Graph Stabilization: A Survey

Karthekeyan Chandrasekaran

Abstract Graph stabilization has raised a family of network design problems that has received considerable attention recently. Stable graphs are those graphs for which the matching game has non-empty core. In the optimization terminology, they are graphs for which the fractional matching linear program has an integral optimum solution. Graph stabilization involves minimally modifying a given graph to make it stable. In this survey, we outline recent developments in graph stabilization and highlight some open problems.

1 Introduction

The increasingly networked structure of human interactions in modern society has raised fascinating and novel game-theoretic questions [11, 37, 41]. Graph models for such networked interactions have been a central topic of research in algorithmic game theory over the last two decades [15, 28]. Fundamental to these interactions is matching, namely players in a network pairing up according to some self-interest. Formally, a matching in a graph is a collection of edges such that each vertex belongs to at most one edge in the collection. Owing to the simplicity of the description and the widespread nature of combinatorial objects that they can model, matchings have attracted much interest in games over networks [6, 8, 9, 18–20, 24, 26, 31, 40]. They play a prominent role in the network bargaining game introduced by Kleinberg and Tardos [31] as a network generalization of Nash's two-player bargaining game [36] as well as the classic cooperative matching game introduced by Shapley and Shubik [40].

An instance of a *cooperative matching game* is specified by an edge-weighted graph. The vertices correspond to players. The weight of an edge between two players represents the profit achievable from the relationship. The value of a coalition of players is the weight of the maximum matching achievable in the subgraph induced by the players in the coalition. The profit of the game is the value of the grand-

K. Chandrasekaran (✉)
University of Illinois, Urbana-Champaign, IL, USA
e-mail: karthe@illinois.edu

T. Fukunaga and K. Kawarabayashi (eds.), *Combinatorial Optimization and Graph Algorithms*, DOI 10.1007/978-981-10-6147-9_2

coalition, namely the maximum weight of a matching in the graph. The central network authority needs to distribute the profit of the game among the players so that the cumulative allocation to any coalition is at least its value. Such profit-sharing allocations are known as *core allocations* [23]. The existence of a core allocation ensures that no coalition can achieve larger profit than their allocated value by acting for themselves and hence, the stability of the grand coalition. Core allocations are desirable for the fundamental reason that they foster cooperation among the players.

Core allocations do not exist for every graph. For instance, consider a cycle on three vertices with unit-weight on the edges: any distribution of the value of the maximum matching in the graph, namely 1, among the three players leaves a pair of players who are cumulatively receiving less than 1, which is the value of the coalition formed by that pair. Graphs which have a core allocation are known as *stable graphs*. Since stable graphs are the only graphs for which the grand coalition of players cooperate, they are particularly attractive from the perspective of the central network authority. For this reason, when faced with an unstable graph, the central network authority is interested in stabilizing it, i.e., modifying it in the *least intrusive fashion* in order to convert it into a stable graph.

In this survey, we will describe recent progress in graph stabilization. The main goals of the survey are to provide game-theoretic interpretations of stabilization models and intermediate structural results, highlight the interplay between graph-theory and mathematical programming as algorithmic techniques towards stabilization, and state some immediate open problems to spur further research. The results to be presented will include stabilization by edge and vertex deletion, edge and vertex addition and edge weight addition [2, 10, 12, 27] (see Table 1 for a summary).

We assume familiarity with integer programming (IP), linear programming (LP), LP-duality (see Schrijver [38]), approximation terminology (see Vazirani [44], Shmoys-Williamson [46]), Edmonds' maximum matching algorithm [13] and basic graph theory (see West [45]). For an elaborate theory of matchings, we refer the reader to the textbooks by Lovász-Plummer [34] and Schrijver [39].

1.1 Definitions

Throughout, we are only interested in undirected graphs. We will denote an edge-weighted graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}_+$ by (G, w). For a vertex $u \in V$, let $\delta(u)$ denote the set of edges incident to u. For a subset $S \subseteq V$, let $E[S]$ denote the set of edges both of whose end vertices are in S, let $G[S]$ denote the subgraph induced by the vertices in S and $N_G(S)$ denote the set of vertices in $V \setminus S$ adjacent to at least one vertex in S. We denote the weighted subgraph induced by a subset $S \subseteq V$ by $(G[S], w)$. For given costs on edges (vertices), the cost of a subset S of edges (vertices) is the sum of the costs on the edges (vertices) in S. A matching in G is a subset M of edges such that each vertex is incident to at most one edge in M. The problem of finding a matching of maximum weight in (G, w) is formulated by the following integer program:

Table 1 Stabilization problems for unit-weight graphs. See Sects. 3, 4 and 5 for model descriptions

Model	Hardness	Approximation
min-EDGE-DEL	$(2-\varepsilon)$-inapprox for $\varepsilon > 0$	$O(\omega)$-approx in ω-sparse graphs, 2-approx in regular graphs
min-EDGE-ADD	P	–
min-Cost-EDGE-ADD	NP-hard	
max-EDGE-SUBGRAPH	NP-hard	$(5/3)$-approx
min-VERTEX-DEL	P	–
min-VERTEX-ADD	P	–
min-Cost-VERTEX-DEL	NP-hard	$(\|C\|+1)$-approx
max-Cost-VERTEX-SUBGRAPH	NP-hard	2-approx
min-EDGE-WT-ADD	$c \log \|V\|$-inapprox for some constant c, $O(\|V\|^{\frac{1}{16}-\varepsilon})$-inapprox for $\varepsilon > 0$ (assuming $O(\|V\|^{1/4-\varepsilon})$-inapprox of Densest k-subgraph)	Solvable in factor-critical graphs, Exact algorithm in graphs G with GED (B, C, D) in time $2^{\|C\|}\text{poly}(\|V(G)\|)$, $\min\{OPT, \sqrt{\|V(G)\|}\}$-approx in graphs G whose GED (B, C, D) has no trivial components in $G[B]$

$$\nu(G, w) := \max\left\{\sum_{e\in E} w_e x_e : \sum_{e\in\delta(v)} x_e \le 1\ \forall\, v \in V,\ x \in \mathbb{Z}_+^E\right\}. \tag{1}$$

Definition 1 A *cooperative matching game instance* is an edge-weighted graph (G, w). The value of a coalition S is $\nu(G[S], w)$ and the value of the game is $\nu(G, w)$. The *core* of an instance (G, w) consists of allocations $y \in \mathbb{R}_+^V$ satisfying $\sum_{u\in V} y_u = \nu(G, w)$ and $\sum_{u\in S} y_u \ge \nu(G[S], w)$ for every $S \subseteq V$. A weighted graph (G, w) is defined to be *stable* if its core is non-empty.

Relaxing the integrality constraints in $\nu(G, w)$, we obtain the maximum weight fractional matching linear program:

$$\nu_f(G, w) := \max\left\{\sum_{e\in E} w_e x_e : \sum_{e\in\delta(v)} x_e \le 1\ \forall\, v \in V,\ x_e \ge 0\ \forall\, e \in E\right\}. \tag{2}$$

The dual LP formulates the minimum fractional w-vertex cover:

$$\tau_f(G,w) := \min\left\{\sum_{u\in V} y_u : y_u + y_v \geq w_{\{u,v\}}\ \forall\,\{u,v\}\in E,\ \ y_v \geq 0\ \forall\, v\in V\right\}. \tag{3}$$

Imposing integrality requirements on the variables of the dual LP formulates the minimum w-vertex cover:

$$\tau(G,w) := \min\left\{\sum_{u\in V} y_u : y_u + y_v \geq w_{\{u,v\}}\ \forall\,\{u,v\}\in E,\ \ y_v \in \mathbb{Z}_+\ \forall\, v\in V\right\}. \tag{4}$$

The fractional problems are relaxations of the integral formulations. Further, by LP duality, the weight of a maximum fractional matching is equal to the weight of a minimum fractional w-vertex cover. Hence, we have the following relation:

$$\nu(G,w) \leq \nu_f(G,w) = \tau_f(G,w) \leq \tau(G,w). \tag{5}$$

We will use $\sigma(G,w) := \nu_f(G,w) - \nu(G,w)$ to denote the additive integrality gap of the fractional matching LP. It is well-known that every basic feasible solution to $\nu_f(G,w)$ is half-integral and moreover, the edges with half-integral components induce vertex disjoint odd cycles [5].

