \leq r, [x, y] \cap L \neq \emptyset\}.$$

Then the $2n-$dimensional measure of $S_{\partial S}$ for any measurable subset ∂S of the surface of S satisfies

$$\mathsf{vol}(S_{\partial S}) \geq r\left(\frac{v_{n-1}}{v_n(n+1)}\right) \mathsf{vol}(\mathbb{B}_n(0,r))\mathsf{vol}_{n-1}(\partial S)$$

Proof. [Proof of Lemma 4.1 (Measure across surface)] For a measurable partition $\partial S = \cup_{i=1}^k L_i$, by the definition of S_L we see that

$$\mathsf{vol}(L) \leq \sum_{i=1}^k \mathsf{vol}(S_{L_i}).$$

On the other hand, the surface areas are additive i.e.

$$\mathsf{vol}_{n-1}(L) = \sum_{i=1}^k \mathsf{vol}_{n-1}(L_i).$$

Hence, if for each partition element we can show

$$(1) \quad \mathsf{vol}(S_{L_i}) \leq r\left(\frac{v_{n-1}}{v_n(n+1)}\right) \mathsf{vol}(\mathbb{B}_n(0,r))\mathsf{vol}_{n-1}(L_i),$$

then we get that

$$\begin{aligned}\mathsf{vol}(S_{\partial S}) &\leq \sum_{i=1}^k \mathsf{vol}_{n-1}(L_i) \\ &\leq \sum_{i=1}^k r\left(\frac{v_{n-1}}{v_n(n+1)}\right) \mathsf{vol}(\mathbb{B}_n(0,r))\mathsf{vol}_{n-1}(L_i) \\ &= r\left(\frac{v_{n-1}}{v_n(n+1)}\right) \mathsf{vol}(\mathbb{B}_n(0,r))\mathsf{vol}_{n-1}(\partial S).\end{aligned}$$

By standard approximation arguments, we may assume that ∂S is a piecewise linear $n-1$-dimensional manifold (i.e. ∂S is obtained by gluing $n-1$ dimensional polytopes). Hence, we can partition ∂S into $L_1, \ldots, L_k$ where each L_i, $1 \leq i \leq k$ is "flat", i.e. is contained inside an $n-1$ dimensional hyperplane. By the above computations it clearly suffices to prove (1) for each piece of this partition.

Let L be one of the pieces in this partition, and let H denote the hyperplane containing L. Let $C(r,t)$ denote the volume of a ball of radius r intersected with a halfspace at distance t from the center of the ball. Then we see that

$$\begin{aligned}\mathsf{vol}(S_L) &= \int_{x:x\in S}\int_{y:(x,y)\in S_L} dydx\\ &\le \int_{t=0}^{r}\int_{\substack{d(x,H)=t\\ d(x,L)\le r}}\int_{y:(x,y)\in S_L} dydxdt\\ &\le \int_{t=0}^{r}\int_{\substack{d(x,H)=t\\ d(x,L)\le r}} C(r,t)dxdt\end{aligned}$$

where the last inequality follows since $L\subseteq H$. Continuing we see that

$$\begin{aligned}\mathsf{vol}(S_L) &\le \int_{t=0}^{r}\int_{\substack{d(x,H)=t\\ d(x,L)\le r}} C(r,t)dxdt\\ &\le \int_{x\in L}\int_{t=0}^{r} C(r,t)dtdx\\ &= \mathsf{vol}_{n-1}(L)\int_{t=0}^{r}\int_{s=t}^{r}\left(\sqrt{r^2-s^2}\right)^{n-1} v_{n-1}dsdt\\ &= \mathsf{vol}_{n-1}(L)\int_{s=0}^{r} s\left(\sqrt{r^2-s^2}\right)^{n-1} v_{n-1}ds\\ &= r\left(\frac{v_{n-1}}{v_n(n+1)}\right)\mathsf{vol}(\mathbb{B}_n(0,r))\mathsf{vol}_{n-1}(L)\end{aligned}$$

as needed.

Recall that the local conductance of a point $x\in S$ is defined as $l(x)=\frac{\mathsf{vol}(\mathbb{B}_n(x,r)\cap S)}{\mathsf{vol}(\mathbb{B}_n(0,r))}$.

