SIAM J. COMPUT.
Vol. 42, No. 6, pp. 2132–2155

DETERMINISTIC ALGORITHMS FOR THE LOVÁSZ LOCAL LEMMA*

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Abstract. The Lovász local lemma (LLL) [P. Erdős and L. Lovász, *Problems and results on 3-chromatic hypergraphs and some related questions*, in Infinite and Finite Sets, Vol. II, A. Hajnal, R. Rado, and V. T. Sós, eds., North–Holland, Amsterdam, 1975, pp. 609–627] is a powerful result in probability theory that informally states the following: the probability that none of a set of bad events happens is positive if the probability of each event is small compared to the number of events that depend on it. The LLL is often used for nonconstructive existence proofs of combinatorial structures. A prominent application is to k-CNF formulas, where the LLL implies that if every clause in a formula shares variables with at most $d \leq 2^k/e - 1$ other clauses, then such a formula has a satisfying assignment. Recently, a randomized algorithm to efficiently construct a satisfying assignment in this setting was given by Moser [*A constructive proof of the Lovász local lemma*, in STOC '09: Proceedings of the 41st Annual ACM Symposium on Theory of Computing, ACM, New York, 2009, pp. 343–350]. Subsequently Moser and Tardos [*J. ACM*, 57 (2010), pp. 11:1–11:15] gave a general algorithmic framework for the LLL and a randomized algorithm within this framework to construct the structures guaranteed by the LLL. The main problem left open by Moser and Tardos was to design an efficient deterministic algorithm for constructing structures guaranteed by the LLL. In this paper we provide such an algorithm. Our algorithm works in the general framework of Moser and Tardos with a minimal loss in parameters. For the special case of constructing satisfying assignments for k-CNF formulas with m clauses, where each clause shares variables with at most $d \leq 2^{k/(1+\epsilon)}/e - 1$ other clauses, for any $\epsilon \in (0, 1)$, we give a deterministic algorithm that finds a satisfying assignment in time $\tilde{O}(m^{2(1+1/\epsilon)})$. This improves upon the deterministic algorithms of Moser and of Moser and Tardos with running times $m^{\Omega(k^2)}$ and $m^{\Omega(d \log d)}$, respectively, which are superpolynomial for $k = \omega(1)$ and $d = \omega(1)$, and upon the previous best deterministic algorithm of Beck, which runs in polynomial time only for $d \leq 2^{k/16}/4$. Our algorithm is the first deterministic algorithm that works in the general framework of Moser and Tardos. We also give a parallel NC algorithm for the same setting, improving upon an algorithm of Alon [*Random Structures Algorithms*, 2 (1991), pp. 367–378].

Key words. probabilistic method, derandomization, satisfiability, parallelization

AMS subject classifications. 68Q25, 68R05

DOI. 10.1137/100799642

1. Introduction.

The Lovász local lemma (LLL) [7] informally states that the probability that none of a set of bad events happens is nonzero if the probability of each event is small compared to the number of events that depend on it (see section 1.1 for details). It is a powerful result in probability theory and is often used in conjunction with the probabilistic method to prove the existence of combinatorial structures. For this, one designs a random process guaranteed to generate the desired structure if none of a set of bad events happens. If the events satisfy the above-

*Received by the editors June 21, 2010; accepted for publication (in revised form) July 22, 2013; published electronically November 21, 2013. A preliminary version of this work appeared in *Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms* (SODA), Austin, TX, pp. 992–1004.

http://www.siam.org/journals/sicomp/42-6/79964.html

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mentioned condition, then the LLL guarantees that the probability that the random process builds the desired structure is positive, thereby implying its existence. For many applications of the LLL, it is also important to find the desired structures efficiently. Unfortunately, the original proof of the LLL [7] does not lead to an efficient algorithm. In many applications of the LLL, the probability of none of the bad events happening is negligible. Consequently, the same random process does not directly provide a randomized algorithm to find the desired structure. Further, in most applications where the LLL is useful (e.g., [9, 12, 14]), the proof of existence of the desired structure is known only through the LLL (one exception to this is [9]). Thus, an efficient algorithm for the LLL would also lead to an efficient algorithm to find these desired structures. Starting with the work of Beck [3], a number of papers, e.g., [1, 5, 15, 16, 21], have sought to make the LLL algorithmic. Before discussing these results in more detail we describe the LLL formally.