The rest of the treatise, except Sects. 5 and 6, will focus on uniform weight graphs, i.e., $w = \mathbb{1}$. When dealing with uniform weight graphs, we will omit the argument w for brevity (even from the ν, τ and σ notations).

2 Characterizing Stable Graphs

Before addressing the stabilization problems, we will discuss alternative and efficient characterizations of stable graphs and identify some well-known subfamilies of stable graphs. We will also illustrate the significance of stable graphs in the contexts of optimization and graph theory.

LP-based Characterization. The following alternative characterization of stable graphs is well-known.

Theorem 1 *[14, 31] A graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}_+$ is stable iff $\nu(G,w) = \nu_f(G,w)$.*

Theorem 1 implies that the algorithmic problem of verifying whether a given edge-weighted graph (G,w) is stable is solvable in polynomial time: we can solve the integer program $\nu(G,w)$ using Edmonds' max weight matching algorithm [16, 17], solve the linear program $\nu_f(G,w)$ using the Ellipsoid algorithm [25] and verify if the two values are equal. Further, *a witness of stability* for a stable graph is a matching M and a feasible fractional w-vertex cover y such that y satisfies complementary slackness conditions with the indicator vector of M. We will refer to such a witness of stability by the tuple (M, y).

In the terminology of discrete optimization, the *additive integrality gap of an instance for an LP* is the magnitude of the difference between the objective values of the optimum integral solution and the optimum LP solution. Using this terminology, an edge weighted graph (G, w) is stable iff the additive integrality gap of the instance (G, w) for the fractional matching LP is zero. Thus, the goal of stabilization is to modify the given instance (G, w) so that the additive integrality gap of the resulting instance (G', w') for the fractional matching LP is zero.

By Egerváry's theorem, if G is bipartite and $w \in \mathbb{Z}_+$, then $\nu(G, w) = \tau(G, w)$. Thus, bipartite graphs with integral edge weights are stable. In particular, unit-weight graphs with $\nu(G) = \tau(G)$ are known as *König-Egerváry graphs* [32, 33, 42] and by Theorem 1, König-Egerváry graphs are stable. Considering unit-weight graphs, we have the following strict containment relation: (see Figs. 1 and 2 illustrating strict containment):

$$\text{Bipartite} \subsetneq \text{König-Egerváry} \subsetneq \text{Stable}$$

Thus, stabilization closely resembles the goal of modifying a given graph to convert it into a König-Egerváry graph or a bipartite graph, both of which have been well-studied [1, 35]. While bipartite graphs are monotonic, i.e., closed under subgraphs, König-Egerváry graphs and stable graphs are not monotonic. For this reason, standard techniques to address graph modification problems to attain a monotonic property, e.g., [3], are not applicable for stabilization.

Graph-theoretic Characterization. We now mention equivalent graph-theoretic characterizations of unit-weight stable graphs. We will need some terminology

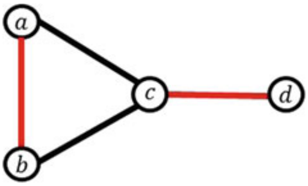

Fig. 1 A König-Egerváry graph that is not bipartite: $\nu(G) = 2$ and $S = \{a, c\}$ is a minimum vertex cover. However, the graph contains an odd cycle and hence is not bipartite

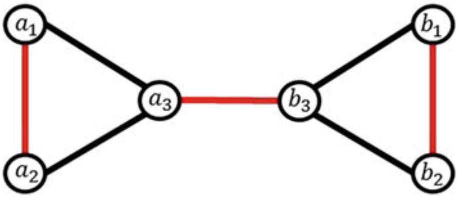

Fig. 2 A stable graph that is not König-Egerváry: $\nu(G) = 3$ and $\nu_f(G) = \tau_f(G) = 3$, since $y_{a_i} := 1/2$, $y_{b_i} := 1/2$ for all $i \in \{1, 2, 3\}$ is a feasible fractional vertex cover. However, $\tau(G) \geq 4$ since any feasible vertex cover S should have at least two vertices from $\{a_1, a_2, a_3\}$ as well as $\{b_1, b_2, b_3\}$

related to matchings. Let M be a matching in a graph $G = (V, E)$. A vertex $u \in V$ is said to be M-exposed if none of the edges in M contain u. A path is called *M-alternating* if it alternates edges from M and $E \setminus M$. An odd cycle of length $2k + 1$ in which exactly k edges are in M, for any positive integer k, is known as an *M-blossom*. The M-exposed vertex of an M-blossom is known as the *base vertex* of the M-blossom. An *M-flower* consists of an M-blossom with an even-length M-alternating path from the base vertex of the blossom to an M-exposed vertex. A matching in G is said to be a maximum matching if it has the maximum cardinality. A vertex $u \in V$ is said to be *inessential* if there exists a maximum matching in G that exposes u and is said to be essential otherwise. The *Gallai-Edmonds decomposition* (GED) [17, 21, 22] of a graph $G = (V, E)$ is a three partitioning (B, C, D) of the vertex set, where B is the set of inessential vertices, $C := N_G(B)$ and $D := V \setminus (B \cup C)$.

Theorem 2 *[4, 31, 43] Let $G = (V, E)$ be a graph with GED (B, C, D). The following are equivalent:*

1. *G is stable.*
2. *The induced subgraph $G[B]$ has no edges, i.e., the set of inessential vertices of G forms an independent set.*
3. *G contains no M-flower for every maximum matching M.*

Moreover, if G is unstable, then G contains an M-flower for every maximum matching M.

The Gallai-Edmonds decomposition (GED) and its properties will play an important role in stabilization. The GED of a graph is unique and can be found efficiently [17]. The GED of a graph contains valuable information that can be used to obtain optimal solutions to $\nu(G)$, $\nu_f(G)$ and $\tau_f(G)$. The following theorem summarizes some of these properties for our purposes. A graph is said to be *factor-critical* if for every vertex $u \in V$, there exists a matching M that exposes only u. A connected component is said to be *trivial* if it has only one vertex, otherwise it is said to be *non-trivial*.