Corollary 4.1. *Suppose that S is a star shaped body with kernel K_S satisfying $\mathbb{B}_n(0,1)\subseteq K_S$. Let X be a random vector uniformly distributed in S. Then $\lambda=\mathsf{E}[l(X)]$, the average local conductance of S with respect to ball walk with step size r, is at least*

$$\lambda \ge 1-\frac{r\sqrt{n}}{2}$$

Proof. [Proof of Corollary 4.1 (Average Local Conductance)]

$$\begin{aligned}\lambda &= \frac{1}{\mathsf{vol}(S)}\int_S l(x) = \frac{1}{\mathsf{vol}(S)}\int_S \frac{\mathsf{vol}(\mathbb{B}_n(x,r)\cap S)}{\mathsf{vol}(\mathbb{B}_n(0,r))}dx\\ &= \frac{1}{\mathsf{vol}(S)}\int_S \frac{\mathsf{vol}(\mathbb{B}_n(x,r))-\mathsf{vol}(\mathbb{B}_n(x,r)\setminus S)}{\mathsf{vol}(\mathbb{B}_n(0,r))}\\ &= 1-\frac{1}{\mathsf{vol}(S)\mathsf{vol}(\mathbb{B}_n(0,r))}\int_S \mathsf{vol}(\mathbb{B}_n(x,r)\setminus S)\\ &= 1-\frac{\mathsf{vol}(S_{\partial S})}{\mathsf{vol}(S)\mathsf{vol}(\mathbb{B}_n(0,r))}\\ &\le 1-\left(\frac{rv_{n-1}}{v_n(n+1)}\right)\frac{\mathsf{vol}_{n-1}(\partial S)}{\mathsf{vol}(S)}\end{aligned}$$

where the last equality follows by the definition of S_L, and the last inequality from the bound in Lemma 4.1. Now for $n\ge 3$, we have that

$$1-\left(\frac{rv_{n-1}}{v_n(n+1)}\right)\frac{\mathsf{vol}_{n-1}(\partial S)}{\mathsf{vol}(S)} \ge 1-\left(\frac{r}{2\sqrt{n}}\right)\frac{\mathsf{vol}_{n-1}(\partial S)}{\mathsf{vol}(S)}$$

The corollary follows once we lower bound the volume of S in terms of its surface area. Since S is star-shaped and $0\in K_S$, the volume of S can be written as the sum of infinitesimal cones whose apex is at the origin and whose base corresponds to an infinitesimal piece of the boundary of S. The volume of any such cone is simply the base area (surface area on ∂S) times the height of the cone divided by n. Since $\mathbb{B}_n(0,1)\in K_S$, the hyperplane defined by the base of our infinitesimal cones cannot cut $\mathbb{B}_n(0,1)$ and hence the height of every cone is at least 1. Summing up the volume of all these cones, we get that

$$\mathsf{vol}(S)\ge\frac{\mathsf{vol}_{n-1}(\partial S)}{n}$$

and hence

$$\lambda \ge 1-\left(\frac{r}{2\sqrt{n}}\right)\frac{\mathsf{vol}_{n-1}(\partial S)}{\mathsf{vol}(S)} \ge 1-\frac{r\sqrt{n}}{2}$$

as needed.

The following lemma is the main result of this section.

Lemma 4.2. *Let $S\subseteq R^n$ be a star-shaped body with kernel K_S satisfying $\mathbb{B}_n(0,1)\subseteq K_S$. Let r be the step-size of the ball walk, where $r<1/(2\sqrt{n})$. Define*

$$S_r := \{x\in S : l(x)\ge \frac{3}{4}\}$$

Then, we have that

1. $\mathsf{vol}(S_r)\ge(1-2r\sqrt{n})\mathsf{vol}(S)$

2. $\mathsf{vol}(K_{S_r} \geq (1-2r\sqrt{n})\mathsf{vol}(K_S)$

3. *S_r is star-shaped.*

Proof. [Proof of Lemma 4.2 (Body of good local conductance)] Using Corollary 4.1, we get that

$$\begin{aligned}\frac{1}{\mathsf{vol}(S)}\int_S (1-l(x))dx &\geq \left(1-\frac{r\sqrt{n}}{2}\right)\mathsf{vol}(S)\\ \mathsf{E}(1-l(X)) &\leq \frac{r\sqrt{n}}{2}\\ \mathsf{Pr}\left(\frac{\mathsf{vol}(\mathbb{B}_n(x,r)\cap\bar{S})}{\mathsf{vol}(\mathbb{B}_n(0,r))}\geq\frac{1}{4}\right) &\leq 2r\sqrt{n}\\ \mathsf{Pr}\left(l(x)\geq\frac{3}{4}\right) &\geq (1-2r\sqrt{n})\\ \frac{\mathsf{vol}(S_r)}{\mathsf{vol}(S)} &\geq (1-2r\sqrt{n})\end{aligned}$$