1.1. The Lovász local lemma. The LLL gives a lower bound on the probability of avoiding a possibly large number of "bad" events that are not "too dependent" on each other. Let $\mathcal{A}$ be a finite set of events in a probability space. Let G be an undirected graph on vertex set $\mathcal{A}$ with the property that every event $A \in \mathcal{A}$ is mutually independent[1] of the set of all events not in its neighborhood. We assume throughout that G does not contain any self-loops. We denote the set of neighbors of an event A by $\Gamma(A)$, i.e., $\Gamma(A) := \{B \in \mathcal{A} \mid \{A, B\} \in E(G)\}$. The general version of the LLL is the following.

THEOREM 1 (see [7, 20]). *For $\mathcal{A}$ and G as defined above, suppose there exists an assignment of reals $x : \mathcal{A} \to (0, 1)$ such that for all $A \in \mathcal{A}$,*

$$\mathsf{Pr}(A) \leq x(A) \prod_{B \in \Gamma(A)} (1 - x(B)).$$

Then the probability of avoiding all events in $\mathcal{A}$ is nonzero. More precisely,

$$\mathsf{Pr}\left(\bigcap_{A \in \mathcal{A}} \overline{A}\right) \geq \prod_{A \in \mathcal{A}} (1 - x(A)) > 0.$$

A simple corollary of the LLL, called symmetric LLL, often suffices in several applications. In this version there is a uniform upper bound p on the probability of each event and a uniform upper bound d on the number of neighbors of each event in the dependency graph. This quantity $|\Gamma(A)|$ is also called the dependency degree of the event A.

COROLLARY 2 (see [7]). *If each event $A \in \mathcal{A}$ occurs with probability at most p and has dependency degree $|\Gamma(A)| \leq d$ such that $d \leq 1/ep - 1$, then the probability that none of the events occur is positive.*

Proof. Setting $x(A) = 1/(d + 1)$ for all events $A \in \mathcal{A}$ shows that the conditions of Theorem 1 are satisfied:

$$\mathsf{Pr}(A) \leq p \leq \frac{1}{e(d+1)} \leq \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^d. \qquad \square$$

The power of the symmetric version is well demonstrated by showing a satisfiability result for k-CNF formulas, i.e., Boolean formulas in conjunctive normal form with

[1] An event A is mutually independent of a set of events $\{B_1, B_2, \ldots\}$ if $\mathsf{Pr}(A) = \mathsf{Pr}(A \mid f(B_1, B_2, \ldots))$ for every function f that can be expressed using finitely many unions and intersections of the arguments.

k variables per clause. This classic application of the LLL will help in understanding our and previous results and techniques and therefore will be a running example in the rest of the paper.

COROLLARY 3. *Every k-CNF formula in which every clause shares variables with at most $2^k/e - 1$ other clauses is satisfiable.*

Proof. To apply the symmetric LLL (i.e., Corollary 2) we choose the probability space to be the product space of each variable being chosen true or false independently with probability 1/2. For each clause C we define an event A_C that is said to occur if and only if clause C is not satisfied. Clearly, two events A_C and $A_{C'}$ are independent unless the clauses C and C' share variables. Now take G to be the graph on the events with edges between events A_C and $A_{C'}$ if and only if C and C' share variables. It is clear that each event A_C is mutually independent of its nonneighbors in G. By assumption each event has at most $d \leq (2^k/e) - 1$ neighbors. Moreover, the probability p that a clause is not satisfied by a random assignment is exactly 2^{-k}. The requirement $ep(d+1) \leq 1$ of Corollary 2 is therefore met, and hence we obtain that the probability that none of the events occur is positive. The satisfiability of the k-CNF formula follows. ∎