Theorem 3 *(e.g. see Schrijver [39]) Let (B, C, D) denote the GED of a graph $G = (V, E)$ with B_1 being the subset of vertices of B which induce trivial components in $G[B]$ and $B_3 := B \setminus B_1$.*

1. *Each connected component in $G[B]$ is factor-critical.*
2. *Every maximum matching M in G contains a perfect matching in $G[D]$ and matches each vertex in C to distinct components in $G[B]$.*
3. *For every maximum matching M in G and every connected component K in $G[B]$, either (i) M exposes one vertex from K and has no edges leaving K or (ii) M does not expose any vertex from K and has exactly one edge leaving K.*
4. *The subset C is a choice of U that achieves the minimum in the Tutte-Berge formula for maximum matching,*

$$\nu(G) = \frac{1}{2} \min_{U \subseteq V} \{|V| + |U| - \textit{number of odd-sized components in } G \setminus U\}$$

and hence C is known as the Tutte set *of G.*

5. *Let M be a maximum matching that also matches the largest number of B_1 vertices. Then M exposes $2\sigma(G)$ vertices from B_3.*
6. *Let Y be a minimum vertex cover in the bipartite graph $H := (B_1 \cup C, \delta_G(B_1))$. Then, $y \in \mathbb{R}_+^V$ defined as follows is a minimum fractional vertex cover in G:*

$$y(v) := \begin{cases} 1 & \text{if } v \in C \cap Y, \\ 0 & \text{if } v \in B_1 \setminus Y, \\ 1/2 & \text{otherwise}. \end{cases}$$

For notational convenience we will denote the GED of a graph by (B, C, D) as well as $(B = (B_1, B_3), C, D)$ where B_1 and B_3 are as defined in Theorem 3 above.

3 Edge Modifications

In this section, we will focus on stabilizing a given graph by edge-deletion and edge-addition.

3.1 Edge Deletion

In the stabilization problem by edge deletion (min-EDGE-DEL), the input is a graph $G = (V, E)$, and the goal is to find a minimum cardinality subset F of edges to delete to stabilize the graph. This is equivalent to enabling non-empty core for a given cooperative matching game instance by blocking the smallest number of relationships. There is always a feasible solution to min-EDGE-DEL since removing all edges of G will give a stable graph. Moreover, enabling non-empty core to a matching game instance by blocking the smallest number of relationships *does not* decrease the value of the game:

Theorem 4 *[10] For every optimum solution F^* to min-EDGE-DEL in G, we have $\nu(G \setminus F^*) = \nu(G)$.*

We sketch an algorithmic proof. Let M be a maximum matching in G with minimum $|M \cap F^*|$. Suppose $M \cap F^* \neq \emptyset$ for the sake of contradiction. Let $H := (G \setminus F^*) + M$. Since M is a maximum matching in H, there exists an M-flower R in H starting at an M-exposed vertex w. If the M-flower R contains an edge from F^*, then we can switch M along an even M-alternating path to obtain a maximum matching M' with fewer edges from F^*. Hence, R is an $(M \setminus F^*)$-flower in H. This $(M \setminus F^*)$-flower R is also present in $G \setminus F^*$. Since $G \setminus F^*$ is stable, by the characterization of stable graphs in Theorem 2, the matching $M \setminus F^*$ is not a maximum matching in $G \setminus F^*$.

Now, consider executing Edmonds' maximum matching algorithm on the graph $G \setminus F$ using $M \setminus F^*$ as the initial matching and construct an $(M \setminus F^*)$*-alternating* *tree* starting from the $(M \setminus F^*)$-exposed vertex w. Then, we will find either an $(M \setminus F^*)$-augmenting path P starting at w or a *frustrated tree* rooted at w. If we find an $(M \setminus F^*)$-augmenting path P starting at w, then the other end-vertex of P should be adjacent to an edge $f \in M \cap F^*$ since M is a maximum matching in G. Switching M along $P + f$ gives another maximum matching M' in G with $|M' \cap F^*| < |M \cap F^*|$, a contradiction. If we find a frustrated tree rooted at w, say with vertex set T, then the final maximum matching M' in $G \setminus F^*$ identified by Edmonds' algorithm has $M' \cap E[T] = M \cap E[T]$. Thus, the M-flower R is also an M'-flower, a contradiction to the stability of $G \setminus F^*$.

Hardness. min-EDGE-DEL is NP-hard by a reduction from the Vertex-Cover problem. In fact, there is an approximation preserving reduction (using a gadget) from the Vertex-Cover problem to min-EDGE-DEL in factor-critical graphs [10]. Consequently, min-EDGE-DEL has no efficient $(2 - \varepsilon)$-approximation algorithm for any $\varepsilon > 0$ even in factor-critical graphs assuming the Unique Games Conjecture [30].

Lower bound. To design approximation algorithms, we first need a lower bound on the cardinality of the optimum solution $|F^*|$. Note that removing an edge can decrease the value of the minimum fractional vertex cover by atmost one. Consequently, removal of an edge can decrease the additive integrality gap for the fractional matching LP by at most one. Now, consider an arbitrary ordering of the edges in the optimum solution F^* and let F_i^* denote the first i edges according to this ordering and $F_0^* = \emptyset$. Then,

$$\sigma(G) - \sigma(G \setminus F^*) = \sum_{i=1}^{|F^*|} \left(\sigma(G - F_i^*) - \sigma(G - F_{i-1}^*)\right) \leq |F^*|.$$

Since $\sigma(G \setminus F^*) = 0$, we have that $|F^*| \geq \sigma(G)$. Properties of GED can be used to tighten this lower bound to $|F^*| \geq 2\sigma(G)$ [10]. We will later see GED in action to obtain this tight lower bound in Sect. 4.1.

Approximations. On the approximation side, min-EDGE-DEL has a 4ω-approximation in ω-sparse graphs and a 2-approximation in regular graphs [10]. A graph is said to be ω-sparse if $|E[S]| \leq \omega|S|$ for all $S \subseteq V$. We discuss the approximation in ω-sparse graphs. The core idea lies in the following lemma that follows from classic results on the structure of extreme point solutions to $\nu_f(G)$ and $\tau_f(G)$ [4, 43].

Lemma 1 *If G is an unstable ω-sparse graph, then there exists an efficient algorithm to find a subset L of edges such that (i) $|L| \leq 4\omega$, (ii) $\nu(G \setminus L) = \nu(G)$, and (iii) $\nu_f(G \setminus L) \leq \nu_f(G) - 1/2$.*

By the above lemma, we can find a small-sized subset L of edges such that the removal of L preserves the cardinality of the maximum matching while decreasing the additive integrality gap for the fractional matching LP by at least $1/2$. Thus, an algorithm to stabilize by edge deletion is to repeatedly apply the lemma until the graph becomes stable. In each application, we remove 4ω edges. The total number of iterations is not more than $2\sigma(G)$ since the additive integrality gap reduces by at least $1/2$ in each iteration. Hence the total number of edges removed is at most $8\omega(\nu_f(G) - \nu(G))$. Combining this with the lower bound mentioned above shows a 4ω-approximation.

An LP-relaxation. The improved 2-approximation for d-regular graphs is obtained by strengthening Lemma 1 to argue that $|L| \le d$ and tightening the lower bound using an LP-relaxation of min-EDGE-DEL. We now discuss this LP. Consider the following IP formulation of the min-EDGE-DEL problem, which we denote as $IP_{\text{edge-del}}$.

$$\min \left\{ \sum_{e \in E} z_e : y_u + y_v + z_{\{u,v\}} \ge 1 \ \forall \{u, v\} \in E, \sum_{u \in V} y_u = \nu(G), y \in \mathbb{R}_+^V, z \in \{0,1\}^E \right\}.$$

The variable $z_{\{u,v\}}$ indicates if the edge $\{u, v\}$ is to be deleted while the y variables correspond to a fractional vertex cover that will be a witness of stability after deleting the support of z. The validity of the formulation follows from Theorem 4. Considering the LP-relaxation of $IP_{\text{edge-del}}$, we obtain the following dual:

$$\max \left\{ \sum_{e \in E} \alpha_e - \gamma\nu(G) : \sum_{e \in \delta(u)} \alpha_e \le \gamma \ \forall\, u \in V, 0 \le \alpha_e \le 1 \ \forall\, e \in E \right\}$$

In d-regular graphs, the solution ($\gamma = d, \alpha_e = 1 \ \forall\, e \in E$) is a feasible dual solution and thus its objective value of $(d/2)\sigma(G)$ is a lower bound on the primal optimum value and in turn on $|F^*|$. Bock et al. [10] exhibit an example of a graph (non-regular) that demonstrates that the integrality gap of the LP-relaxation of $IP_{\text{edge-del}}$ (in fact, a stronger LP with more valid constraints) is $\Omega(|V|)$.