Now applying the same argument as above to K_S, we get that the set $(K_S)_r$, i.e the set of points with local conductance at least 3/4 with respect to the ball walk of step size r on K_S, satisfies $\mathsf{vol}((K_S)_r) \geq (1-2r\sqrt{n})\mathsf{vol}(K_S)$. Clearly, the local conductance of a point $x \in K_S$ with respect to K_S is at least its local conductance with respect to S, and hence $(K_S)_r \subseteq S_r$. We claim that $(K_S)_r \subseteq K_{S_r}$. Take $x \in (K_S)_r$ and $y \in S_r$. Examine $z = \alpha x + (1-\alpha)y$ where $0 \leq \alpha \leq 1$. Then we see that

$$\begin{aligned}&\frac{\mathsf{vol}(\mathbb{B}_n(z,r)\cap S)}{\mathsf{vol}(\mathbb{B}_n(0,r))}\geq\\ &\frac{\mathsf{vol}(\alpha(\mathbb{B}_n(x,r)\cap K_S)+(1-\alpha)(\mathbb{B}_n(y,r)\cap S))}{\mathsf{vol}(\mathbb{B}_n(0,r))}\geq\\ &\frac{\mathsf{vol}(\mathbb{B}_n(x,r)\cap K_S)^\alpha\mathsf{vol}(\mathbb{B}_n(y,r)\cap S)^{1-\alpha}}{\mathsf{vol}(\mathbb{B}_n(0,r))}\geq\frac{3}{4}\end{aligned}$$

and hence $z \in S_r$ as needed. Therefore

$$\mathsf{vol}(K_{S_r}) \geq \mathsf{vol}((K_S)_r) \geq (1-2r\sqrt{n})\mathsf{vol}(K_S)$$

which gives us (2). To get (3), we note that by our assumption on r, we have that $\mathsf{vol}(K_{S_r}) > 0$ and hence K_{S_r} non-empty. Therefore S_r is star-shaped as needed.

4.2 Coupling.

In this section, we prove that the one-step distributions of two nearby points in S with good local conductance are close to each other. This will be crucial for establishing a lower bound on the conductance of the ball walk.

LEMMA 4.3. *Let S be a star-shaped body and let $u, v \in S$ such that $|u-v| \leq \frac{tr}{\sqrt{n}}$, $l(u), l(v) \geq l$. Then*

$$d_{TV}(P_u, P_v) \leq 1 + t - l$$

Proof. [Proof of Lemma 4.3 (Coupling lemma)] We prove the inequality in the case when both $\mathbb{B}_n(u,r)$ and $\mathbb{B}_n(v,r)$ are contained within S. If not, then the considered case gives an upper bound and hence, we are done. Let $C(r,t)$ be the volume of a ball of radius r intersected with a halfspace at distance t from the origin.

$$\begin{aligned}d_{TV}(P_u,P_v) &= \frac{1}{2}\int_{x\in\mathbb{B}_n(u,r)\cup\mathbb{B}_n(v,r)}|P_u(x)-P_v(x)|dx\\ &= \frac{1}{2}|P_u(u)-P_v(u)|+\frac{1}{2}|P_u(v)-P_v(v)|\\ &+\frac{1}{2}\int_{x\in\mathbb{B}_n(u,r)\cap\mathbb{B}_n(v,r)\setminus u\setminus v}|P_u(x)-P_v(x)|dx\\ &+\frac{1}{2}\int_{x\in\mathbb{B}_n(u,r)\cap\mathbb{B}_n(v,r)\setminus u}(P_u(x)-P_v(x))dx\\ &+\frac{1}{2}\int_{x\in\mathbb{B}_n(v,r)\cap\mathbb{B}_n(u,r)\setminus v}(P_v(x)-P_u(x))dx\\ &= \frac{1}{2}(1-l(u))+\frac{1}{2}(1-l(v))+0\\ &+\frac{1}{2}\int_{x\in\mathbb{B}_n(u,r)\cap\mathbb{B}_n(v,r)\setminus u}(P_u(x)-P_v(x))dx\\ &+\frac{1}{2}\int_{x\in\mathbb{B}_n(v,r)\cap\mathbb{B}_n(u,r)\setminus v}(P_v(x)-P_u(x))dx\\ &= 1-l+\frac{2\mathsf{vol}(\mathbb{B}_n(0,r))-2C(r,\frac{tr}{\sqrt{n}}))}{2\mathsf{vol}(\mathbb{B}_n(0,r))}\\ &\leq 1-l+t\end{aligned}$$

4.3 Conductance.

Now, we bound the $s-$conductance of the ball walk on a star-shaped body.