1.2. Previous work. Algorithms for the LLL are often targeted toward one of two model problems: k-CNF formula satisfiability and k-uniform hypergraph 2-coloring. Interesting in their own right, these problems capture the essence of the LLL without many technicalities. Moreover, algorithms for these problems usually lead to algorithms for more general applications of the LLL [5, 6, 14]. As shown in section 1.1, for the k-CNF formula satisfiability problem, the LLL implies that every k-CNF formula in which each clause shares variables with at most $2^k/e - 1$ other clauses has a satisfying assignment. Similarly, it can be shown that the vertices of a k-uniform hypergraph, in which each edge shares variables with at most $2^k/e - 1$ other edges, can be colored using two colors so that no edge is monochromatic. The algorithmic objective is to efficiently find such a 2-coloring (or a satisfying assignment in the case of k-CNF).

This question was first addressed by Beck in his seminal paper [3], where he gave an algorithm for the hypergraph 2-coloring problem with dependency degree $O(2^{k/48})$. More precisely, he gave a polynomial-time deterministic algorithm to find a 2-coloring of the vertices of every k-uniform hypergraph in which each edge shares vertices with $O(2^{k/48})$ other edges such that no edge is monochromatic. Molloy and Reed [14] showed that the dependency degree of this algorithm can be improved to $2^{k/16}/4$. In the same volume in which Beck's paper appeared, Alon [1] gave a randomized parallel version of Beck's algorithm that outputs a valid 2-coloring when the dependency degree is at most $2^{k/500}$ and showed that this algorithm can be derandomized.[2] Since then, tremendous progress has been made on randomized LLL algorithms. Nonetheless, prior to our work, Beck's and Alon's algorithms remained the best deterministic and parallel algorithms for the (symmetric) LLL.

For randomized algorithms and algorithms that require k to be a fixed constant, a long line of work improved the maximum achievable dependency degree and the generality of the results, culminating in the work of Moser and Tardos [18], who provided a simple randomized (parallel) algorithm for the general LLL. These results are summarized in Table 1, and we discuss them next.

[2] Alon did not attempt to optimize the exponent, but Srinivasan [21] states that optimizing the bound would still lead to an exponent with several tens in the denominator.

TABLE 1
Maximum dependency degrees achieved for k-CNF formulas by previous randomized, deterministic and parallel algorithms.

	Max. Dep. Deg. d	Det.	Par.	Remark
Beck [3]	$O(2^{k/48})$	X		
Molloy and Reed [14]	$O(2^{k/16})$	X		prev. best det. algorithm
Alon [1]	$O(2^{k/500})$	X	X	prev. best det. par. algorithm
	$O(2^{k/8})$	X	X	efficient only for constant k,d
Srinivasan [21]	$O(2^{k/4})$			
	$O(2^{k/10.3})$		X	
Moser [16]	$O(2^{k/2})$			
	$O(2^{k/2})$	X		efficient only for constant k,d
Moser [17]	$O(2^k)$			
	$O(2^k)$	X		efficient only for constant k,d
Moser and Tardos [18]	$(2^k/e-1)$			
	$(1-\epsilon)\cdot(2^k/e-1)$		X	
	$(1-\epsilon)\cdot(2^k/e-1)$	X	X	efficient only for constant k,d
Our work	$(2^{k/(1+\epsilon)}/e-1)$	X	X	