Open Problem 1. What is the approximability of min-EDGE-DEL in factor-critical graphs? It is impossible to obtain a $(2 - \varepsilon)$-approximation subject to the unique games conjecture as mentioned above. On the other hand, the main bottleneck in designing approximation algorithms is the lack of a good lower bound. The current known lower bound arguments (discussed above) only show that at least one edge needs to be deleted from a factor-critical graphs to stabilize it. However, there exist factor-critical graphs from which $\Omega(|V|)$ edges need to be deleted for stabilizing [10].

Open Problem 2. In the max-EDGE-SUBGRAPH problem, the input is a graph $G = (V, E)$ and the goal is to find a subgraph $F \subseteq E$ with the largest number of edges such that (V, F) is stable. Even though max-EDGE-SUBGRAPH is equivalent to min-EDGE-DEL from the perspective of exact solvability, they differ in the approximability (similar to min-BIPARTIZATION and max-CUT). max-EDGE-SUBGRAPH is NP-hard and we can obtain a $(5/3)$-approximation: Mishra et al. [35] give an algorithm to find a subset of edges of cardinality at least $5|E|/3$ such that the induced subgraph is König-Egerváry. However, stable graphs are a super-family of König-Egerváry and it might be possible to improve on this approximation factor. What is the approximability of max-EDGE-SUBGRAPH?

3.1.1 Edge Deletion to Stabilize a Given Matching

In cooperative matching game instances, the central network authority might be interested in stabilizing so that a chosen maximum matching is preserved. Formally, in min-EDGE-DEL-M-STAB, the input is a graph $G = (V, E)$ and a maximum matching M in G, while the goal is to find a subset F of edges that is disjoint from M whose deletion stabilizes the graph. This is to be viewed as stabilizing a particular matching M in the graph. There is always a feasible solution to min-EDGE-DEL-M-STAB since we may delete all non-matching edges.

Connections to min-EDGE-DEL. Edge deletion to stabilize a chosen matching is also of interest as an intermediate problem towards the approximability of min-EDGE-DEL. By Theorem 4, we know that some maximum matching M survives the deletion of the optimal solution to min-EDGE-DEL. If we can find such a matching M (that survives the deletion of the optimal solution to min-EDGE-DEL), then we may simply solve (approximate) min-EDGE-DEL-M-STAB with respect to M to obtain an (approximately) optimal solution for min-EDGE-DEL.

We currently do not know how to identify a matching M that survives the deletion of the optimum solution to min-EDGE-DEL. It is known that not every maximum matching may survive the deletion of the optimum solution to min-EDGE-DEL. Bock et al. [10] show an example with two matchings M and M' where the optimum to min-EDGE-DEL-M-STAB for matchings M and M' differ by a factor of $\Omega(|V|)$, so arbitrary choices of maximum matching M to stabilize will not lead to good approximations for min-EDGE-DEL.

Approximability. min-EDGE-DEL-M-STAB is NP-hard in weighted graphs [8]. For unit-weight graphs, we know tight approximability results for min-EDGE-DEL-M-STAB. By an approximation preserving reduction from the Vertex Cover problem [10], min-EDGE-DEL-M-STAB is NP-hard and has no efficient $(2 - \epsilon)$-approximation algorithm for any $\varepsilon > 0$ subject to Unique Games Conjecture even in factor-critical graphs. On the other hand, we can obtain a 2-approximation by an LP-based algorithm [10].

We briefly describe this LP-based algorithm. The LP is a modification of the LP-relaxation of $IP_{\text{edge-del}}$ where we impose the complementary slackness conditions with M explicitly:

$$
\begin{aligned}
\min \sum_{e \in E} z_e& \\
y_u + y_v &= 1 \ \forall \{u, v\} \in M \\
y_u + y_v + z_{\{u,v\}} &\geq 1 \ \forall \{u, v\} \in E \setminus M, \ u, v \in V(M) \\
y_v + z_{\{u,v\}} &\geq 1 \ \forall \{u, v\} \in E \setminus M, \ v \in V(M), \ u \notin V(M) \\
y, z &\geq 0.
\end{aligned}
$$

If G is bipartite, then the constraint matrix of the above LP is *totally unimodular* and hence all extreme point optimum solutions to the LP are integral. If G is non-bipartite, then a standard construction transforms the graph to a bipartite graph whose integral optimum solution can be used to obtain a 2-approximate solution for G [10].

3.2 Edge Addition

In the stabilization problem by edge addition (min-EDGE-ADD), the input is a graph $G = (V, E)$, and the goal is to find a minimum cardinality subset $F \subseteq \binom{|V|}{2} \setminus E$ of non-edges of G to add to stabilize the graph. This is equivalent to enabling non-empty core for a given cooperative matching game instance by introducing the smallest number of new relationships.

Feasibility. Note that min-EDGE-ADD may not have a feasible solution: For instance, consider a triangle which has $\nu_f(G) = 3/2$ while $\nu(G) = 1$ and there are no more non-edges to add. More generally, we have the following family of infeasible instances.

Lemma 2 *[27] If $|V|$ is odd and $\nu_f(G) = |V|/2$, then min-EDGE-ADD has no feasible solution.*

The lemma follows from the following observation: Since $\nu_f(G) = |V|/2 = \tau_f(G)$, the addition of any new non-edges to G will not decrease the optimum value of the fractional vertex cover and hence the fractional matching value will be at least $|V|/2$. On the other hand, since $|V|$ is odd, the size of the maximum matching after the addition of any new edges to G can be at most $(|V| - 1)/2$. Thus, adding new edges can at best reduce the additive integrality gap for the fractional matching LP to $1/2$ but not to zero.

Efficient Solvability. In fact, the family of graphs satisfying the hypothesis of Lemma 2 are the only unstable graphs that cannot be stabilized by adding edges. The remaining graphs can be stabilized by adding edges.

Theorem 5 *[27] If either $|V|$ is even or $\nu_f(G) < |V|/2$, then an optimum solution to min-EDGE-ADD can be found efficiently and its cardinality is equal to $\lceil\sigma(G)\rceil$.*

We sketch a proof assuming $2\sigma(G)$ is odd (the other case is similar). Since $2\sigma(G)$ is odd, we have $\nu_f(G) < |V|/2$ (otherwise, $|V| = 2\nu_f(G) = 2|M| + 2\sigma(G)$ by Theorem 3 and hence, $|V|$ is odd, a contradiction). A lower bound of $\lceil\sigma(G)\rceil$ on the size of the optimal solution follows from the observation that the addition of a single edge can decrease the additive integrality gap for the fractional matching LP by at most one (similar to the lower bound in Sect. 3.1).