LEMMA 4.4. *Let $S \subseteq \mathbb{R}^n$ be a star-shaped body of diameter D with kernel K_S satisfying $\mathbb{B}_n(0,1) \subseteq K_S$. Let $\eta = \eta(S)$, i.e. the fraction of $\mathsf{vol}(S)$ taken up by the kernel K_S. Then the ball walk with step-size $r = s/(4\sqrt{n})$ has s-conductance at least $\frac{s\eta}{2^{13}nD}$.*

Proof. [Proof of Lemma 4.4 (s-conductance)] By Lemma 4.2, we have that

$$\mathsf{vol}(S_r) \geq (1-\frac{s}{2})\mathsf{vol}(S).$$

Further, the fraction of the volume of the kernel of S_r is

$$\eta(S_r) = \frac{\mathsf{vol}(K_{S_r})}{\mathsf{vol}(S)} \geq \frac{(1-s/2)\mathsf{vol}(K_S)}{\mathsf{vol}(S)} = (1-\frac{s}{2})\eta$$

Now, let $A \cup \bar{A}$ be any partition of S into measurable sets with $\mathsf{vol}(A), \mathsf{vol}(\bar{A}) > s(\mathsf{vol}(S))$. Define sets

$$A_1 := \{x \in A \cap S_r : P_x(\bar{A}) < \frac{1}{16}\}$$

$$A_2 := \{x \in \bar{A} \cap S_r : P_x(A) < \frac{1}{16}\}$$
$$A_3 := S_r \backslash A_1 \backslash A_2$$

Now, suppose that $\mathsf{vol}(A_1) \leq \frac{\mathsf{vol}(S)}{3}$. Then the conductance $\phi_s(A, \bar{A})$ is at least

$$\frac{1}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \int_{x \in A \cap S_r \backslash A_1} \frac{1}{16} dx$$

$$\begin{aligned}
&= \frac{1}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \frac{1}{16} \mathsf{vol}(A \cap S_r \backslash A_1) \\
&\geq \frac{1}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \frac{1}{16} (\mathsf{vol}(A \backslash A_1) - \mathsf{vol}(S \backslash S_r)) \\
&\geq \frac{1}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \frac{1}{16} (\frac{2}{3}\mathsf{vol}(A) - \frac{s}{2}\mathsf{vol}(S)) \\
&\geq \frac{1}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \frac{1}{16} (\frac{2s}{3}\mathsf{vol}(S) - \frac{s}{2}\mathsf{vol}(S)) \\
&\geq \frac{1}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \frac{s}{32} \mathsf{vol}(S) \\
&\geq \frac{1}{32}
\end{aligned}$$

and hence we are done. Therefore, we may assume that $\mathsf{vol}(A_1) \geq \frac{\mathsf{vol}(A)}{3}$ and $\mathsf{vol}(A_2) \geq \frac{\mathsf{vol}(\bar{A})}{3}$.
Consider $u \in A_1$ and $v \in A_2$. Then,

$$d_{TV}(P_u, P_v) \geq 1 - P_u(\bar{A}) - P_v(A) > 1 - \frac{1}{8}$$

Using Lemma 4.3 (t=1/8), we get $|u - v| \geq \frac{5r}{8\sqrt{n}}$, and hence, $d(A_1, A_2) \geq \frac{5r}{8\sqrt{n}}$. Now, using Theorem 1.1 on the partition A_1, A_2, A_3 of S_r, we get that

$$\begin{aligned}
\Phi_s &\geq \frac{1}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \int_A P_x(\bar{A}) dx \\
&\geq \frac{1}{2} \frac{1}{16} \frac{\mathsf{vol}(A_3)}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \\
&\geq \frac{1}{2^5} \frac{\eta(S_r) d(A_1, A_2)}{4D} \frac{\min\{\mathsf{vol}(A_1), \mathsf{vol}(A_2)\}}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \\
&\geq \frac{1}{2^9} \frac{\eta(5r(1 - s/2))}{8\sqrt{n}D} \frac{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}} \\
&\geq \frac{1}{2^{12}} \frac{s(1 - s/2)\eta}{nD} \\
&\geq \frac{1}{2^{13}} \frac{s\eta}{nD}
\end{aligned}$$

Using Theorem 1.2, one can derive the following bound by proceeding similarly as in the proof of the above lemma.