Alon [1] gave an algorithm that is efficient for a dependency degree of $O(2^{k/8})$ if one assumes that k, and therefore also the dependency degree, is bounded above by a fixed constant. Molloy and Reed [15] generalized Alon's method to give efficient algorithms for a certain set-system model for applications of the symmetric form of the LLL. Czumaj and Scheideler [5, 6] consider the algorithmic problem for the asymmetric version of the LLL. The asymmetric version of the LLL addresses the possibility of 2-coloring the vertices of nonuniform hypergraphs with no monochromatic edges. The next improvement in increasing the dependency degree threshold was due to Srinivasan [21]. He gave a randomized algorithm for hypergraph 2-coloring when the dependency degree is at most $2^{k/4}$. Moser [16] improved the dependency degree threshold to $O(2^{k/2})$ using a variant of Srinivasan's algorithm. Later, Moser [17] achieved a significant breakthrough, improving the dependency degree threshold to 2^{k-5} using a much simpler randomized algorithm. Moser and Tardos [18] closed the small constant-factor gap to the optimal dependency degree $2^k/e$ guaranteed by the general LLL.

More importantly, Moser and Tardos [18] gave an algorithmic framework for the general version of the LLL (discussed in section 2.1) that minimally restricts the abstract LLL setting to make it amenable to algorithmic considerations. In this framework they gave an efficient randomized algorithm for computing the structures implied by the LLL. The importance of the framework stems from the fact that it captures most of the LLL applications, thus directly providing algorithms for these applications. Moser [16, 17] and Moser and Tardos [18] also gave a derandomization of their algorithms, obtaining an algorithm that runs in $m^{O((1/\epsilon)d\log d)}$ time, where d is the maximum dependency degree and m is the number of events. For the simpler k-CNF problem, the running time of the deterministic algorithms can be improved to $m^{O(k^2)}$. Nonetheless, this running time is polynomial only under the strong condition that k and the dependency degree are bounded by a fixed constant.

The main question that remained open was how to obtain deterministic algorithms that go beyond the initial results of Beck [3] and that are efficient for unbounded dependency degrees. We address this question by giving new derandomizations of the Moser–Tardos algorithm. We give a derandomization that works efficiently for the

general version of the LLL in the aforementioned algorithmic framework of Moser and Tardos [18], assuming a mild ϵ-slack in the LLL conditions. As a corollary, we obtain an algorithm that runs in time $\tilde{O}(m^{2(1+(1/\epsilon))})$ to find a satisfying assignment for a k-CNF formula with m clauses such that no clause shares variables with more than $2^{k/(1+\epsilon)}/e$ other clauses for any $\epsilon > 0$. We note that our ϵ-slack assumption is in the exponent as opposed to the multiplicative slackness in the Moser and Tardos results (see Table 1). We also extend the randomized parallel algorithm of Moser and Tardos to obtain an efficient deterministic parallel algorithm under the same assumption, thereby improving over Alon's algorithm with a dependency degree of $O(2^{k/500})$.

Organization. In section 2, we describe the algorithmic framework of Moser and Tardos for the LLL and their algorithm. In section 3, we state our results and their implications for the k-CNF problem. In section 4, we give an informal description of the new ideas in the paper. In section 5, we formally define the major ingredient in our derandomization: the partial witness structure. In section 6, we give our sequential deterministic algorithm and analyze its running time. Finally, in section 7, we present our parallel algorithm and its running time analysis.

2. Preliminaries.

2.1. Algorithmic framework.

To get an algorithmic handle on the LLL, we move away from the abstract probabilistic setting of the original LLL. We impose some restrictions on the representation and form of the probability space under consideration. In this paper we follow the algorithmic framework for the LLL due to Moser and Tardos [18]. We describe the framework in this section.

The probability space is given by a finite collection of mutually independent discrete random variables $\mathcal{P} = \{P_1, \ldots, P_n\}$. Let D_i be the domain of P_i, which is assumed to be finite. Every event in a finite collection of events $\mathcal{A} = \{A_1, \ldots, A_m\}$ is determined by a subset of $\mathcal{P}$. We define the *variable set* of an event $A \in \mathcal{A}$ as the unique minimal subset $S \subseteq \mathcal{P}$ that determines A and denote it by $\mathrm{vbl}(A)$.