Next we find a collection of $\lceil\sigma(G)\rceil$ edges whose addition stabilizes the graph. For this, consider the GED $(B = (B_1, B_3), C, D)$ of G and a maximum matching M that matches the largest number of B_1 vertices. By Theorem 3, the matching M exposes $2\sigma(G)$ vertices from B_3 with at most one from each component of $G[B_3]$. If there is no M-exposed vertex in B_1, then $|V| = 2|M| + 2\sigma(G) = 2\nu_f(G)$ contradicting $\nu_f(G) < |V|/2$. Let s be an M-exposed vertex from B_1. Let F^* denote an arbitrary pairing of the exposed nodes from B_3 and s. Then $|F^*| = \lceil\sigma(G)\rceil$. We will prove that $G + F^*$ is stable by showing that $\sigma(G + F^*) = 0$. For the integral optimum, we have $\nu(G + F^*) \geq \nu(G) + |F^*|$ since $M \cup F^*$ is a matching in $G + F^*$. It remains to bound the fractional optimum $\nu_f(G + F^*)$. Consider the fractional vertex cover y of G obtained using GED (Theorem 3). Increase $y(s)$ by $1/2$. Now, all end vertices u of F^* have $y_u \geq 1/2$. So y is also a feasible fractional vertex cover in $G + F^*$ and $\tau_f(G + F^*) \leq \tau_f(G) + 1/2$. Hence $\sigma(G + F^*) = \nu_f(G + F^*) - \nu(G + F^*) \leq \tau_f(G + F^*) - \nu(G) - |F^*| \leq \tau_f(G) + 1/2 - \nu(G) - |F^*| = 0$.

As a consequence of Theorem 5, enabling non-empty core to a matching game instance G by introducing the smallest number of new relationships will increase the value of the game by exactly $\lceil\sigma(G)\rceil$.

Polyhedral Description. The efficient solvability and the tight characterization of the optimal solution to min-EDGE-ADD raises the question of whether there exists a polyhedral description of the characteristic vectors of non-edges of a graph whose addition stabilizes the graph. This is unlikely to exist since the following min-Cost-EDGE-ADD problem is NP-hard [27]: In the min-Cost-EDGE-ADD problem, we are given a graph $G = (V, E)$ with costs $c : \binom{|V|}{2} \setminus E \to \mathbb{R}_+$ on the non-edges and the goal is to find a subset $F \subseteq \binom{|V|}{2} \setminus E$ of minimum cost whose addition stabilizes the graph.

Open Problem. What is the approximability of min-Cost-EDGE-ADD problem? We only know that the problem is NP-hard [27].

4 Vertex Modifications

In this section, we will focus on stabilizing a given graph by vertex-deletion and vertex-addition.

4.1 Vertex Deletion

In the stabilization problem by vertex deletion (min-VERTEX-DEL), the input is a graph $G = (V, E)$, and the goal is to find a minimum cardinality subset S of vertices whose removal stabilizes the graph. This is equivalent to enabling non-empty core for a given cooperative matching game instance by blocking the smallest number of players. There is always a feasible solution to min-VERTEX-DEL since removing all but one vertex will give a stable graph. Enabling non-empty core for a matching game instance by blocking the smallest number of players does not decrease the value of the game (similar to min-EDGE-DEL):

Theorem 6 *[2] For every optimum solution S^* to min-VERTEX-DEL in G, we have $\nu(G \setminus S^*) = \nu(G)$.*

Efficient Solvability. In contrast to min-EDGE-DEL, stabilization by minimum vertex deletion is solvable efficiently.

Theorem 7 *[2, 27] An optimum solution to min-VERTEX-DEL can be found efficiently and its cardinality is equal to $2\sigma(G)$.*

We first outline the lower bound using the properties of GED mentioned in Theorem 3. Let S^* be an optimum, $H := G \setminus S^*$, $(B = (B_1, B_3), C, D)$ be the GED of G and M be a maximum matching that matches the largest number of B_1 vertices. Then M exposes one vertex from $2\sigma(G)$ components in $G[B_3]$. Let $K_1, \ldots, K_r$ be these components.

For each such component K_i, we claim that at least one vertex $u_i \in V(K_i)$ becomes essential in H: otherwise, every vertex $u \in V(K_i) \setminus S^*$ is inessential in H. If $|S^* \cap V(K_i)| \geq 1$, then every maximum matching N in H necessarily exposes two vertices in $V(K_i)$ and hence by Property 3 of GED (Theorem 3) applied to G, we have that N is not a maximum matching in G. Hence, $\nu(H) = |N| < \nu(G)$, a contradiction to Theorem 6. If $S^* \cap V(K_i) = \emptyset$, then H is unstable, a contradiction.

Since u_i is essential in H, a maximum matching N in H will match u_i. We may assume without loss of generality that u_i is M-exposed. By Theorem 6, the graph $M \Delta N$ is a disjoint union of even paths and even cycles. The end-vertices of the paths that start at u_i must necessarily be in S^*, otherwise we may switch the matching N along this path to obtain a maximum matching N' in H, where u_i is N'-exposed, contradicting that u_i is essential in H.

The above proof technique is essentially a charging argument: for each M-exposed component in $G[B_3]$, there is a unique vertex in S^*. Ito et al. [27] present an alternative proof of the lower bound based on GED, but without using Theorem 6.

For the upper bound, consider a maximum matching M that matches the largest number of B_1 vertices. Then M exposes one vertex from $2\sigma(G)$ components in $G[B_3]$. For each such component, pick an arbitrary vertex into S^*. Then $|S^*| = 2\sigma(G)$ and $\nu(G - S^*) = \nu(G)$. In order to show that $G - S^*$ is stable, we will bound the fractional optimum value $\nu_f(G - S^*)$. Consider the fractional vertex cover y of G obtained using GED (Theorem 3). For each vertex $u \in S^*$, we have $y_u = 1/2$. Now projecting y to the remaining graph $G - S^*$ gives a feasible fractional vertex cover in $G - S^*$ with value $\tau_f(G) - |S^*|/2$. Hence, $\sigma(G - S^*) \leq \tau_f(G) - |S^*|/2 - \nu(G) = 0$.

4.2 Min Cost Vertex Deletion

The efficient solvability and the tight characterization of the optimal solution to min-VERTEX-DEL raises the next natural question of whether there exists a polyhedral description of the characteristic vectors of vertices whose deletion stabilizes the graph. However this is unlikely to exist since the minimum cost version of the stabilization by vertex deletion is NP-hard. We elaborate on the approximability of this problem in this section.

In the stabilization problem by min cost vertex deletion (min-Cost-VERTEX-DEL), the input is a graph $G = (V, E)$ with vertex-costs $c : V \to \mathbb{R}_+$, and the goal is to find a subset S of vertices of minimum cost $\sum_{u\in S} c_u$ whose removal stabilizes the graph. This is equivalent to enabling non-empty core for a given cooperative matching game instance by blocking the least cost set of players. The motivation is that not all players are equally important and hence the cost of blocking a player needs to be taken into account.

Approximation. min-Cost-VERTEX-DEL is NP-hard [2, 27] and has a $(|C| + 1)$-approximation [2], where (B, C, D) is the GED of the given graph G. We now discuss the approximation. Let $(B = (B_1, B_3), C, D)$ be the GED of the given graph G. The following structural properties of the optimal solution follow from Theorem 6.

Lemma 3 *Let S^* be an optimum to min-Cost-VERTEX-DEL. Then (i) S^* consists of only vertices in B, (ii) S^* contains at most one vertex from each component in $G[B]$ and (iii) if S^* contains a vertex from a component in $G[B]$, then that vertex is the least cost vertex in that component.*

The structural observations in Lemma 3 simplify the problem: replace the given instance G by contracting the non-trivial components in $G[B]$ and giving them a cost equal to the least cost vertex in the component. Let B_3 denote the vertex set of the contracted components. Delete D and the edges in $E[C]$. Denote the resulting bipartite graph as $G_b = (B \cup C, F)$. Now consider the following B_3-essentializer problem.