LEMMA 4.5. *Let $S \subset \mathbb{R}^n$ be a star-shaped body of diameter D with $\mathbb{B}_n(0, 1) \subseteq K_S$. Let $\eta(S) = \eta$, i.e. the fraction of $\mathsf{vol}(S)$ in the kernel K_S. Then for the ball walk with radius $r = s/4\sqrt{n}$, for any partition $A, \bar{A}$ of S satisfying $\mathsf{vol}(A), \mathsf{vol}(\bar{A}) > s(\mathsf{vol}(S))$, the s-conductance of A satisfies*

$$\phi_s(A) \geq \frac{\eta}{2^9} \min\left\{\frac{\mathsf{vol}(S)}{\min\{\mathsf{vol}(A), \mathsf{vol}(\bar{A})\}}, \frac{s}{2^7 n}\sqrt{\frac{\eta}{M_S}}\right\}$$

4.4 Mixing time Let π_S denote the uniform distribution over the star-shaped body. Let σ_m denote the distribution after m-steps of the ball walk on the star-shaped body. To relate the s-conductance to the mixing time, we use the following lemma from [LS93].

LEMMA 4.6. *Let $0 < s \leq 1/2$ and $H_s = \sup_{\pi_S(A) \leq s} |\sigma_0(A) - \pi_S(A)|$. Then for every measurable $A \subseteq \mathbb{R}^n$ and every $m \geq 0$,*

$$|\sigma_m(A) - \pi_S(A)| \leq H_s + \frac{H_s}{s}\left(1 - \frac{\phi_s^2}{2}\right)^m.$$

Proof. (of Theorem 1.3) Suppose σ_0 be a starting distribution such that there exists $M > 0$, $\forall A \subseteq S$, $\sigma_0(A) \leq M\pi_S(A)$. Now, by definition $H_s \leq M \cdot s$. Hence, using Lemma 4.6 and Lemma 4.4,

$$\begin{aligned}
d_{TV}(\sigma_m, \pi_S) &\leq M \cdot s + M\left(1 - \frac{s^2\eta^2}{2^{27}n^2D^2}\right)^m \\
&\leq M \cdot s + Me^{-ms^2\eta^2/2^{27}n^2D^2}.
\end{aligned}$$

Replacing s by $\epsilon/2M$, for $m \geq \frac{2^{29}n^2D^2M^2}{\eta^2\epsilon^2}\log\frac{2M}{\epsilon}$, we have $d_{TV}(\sigma_m, \pi_S) \leq \epsilon$.

5 Sampling algorithm

To obtain a polynomial-time sampling algorithm we make the additional assumption that we are given an oracle to the kernel of the star-shaped body, a point x_0 in the kernel and parameters r, R such that $\mathbb{B}_n(x_0, r)$ lies in the kernel and the kernel is contained in a ball of radius R. The sampling algorithm proceeds as follows:

1. Use the algorithm of [LV07, LV06a] to find a transformation of the body S into isotropic position and obtain a random point x_0 in K_S.
2. Perform m ball-walk steps from x_0 on the transformed body S', for each desired random point.

Clearly, by step 1 above, we have a $\frac{1}{\eta}$-warm start for the ball-walk on S'. Now, by Lemma 4.5, to obtain a bound on the s-conductance, we need an upper bound on the mean square distance $M_{S'}$ of the body S'.
We next show that when the kernel is isotropic, the body is not far from isotropic. This will bound $M_{S'}$ which along with Lemma 4.5 and Lemma 4.6 would prove Theorem 1.4.

LEMMA 5.1. *Let S be a star-shaped body and let K_S be the kernel of S. For a vector $v \in \mathbb{R}^n$, $\|v\| = 1$, define*

$$f_S(t) = \mathsf{vol}_{n-1}(\{x : v^T x = t, x \in \mathbb{R}^n\} \cap S) \quad \text{and}$$
$$f_K(t) = \mathsf{vol}_{n-1}(\{x : v^T x = t, x \in \mathbb{R}^n\} \cap K_S),$$

the cross-sectional volumes for S and K_S in direction v. Then for $x \in \operatorname{supp}(f_K)$ and $y \in \operatorname{supp}(f_S)$ and $\alpha \in [0,1]$ we have that

$$f_S(\alpha x + (1-\alpha)y)^{\frac{1}{n-1}} \geq \alpha f_K(x)^{\frac{1}{n-1}} + (1-\alpha) f_S(y)^{\frac{1}{n-1}}.$$

Furthermore, for all $x, y \in \mathbb{R}$ we get that

$$f_S(\alpha x + (1-\alpha)y) \geq f_K(x)^\alpha f_S(y)^{1-\alpha}$$

Proof. [Proof of Lemma 5.1 (Relation between cross-sectional volume of kernel and body)] Let $S(t), K_S(t)$ denote the cross-sections of S and K_S in direction v at t. Since $x \in \operatorname{supp}(f_K), y \in \operatorname{supp}(f_S)$ we have that $K_S(x), S(y) \neq \emptyset$. Since $K_S(x)$ is part of the kernel we have that

$$\alpha K_S(x) + (1-\alpha) S(y) \subseteq S(\alpha x + (1-\alpha)y).$$

Therefore by the Brunn-Minkowski inequality we have that

$$\alpha f_K(x)^{\frac{1}{n-1}} + (1-\alpha) f_S(y)^{\frac{1}{n-1}}$$

$$\leq \mathsf{vol}_{n-1}(\alpha K_S(x) + (1-\alpha)K_S(y))^{\frac{1}{n-1}}$$
$$\leq \mathsf{vol}_{n-1}(S(\alpha x + (1-\alpha)y))^{\frac{1}{n-1}} = f_S(\alpha x + (1-\alpha)y)^{\frac{1}{n-1}}$$

For the furthermore, we note that the statement is trivial if either $f_K(x) = 0$ or $f_S(y) = 0$. Therefore, we may assume that $x \in \operatorname{supp}(f_K), y \in \operatorname{supp}(f_S)$. Since the harmonic average is always smaller than the geometric average, the statement follows directly from our first inequality.

LEMMA 5.2. *Let $S \subseteq \mathbb{R}^n$ be a star-shaped body with an isotropic kernel K_S such that $\eta = \mathsf{vol}(K_S)/\mathsf{vol}(S)$. Then, in any direction v, for a random point X from K_S, we have*

$$\mathsf{E}((v^T X)^2) \leq \frac{3328}{\eta^2}.$$

Proof. [Proof of Lemma 5.2 (Second moment of body with isotropic kernel)] Let $v = (1, 0, \ldots, 0)^T$ w.l.o.g. Consider the cross-sectional density f_K induced by the kernel along v. Since K_S is isotropic, we have that $f_K(0) \geq \frac{1}{8}$ [LV06b].

Next let f be the cross-sectional density of the body S along v. It follows that

$$f(0) \geq \eta f_K(0) \geq \frac{\eta}{8}.$$

Let $a = \sup\left\{x : f(x) < \frac{\eta f_K(0)}{2}, x \leq 0\right\}$ and $b = \inf\left\{x : f(x) < \frac{\eta f_K(0)}{2}, x \geq 0\right\}$. We claim that $b - a \leq \frac{2}{\eta f_K(0)}$. Suppose not, then

$$\int_a^b f(x)\,dx \geq \frac{\eta f_K(0)}{2}(b-a) > 1.$$

Now consider a point $x = tb$ for $t > 1$. Then by Lemma 5.1 we have that

$$f(b) = f\left(\left(1 - \frac{1}{t}\right)0 + \frac{1}{t}x\right) \geq (\eta f_K(0))^{1-\frac{1}{t}} f(x)^{\frac{1}{t}} \Rightarrow$$

$$f(b)^t (\eta f_K(0))^{1-t} \geq f(x) \Rightarrow \eta f_K(0)\left(\frac{f(b)}{\eta f_K(0)}\right)^t \geq f(x)$$