> In the min-Cost-ESSENTIALIZER problem, we are given a bipartite graph $G_b = (B \cup C, F)$, with costs on the B-vertices, $c : B \to \mathbb{R}_+$ and a subset $B_3 \subseteq B$ of vertices. The goal is to find a minimum cost subset $S \subseteq B$ of vertices to delete such that every vertex in $B_3 \setminus S$ becomes essential in $G_b \setminus S$.

There is an approximation preserving reduction from min-Cost-VERTEX-DEL to min-Cost-ESSENTIALIZER using the above construction: if S is a feasible solution to min-Cost-ESSENTIALIZER such that $\nu(G_b \setminus S) = \nu(G)$, then it gives a feasible solution to min-Cost-VERTEX-DEL of the same cost (by mapping $u \in S$ to the least cost vertex in the component that was contracted to u to obtain G_b). So the goal boils down to finding an approximately optimum solution S to min-Cost-ESSENTIALIZER such that $\nu(G_b \setminus S) = \nu(G_b)$. Ahmadian et al. [2] give a $(|C| + 1)$-approximation for min-Cost-ESSENTIALIZER by an LP-based algorithm.

An LP-relaxation. The IP formulation for min-Cost-ESSENTIALIZER by Ahmadian et al. [2] differs notably from the IP formulations that have appeared in the stabilization literature. We discuss this formulation and the LP-based $(|C| + 1)$-approximation. For each $v \in B$, introduce two indicator variables: z_v indicates if

$v \in S$ (i.e., to be deleted) and y_v indicates if v will be essential in $G_b \setminus S$. For each $u \in C$, introduce x_u to indicate if u will be matched to inessential nodes in every maximum matching of $G_b \setminus S$.

$$\min \sum_{u \in B} c_u z_u$$

$$y_v + z_v \geq 1 \; \forall \; v \in B_3 \tag{6}$$

$$x_u + y_v + z_v \geq 1 \; \forall \; \{u, v\} \in F, \;\; u \in C, \;\; v \in B_1 \tag{7}$$

$$y(B) + x(C) = |C| \tag{8}$$

$$y(N_{G_b}(A)) \geq |A| - x(A) \; \forall \; A \subseteq C \tag{9}$$

$$y, z \in \{0, 1\}^B, \;\; x \in \{0, 1\}^C$$

Constraint (6) formulates that each $v \in B_3$ is either deleted or becomes essential. Constraint (7) formulates that if $v \in B_1$ is not deleted, then either v is essential or the neighbor u is a vertex that is matched to an inessential vertex in a maximum matching in $G_b \setminus S$. Constraint (8) formulates that for each vertex in C, either it is matched to an inessential vertex in a maximum matching in $G_b \setminus S$ or there exists a unique vertex in B that is essential. Constraint (9) formulates Hall's condition: there exists a matching between $C \setminus \text{Support}(x)$ and $\text{Support}(y)$.

The approximation algorithm is to find an extreme point optimum to the LP-relaxation of the above IP and perform threshold rounding: set $S := \{v : z_v \geq 1/(|C| + 1)\}$. The approximation factor follows immediately. Ahmadian et al. exploit the properties of the extreme point solution to argue that the solution S is indeed a B_3-essentializer. Recall that we obtain a solution for min-Cost-VERTEX-DEL only if $\nu(G_b \setminus S) = \nu(G_b)$. So, in order to ensure that this condition is satisfied, we repair the solution S without losing feasibility: repeatedly remove vertices from S and add back into the graph if this operation increases the cardinality of the maximum matching. Tighter valid inequalities or alternate algorithmic techniques are needed to improve on the approximation factor since the integrality gap of the LP is $\Omega(|C|)$ [2].

Open Problem 1. Can we design an approximation algorithm for min-Cost-VERTEX-DEL whose approximation factor is independent of the size of the Tutte set (e.g., a constant factor approximation)? A first step is to consider input instances where each vertex has only one of two possible costs. Even this case is NP-hard [2], but no constant-factor approximation is known.

Open Problem 2. In the max-Cost-VERTEX-SUBGRAPH, the input is a graph $G = (V, E)$ with vertex costs $c : V \to \mathbb{R}_+$ and the goal is to find a subset of vertices $U \subseteq V$ with maximum cost such that the induced subgraph $G[U]$ is stable. This is equivalent to min-Cost-VERTEX-DEL from the perspective of exact solvability but not approximability. Ahmadian et al. [2] give a 2-approximation following the above reduction to min-Cost-ESSENTIALIZER and based on the associated LP. What is the approximability of max-Cost-VERTEX-SUBGRAPH?

4.3 Vertex Addition

In the stabilization problem by vertex addition (min-VERTEX-ADD), the input is a graph $G = (V, E)$, and the goal is to find a minimum cardinality subset S of vertices *along with some edges to* V whose addition stabilizes the graph. This is equivalent to enabling non-empty core for a given cooperative matching game instance by introducing the smallest number of new players with some relationships to the original players. There is always a feasible solution to min-VERTEX-ADD since we may pick an arbitrary maximum matching M and for each M-exposed vertex u, we can add a new vertex v that is adjacent to u. In fact, min-VERTEX-ADD has a tight characterization similar to min-VERTEX-DEL and min-EDGE-ADD.

Theorem 8 *[27] An optimum solution to min-VERTEX-ADD can be found efficiently and its cardinality is equal to* $2\sigma(G)$.

The proof technique is similar to the one discussed for min-EDGE-ADD (Theorem 5). In particular, the algorithm adds exactly $2\sigma(G)$ vertices with exactly one edge from each of these vertices, so the algorithm is also optimal if the goal is to minimize the number of edges adjacent to the newly added vertices. As a consequence of Theorem 8, optimal ways to enable non-empty core to a matching game instance G by introducing new players will increase the profit of the game by exactly $2\sigma(G)$.

5 Edge Weight Addition

In the stabilization problem by edge weight addition (min-EDGE-WT-ADD), the input is an edge-weighted graph $(G = (V, E), w')$. The goal is to find an increase in the edge weights $w : E \to \mathbb{R}_+$ so that the resulting edge-weighted graph $(G, w' + w)$ is stable and moreover the total increase in edge-weights $\sum_{e \in E} w_e$ is minimized. This is equivalent to enabling non-empty core for a given cooperative matching game instance by minimally increasing the weights on the edges.

In understanding the complexity of min-EDGE-WT-ADD, it is imperative to first address input graphs with uniform edge-weights, i.e., $w' = \mathbb{1}$. These are also the input graphs discussed in the previous sections. In the rest of this section, we will assume that the inputs are unit-weight graphs.

min-EDGE-WT-ADD is a continuous optimization problem as opposed to the stabilization problems considered in Sects. 3 and 4 which are discrete optimization problems. Note that min-EDGE-WT-ADD is not a continuous version of min-EDGE-ADD since the edge weights are allowed to increase only on the edges of the given graph.

There is always a feasible solution to min-EDGE-WT-ADD: consider a maximum matching M in G and increase the weights on the matching edges by one unit, i.e., set $w(e)$ to be 1 for $e \in M$ and 0 for $e \in E \setminus M$. Then, the characteristic vector of M is an optimum solution to $\nu(G, \mathbb{1} + w)$ while $y_v = 1$ for all vertices $v \in V(M)$ is a feasible fractional $(\mathbb{1}+w)$-vertex cover. Since $\nu(G, \mathbb{1}+w) = \tau_f(G, \mathbb{1}+w)$, we have that (M, y) is a witness of stability for $(G, \mathbb{1}+w)$.

The following result shows that every optimal edge-weight addition w to stabilize a unit-weight graph preserves the number of matching edges in the maximum $(\mathbb{1}+w)$-weight matching. Thus, enabling non-empty core for a given uniform-weight cooperative matching game instance by minimally increasing the weights does not decrease the number of matching edges in the grand-coalition.