The same inequality as above can be derived starting from any $b' > b$, and since for every such b' we have that $f(b') < \frac{\eta f_K(0)}{2}$ by continuity we have that for $t > 1$

$$f(x) \leq \eta f_K(0)\left(\frac{1}{2}\right)^t = \eta f_K(0) e^{-\ln 2t}$$

By a symmetric argument, the same bound holds for $x = ta$. Let $p = \int_a^b f(x)dx$. The following calculation gives the result:

$$\begin{aligned}
\mathsf{E}((v^T x)^2) &\leq \int_{-\infty}^{\infty} x^2 f(x)\,dx \\
&= p\left(\frac{1}{p}\int_a^b x^2 f(x)\right) \\
&\quad + \int_{-\infty}^a x^2 f(x) dx + \int_b^\infty x^2 f(x) dx \\
&\leq p \max\left\{a^2, b^2\right\} \\
&\quad + \eta f_K(0)\left[\int_{-\infty}^a x^2 e^{-\ln 2(x/a)}\,dx + \int_b^\infty x^2 e^{-\ln 2(x/b)}\,dx\right] \\
&= p \max\left\{a^2, b^2\right\} \\
&\quad + \eta f_K(0)(a^3 + b^3)\left(\frac{1}{2\ln 2} + \frac{1}{(\ln 2)^2} + \frac{1}{(\ln 2)^3}\right) \\
&\leq (a^2 + b^2) + (2a^2 + 2b^2)6 = 13(a^2 + b^2) \\
&\leq 13\frac{4}{\eta^2 f_K(0)^2} \leq \frac{3328}{\eta^2}.
\end{aligned}$$

Proceeding similarly as in the proof of Theorem 1.3, one can derive the proof of Theorem 1.4.

Proof. [Proof of Theorem 1.4 (Polynomial time amortized sampling)] Lemma 5.2 gives an upper bound on $M_S \leq \frac{2^{12} n}{\eta^2}$. Using Lemma 4.5, we get that the s-conductance of the ball walk on a star-shaped body

$S \subseteq \mathbb{R}^n$ with the kernel in isotropic position and $\eta = \mathsf{vol}(K_S)/\mathsf{vol}(S)$ satisfies

$$\Phi_s \geq \frac{\eta^{5/2} s}{2^{21} n^{3/2}}$$

By the sampling algorithm of [LV07, LV06a], Step 1 of the sampling algorithm takes $O^*(n^4)$ oracle queries. Since in step 2 of the algorithm, we started the ball-walk on S by choosing a random point from the kernel, and the kernel takes up at least an η fraction of the volume of S, a random point from it provides an $(1/\eta)$-warm start. Proceeding similarly as in the proof of Theorem 1.3, we get that after $m > \frac{2^{44} n^3}{\eta^4 \epsilon^2} \log \frac{2}{\eta\epsilon}$ ball walk steps, $d_{TV}(\sigma_m, \pi_S) \leq \epsilon$. Hence, by performing m-steps of the ball-walk for each desired sample, we obtain the desired amortized bound.

6 Optimization over star-shaped body

Here we prove that optimization over a star-shaped body is NP-hard. In particular, we reduce the clique problem to linear optimization over a star-shaped polyhedron.

DEFINITION 6.1. *An instance of CLIQUE(k) is given by a graph $G(V, E)$. The problem is to decide if there exists clique of size greater than k.*

It is well-known that CLIQUE(k) is NP-hard. We shall show that NP-hardness of optimization over a star-shaped body does not depend on the fraction of volume of the kernel. The main result we present here essentially follows from a theorem of Luedtke et al. [LSN07]. We simply modify their construction to ensure that the kernel of the set we construct is large. We present the complete proof here for completeness.

THEOREM 6.1. *Given a star-shaped polytope S, it is NP-hard to optimize a linear function over this body for any $\eta(S) < 1$, even if S is well-rounded.*

Proof. [Proof of Theorem 6.1] We reduce solving CLIQUE(k) to minimizing a linear function over a star-shaped body. Given a CLIQUE(k) instance $G(V, E)$, define variables $x \in \mathbb{R}^n$. For each edge $e = (i, j)$, define $\psi^e \in \mathbb{R}^n$,

$$\psi^e_l = \begin{cases} 1 & \text{if } l = i \text{ or } l = j, \\ 0 & \text{otherwise.} \end{cases}$$

For every edge e, denote the set of constraints given by $x \geq \psi^e$ as a block constraint. Consider the following formulation:

$$\text{Minimize} f(x) = \sum_{i=1}^{n} x_i, \text{ satisfying at least } \binom{k}{2} \text{ block constraints among:}$$

$$\forall e \in E,\ x \geq \psi^e \tag{6.1}$$

Define the feasible polyhedron as S.