Theorem 9 *[12] For every optimal solution w^* to min-EDGE-WT-ADD, every maximum $(\mathbb{1}+w^*)$-weight matching M^* has the same cardinality as $\nu(G)$. Moreover, $w^*(e) = 0$ on all edges $e \in E \setminus M^*$ and $0 \le w^*(e) \le 1$ for all edges $e \in M^*$.*

Thus, a maximum cardinality matching becomes a maximum weight matching after stabilizing by minimum edge weight addition. The structural properties of the optimum given in Theorem 9 can be used to solve min-EDGE-WT-ADD in factor-critical graphs.

Efficient algorithm in factor-critical graphs. Let M^* denote the maximum weight matching after stabilizing a factor-critical graph G by minimum edge weight addition. By Theorem 9, M^* exposes exactly one vertex. By guessing this vertex $a \in V$, we can find an optimum solution to min-EDGE-WT-ADD as follows: Find a maximum cardinality matching M exposing a. Solve the minimum fractional vertex cover LP with the additional constraint that $y_a = 0$. Now, set $w_{\{u,v\}} = y_u + y_v - 1$ for each matching edge $\{u, v\} \in M$ and $w_e = 0$ for non-matching edges. It is a simple exercise to show that (M, y) is a witness of stability for $(G, \mathbb{1}+w)$.

We will show that w is an optimum. Consider an optimum solution w^* to min-EDGE-WT-ADD. Let (M^*, y^*) be a witness of stability for $(G, \mathbb{1}+w^*)$. By complementary slackness conditions, $y_a^* = 0$ and $w^*(\{u, v\}) = y_u^* + y_v^* - 1$ for matching edges $\{u, v\} \in M^*$. We have the lower bound by the following:

$$\sum_{e\in E} w^*(e) = \sum_{\{u,v\}\in M^*} y_u^* + y_v^* - 1 = \sum_{u\in V} y_u^* - |M^*| = \sum_{u\in V} y_u - |M| = \sum_{e\in E} w(e).$$

The above argument shows that the precise choice of the matching M^* does not influence the optimum edge weight addition in factor-critical graphs; instead, the vertex exposed by M^* completely determines the optimum edge weights.

Arbitrary graphs: reducing to discrete decision domain. The following stronger structural properties of the optimum solution are helpful in the investigation of the complexity of min-EDGE-WT-ADD for arbitrary graphs.

Theorem 10 *[12] There exists an optimal solution w^* to min-EDGE-WT-ADD that is half-integral with a witness (M^*, y^*) of stability of $(G, \mathbb{1}+w^*)$ such that $y^* \in \{0, 1/2, 1\}^V$ with Support(y^*) containing the Tutte set.*

Theorem 10 simplifies the continuous decision domain of min-EDGE-WT-ADD to the discrete decision domain: the goal is to determine the half-integral weight increase for each edge.

In [12], we take an alternative perspective of the decision domain: we show an efficient algorithm to find the optimum w^* if we know the values of y^* for the Tutte vertices (the algorithm crucially relies on the solvability of min-EDGE-WT-ADD in factor-critical graphs). On the one hand, this immediately implies an algorithm to solve min-EDGE-WT-ADD exactly in time that is exponential only in the size of the Tutte set. On the other hand, the algorithm suggests that the complexity of min-EDGE-WT-ADD essentially lies in deciding whether the vertex-cover value y^*_v on each Tutte vertex v is $1/2$ or 1. We use this alternative discrete perspective to reduce the SET-COVER problem to min-EDGE-WT-ADD in an approximation preserving fashion (up to constant-factors). Consequently, min-EDGE-WT-ADD is inapproximable to a factor better than $c \log |V|$ for some constant c [12]. This is the only known super-constant inapproximability result for any of the stabilization models.

Yet another discrete perspective shows that min-EDGE-WT-ADD is possibly inapproximable to a much larger factor than $c \log |V|$: Suppose the GED (B, C, D) has only non-trivial components in $G[B]$. Then, knowledge of the factor-critical components of $G[B]$ that are exposed by M^* is sufficient to solve the problem exactly. However, there is a reduction from DENSEST k-SUBGRAPH to the problem of determining the M^*-exposed factor-critical components of $G[B]$ and in turn to min-EDGE-WT-ADD [12]. In the DENSEST k-SUBGRAPH, we are given a graph $H = (U, F)$, a positive integer k and the goal is to find a subset S of k vertices such that the induced subgraph $H[S]$ has the largest number of edges. DENSEST k-SUBGRAPH is believed to be a difficult problem [29] and possibly inapproximable to a factor better than the current best known $O(|V|^{1/4})$ [7]. The reduction from DENSEST k-SUBGRAPH suggests that min-EDGE-WT-ADD is unlikely to be approximable to a factor better than $O(|V|^{1/16})$.

In the positive direction, we have an algorithm that attains a $\min\{OPT, \sqrt{|V|}\}$-approximation factor for min-EDGE-WT-ADD in graphs G whose GED (B, C, D) has only non-trivial components in $G[B]$ [12].

Open Problem. Can we obtain an $O(\sqrt{|V|})$-approximation for min-EDGE-WT-ADD? Currently, we have a $O(\sqrt{|V|})$-approximation for min-EDGE-WT-ADD only in graphs whose GED (B, C, D) has no trivial components in $G[B]$.

6 Further Open Problems

We conclude with further open problems related to alternative stabilization models and weighted graphs.

Unit-weight graphs. For unit-weight graphs, we have the following stabilization problem in addition to the open problems mentioned in Sects. 3, 4, and 5.

1. Similar to edge weight addition, we may also consider edge weight deletion (min-EDGE-WT-DEL). The input here is a graph $G = (V, E)$ with unit weights on the

edges and the goal is to reduce the weights on the edges $w : E \to \mathbb{R}_+$ so that the resulting weighted graph $(G, \mathbb{1} - w)$ is stable and moreover the total decrease in edge-weights $\sum_{e \in E} w_e$ is minimized. This is equivalent to enabling non-empty core for a given cooperative matching game instance by minimally penalising the profits on some of the relationships. The complexity of this problem is open and this is perhaps solvable efficiently by an appropriate LP-formulation. Note that min-EDGE-WT-DEL is the continuous version of min-EDGE-DEL.

Weighted graphs. Given the limited knowledge of stabilization in unit-weight graphs, the following problems are realistic goals towards stabilizing weighted graphs $(G = (V, E), w : E \to \mathbb{R}_+)$.

1. In the min-EDGE-ADD problem in weighted graphs, we are given weights $w' : \binom{V}{2} \setminus E \to \mathbb{R}_+$ on the non-edges and the goal is to find a minimum cardinality subset of edges to add so that the resulting graph becomes stable. If w and w' are unity, then we have a thorough understanding of min-EDGE-ADD while the complexity of min-EDGE-ADD for arbitrary weights w and w' is open.
2. In the min-VERTEX-DEL problem in weighted graphs, the goal is to delete the smallest subset of vertices from G so that the remaining graph, with the given weights w on the surviving edges, becomes stable. If w is unity, then min-VERTEX-DEL is solvable efficiently. What is the complexity of min-VERTEX-DEL in weighted graphs?

Acknowledgements I would like to thank my collaborators in [10, 12] and the authors of [2, 27] for sharing their insights and results at various conference venues. Thanks also to Chandra Chekuri, Britta Peis and Corinna Gottschalk for helpful suggestions on previous drafts of the article.