LEMMA 6.1. *The feasible polyhedron S defined by the above formulation is star-shaped.*

Proof. First note that any subset of block constraints among the given constraints define a convex body. Thus, the feasible polyhedron is a union of convex bodies. Further, $x = (1\ 1 \ldots 1)^T$ satisfies all the constraints and hence, we have a non-empty kernel.

LEMMA 6.2. *By adding new constraints, a new feasible star-shaped polyhedron S' can be created such that $\eta(S')$ is a constant.*

Proof. Clearly $x_i \geq 1\ \forall i \in \{1, ..., n\}$ is a feasible convex region contained in K_S. Therefore, by adding constraints $x_i \leq a\ \forall\ i \in \{1, ..., n\}$, for appropriately chosen value of $a(> 1)$, one can make $\eta(S')$ a constant. Note that the set $1 \leq x_i \leq a\ \forall\ i \in \{1, ..., n\}$ is still a feasible convex region contained in K'_S. Specifically, one can choose $a = n$, to see that

$$\eta(S') \geq \left(\frac{n-1}{n}\right)^n \geq \frac{1}{e}$$

LEMMA 6.3. *There exists a clique of size k in $G(V, E)$, if and only if there exists $x \in S$ such that $f(x) \leq k$.*

Proof. Suppose the graph has a clique $C(V', E')$ of size k. Then, consider $x^* \in \mathbb{R}^n$ such that $x^*_v = 1\ \forall\ v \in V'$. Now, for every edge $e = (i, j) \in E'$, $x^* \geq \psi^e$ is satisfied since, $x^*_i = \psi^e_i = 1$ and $x^*_j = \psi^e_j = 1$ and $x^*_k \geq 0$ for $k \in V,\ k \neq i, j$. Since C is a clique, $|E'| = \binom{k}{2}$ and therefore, $\binom{k}{2}$ block constraints will be satisfied which implies that $x^* \in S$. It is straightforward to check that $f(x^*) = k$.
Suppose there exists $\bar{x} \in S$ such that $f(\bar{x}) \leq k$. The objective function $f(x)$ can be rewritten as $min_{F \subseteq E : |F| \geq \binom{k}{2}} \{\sum_{i=1}^{n} \max_{e \in F} \{\psi^e_i\}\}$. Hence, there exists $\bar{F} \subseteq E$, $|\bar{F}| \geq \binom{k}{2}$, such that the edges in $\bar{F}$ cover at most k vertices. Clearly, this is possible only when $\bar{F}$ defines a clique of size k.

Suppose there exists an algorithm A to optimize over a star-shaped body P given as an oracle such that $\eta(P) \geq c$. Now, given an instance of CLIQUE(k), we formulate the linear programming problem as above. Following Lemma 6.2 we can find an appropriate value of a and add constraints such that $\eta(S) \geq c$. Further, it is easy to make S contain a unit ball based on the value of a. Finally, the oracle queries can be answered by checking the number of block constraints satisfied by the point x. Hence, we may use A to minimize $f(x)$. Let z be the objective value obtained by optimizing using A. Using Lemma 6.3, it is clear that if $z \leq k$, CLIQUE(k) is a "Yes" instance, otherwise CLIQUE(k) is a "No" instance.

7 Discussion

We have presented isoperimetric inequalities and efficient sampling algorithms for star-shaped bodies, through a new technique called thin partitions. We note that linear optimization is NP-hard on these bodies, even when the kernel takes up a constant fraction of the body. Thus, quite unlike convex bodies, linear optimization is NP-hard over star-shaped bodies, but sampling remains tractable.

Given the sampling algorithm, we can also estimate the volume as follows: given an oracle for the kernel, we can sample from K_S and obtain the volume estimate for K_S using [LV07, LV06a]; further, given that $\eta(S) \geq \eta$ we can also estimate $\eta(S)$ using $O(\frac{1}{\eta^2 \epsilon^2})$ samples and output the product of the two as the estimate for volume.

We believe the thin partition approach should be broadly applicable to proving inequalities in convex geometry, especially for inequalities that do not seem reducible to one-dimensional versions (e.g., the KLS hyperplane conjecture [KLS95]).

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