References

1. A. Agarwal, M. Charikar, K. Makarychev, Y. Makarychev, O($\sqrt{\log n}$) approximation algorithms for min UnCut, min 2CNF deletion, and directed cut problems, in *Proceeding 37th Annual ACM Symposium Theory Computing* (2005), pp. 573–581,
2. S. Ahmadian, H. Hosseinzadeh, L. Sanitá, Stabilizing network bargaining games by blocking players, in *Proceedings Integer Programming and Combinatorial Optimization: 18th International Conference, IPCO 2016* (2016), pp. 164–177
3. N. Alon, A. Shapira, B. Sudakov, Additive approximation for edge-deletion problems, in *Proceedings 46th Annual IEEE Symposium Foundations Computer Science* (2005), pp. 419–428
4. E. Balas, Integer and fractional matchings. N.-Holl. Math. Stud. **59**, 1–13 (1981)
5. M.L. Balinski, On maximum matching, minimum covering and their connections, in *Proceedings of the Princeton Symposium on Mathematical Programming* (1970)
6. M.H. Bateni, M.T. Hajiaghayi, N. Immorlica, H. Mahini, The cooperative game theory foundations of network bargaining games, in *Proceeding 37th International Colloquium on Automata, Languages and Programming* (Springer, 2010), pp. 67–78
7. A. Bhaskara, M. Charikar, E. Chlamtac, U. Feige, A. Vijayaraghavan, Detecting high log-densities: an $O(n^{1/4})$ approximation for densest k-subgraph, in *Proceeding 42nd Annual ACM Symposium Theory Computing* (2010), pp. 201–210

8. P. Biró, M. Bomhoff, P. A. Golovach, W. Kern, D. Paulusma, Solutions for the stable roommates problem with payments, in *Graph-Theoretic Concepts in Computer Science*. Lecture Notes in Computer Science, vol. 7551 (2012), pp. 69–80
9. P. Biró, W. Kern, D. Paulusma, On solution concepts for matching games, in *Theory and Applications of Models of Computation* (Springer, 2010), pp. 117–127
10. A. Bock, K. Chandrasekaran, J. Könemann, B. Peis, L. Sanità, Finding small stabilizers for unstable graphs. Math. Program. **154**(1), 173–196 (2015)
11. G. Chalkiadakis, E. Elkind, M. Wooldridge, Computational aspects of cooperative game theory. Synth. Lect. Artif. Intell. Mach. Learn. **5**(6), 1–168 (2011)
12. K. Chandrasekaran, C. Gottschalk, J. Könemann, B. Peis, D. Schmand, A. Wierz, Additive stabilizers for unstable graphs (2016), arXiv: abs/1608.06797
13. W. Cook, W. Cunningham, W. Pulleyblank, A. Schrijver, *Combinatorial Optimization (Chapter 5)* (Wiley, 1998)
14. X. Deng, T. Ibaraki, H. Nagamochi, Algorithmic aspects of the core of combinatorial optimization games. Math. Oper. Res. **24**(3), 751–766 (1999)
15. D. Easley, J. Kleinberg, *Networks, Crowds, and Markets: Reasoning About a Highly Connected World* (Cambridge University Press, 2010)
16. J. Edmonds, Maximum matching and a polyhedron with 0, l-vertices. J. Res. Natl. Bur. Stand. B **69**(1965), 125–130 (1965)
17. J. Edmonds, Paths, trees, and flowers. Can. J. Math. **17**(3), 449–467 (1965)
18. K. Eriksson, J. Karlander, Stable outcomes of the roommate game with transferable utility. Int. J. Game Theory **29**(4), 555–569 (2000)
19. L. Farczadi, K. Georgiou, J. Könemann, Network bargaining with general capacities, in *Proceeding 21st Annual European Symposium on Algorithms* (2013), pp. 433–444
20. D. Gale, L. Shapley, College admissions and the stability of marriage. Am. Math. Mon. **69**(1), 9–14 (1962)
21. T. Gallai, Kritische Graphen II. Magy. Tud. Akad. Mat. Kutató Int. Közl **8**, 373–395 (1963)
22. T. Gallai, Maximale Systeme unabhängiger Kanten [German]. Magy. Tud. Akad. Mat. Kutató Int. Közl **9**, 401–413 (1964)
23. D. Gillies, Solutions to general nonzero sum games. Ann. Math. Stud. **IV**(40), 47–85 (1959)
24. D. Granot, A note on the room-mates problem and a related revenue allocation problem. Manag. Sci. **30**, 633–643 (1984)
25. M. Grötschel, L. Lovász, A. Schrijver, Geometric algorithms and combinatorial optimization, in *Algorithms and Combinatorics*, vol. 2, 2nd edn. (Springer, 1993)
26. J.W. Hatfield, P.R. Milgrom, Matching with contracts. Am. Econ. Rev. **95**(4), 913–935 (2005)
27. T. Ito, N. Kakimura, N. Kamiyama, Y. Kobayashi, Y. Okamoto, Efficient stabilization of cooperative matching games, in *Proceedings of the 2016 International Conference on Autonomous Agents and Multiagent Systems, AAMAS 2016* (2016), pp. 41–49
28. M.O. Jackson, *Social and Economic Networks* (Princeton University Press, Princeton, 2008)
29. S. Khot, Ruling out PTAS for graph min-bisection, dense k-subgraph, and bipartite clique. SIAM J. Comput. **36**(4), 1025–1071 (2005)
30. S. Khot, O. Regev, Vertex cover might be hard to approximate to within 2- ε. J. Comput. Syst. Sci. **74**(3), 335–349 (2008)
31. J. Kleinberg, É. Tardos, Balanced outcomes in social exchange networks, in *Proceedings 40th Annual ACM Symposium Theory Computing* (2008), pp. 295–304
32. E. Korach, Flowers and trees in a ballet of K_4, or a finite basis characterization of non-König-Egerváry graphs. Technical Report No. 115 (IBM Israel Scientific Center, 1982)
33. E. Korach, T. Nguyen, B. Peis, Subgraph characterization of red/blue-split graph and König Egerváry graphs, in *Proceedings, ACM-SIAM Symposium on Discrete Algorithms* (2006), pp. 842–850
34. L. Lovász, M. Plummer, *Matching Theory* (North Holland, 1986)
35. S. Mishra, V. Raman, S. Saurabh, S. Sikdar, C.R. Subramanian, The complexity of König subgraph problems and above-guarantee vertex cover. Algorithmica **61**(4), 857–881 (2011)
36. J. Nash Jr, The bargaining problem. Econom. J. Econom. Soc. 155–162 (1950)

37. N. Nisan, T. Roughgarden, É. Tardos, V. Vazirani, *Algorithmic Game Theory* (Cambridge University Press, Cambridge, UK, 2007)
38. A. Schrijver, *Theory of Linear and Integer Programming* (Wiley, 1998)
39. A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, vol. 24 (Springer, Berlin, Germany, 2003)
40. L.S. Shapley, M. Shubik, The assignment game I: The core. Int. J. Game Theory **1**(1), 111–130 (1971)
41. Y. Shoham, K. Leyton-Brown, *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations* (Cambridge University Press, 2008)
42. F. Sterboul, A characterization of the graphs in which the transversal number equals the matching number. J. Comb. Theory Ser. B (1979), pp. 228–229
43. J.P. Uhry, Sur le problème du couplage maximal. Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle **9**(3), 13–20 (1975)
44. V. Vazirani, *Approximation Algorithms* (Springer Inc., New York, 2001)
45. D. West, *Introduction to Graph Theory* (Prentice Hall, 2001)
46. D. Williamson, D. Shmoys, *The Design of Approximation Algorithms* (Cambridge University Press, 2011)