The *dependency graph* $G = G_{\mathcal{A}}$ of the collection of events $\mathcal{A}$ is a graph on vertex set $\mathcal{A}$. The graph $G_{\mathcal{A}}$ has an edge between events $A, B \in \mathcal{A}$, $A \neq B$ if $\mathrm{vbl}(A) \cap \mathrm{vbl}(B) \neq \emptyset$. For $A \in \mathcal{A}$ we denote the neighborhood of A in G by $\Gamma(A) = \Gamma_{\mathcal{A}}(A)$ and define $\Gamma^+(A) = \Gamma(A) \cup \{A\}$. Note that events that do not share variables are independent.

It is useful to think of $\mathcal{A}$ as a family of "bad" events. The objective is to find a point in the probability space or, equivalently, an evaluation of the random variables from their respective domains, for which none of the bad events happens. We call such an evaluation a *good evaluation*.

Moser and Tardos [18] gave a constructive proof of the general version of the LLL in this framework (Theorem 4) using Algorithm 1, presented in the next section. This framework captures most known applications of the LLL.

2.2. The Moser–Tardos algorithm.

Moser and Tardos [18] presented the very simple Algorithm 1 to find a good evaluation.

Observe that if the algorithm terminates, then it outputs a good evaluation. The following theorem from [18] shows that the algorithm is efficient if the LLL conditions are met.

THEOREM 4 (see [18]). *Let $\mathcal{A}$ be a collection of events as defined in the algorithmic framework defined in section* 2.1. *If there exists an assignment of reals $x : \mathcal{A} \to (0, 1)$*

ALGORITHM 1 (sequential Moser–Tardos algorithm).
1. For every $P \in \mathcal{P}$, $v_P \leftarrow$ a random evaluation of P.
2. While $\exists A \in \mathcal{A}$ such that A happens on the current evaluation $(P = v_P : \forall P \in \mathcal{P})$, do
   (a) Pick one such A that happens (any arbitrary choice would work).
   (b) Resample A: For all $P \in \mathrm{vbl}(A)$, do
      • $v_P \leftarrow$ a new random evaluation of P.
3. Return $(v_P)_{P \in \mathcal{P}}$.

such that for all $A \in \mathcal{A}$,

$$\mathsf{Pr}(A) \le x'(A) := x(A) \prod_{B \in \Gamma(A)} (1 - x(B)),$$

then the expected number of resamplings done by Algorithm 1 *is at most* $\sum_{A \in \mathcal{A}} (x(A)/(1 - x(A)))$.

3. Results. This section formally states the new results established in this paper.

If an assignment of reals as stated in Theorem 4 exists, then we use such an assignment to define the following parameters:[3]

- $x'(A) := x(A) \prod_{B \in \Gamma(A)} (1 - x(B))$.
- $D := \max_{P_i \in \mathcal{P}} \{|D_i|\}$.
- $M := \max\{n, 4m, 2 \sum_{A \in \mathcal{A}} \frac{2|\mathrm{vbl}(A)|}{x'(A)} \cdot \frac{x(A)}{1-x(A)}, \max_{A \in \mathcal{A}} \frac{1}{x'(A)}\}$.
- $w_{min} := \min_{A \in \mathcal{A}} \{-\log x'(A)\}$.
- $\gamma = \frac{\log M}{\epsilon}$.

For the rest of this paper, we will use these parameters to express the running time of our algorithms.

Our sequential deterministic algorithm assumes that for every event $A \in \mathcal{A}$, the conditional probability of occurrence of A under any partial assignment to the variables in $\mathrm{vbl}(A)$ can be computed efficiently. This is the same complexity assumption as used in Moser and Tardos [18]. It can be further weakened to use pessimistic estimators.

THEOREM 5. *Let the time needed to compute the conditional probability* $\mathsf{Pr}[A \mid$ *for all* $i \in I : P_i = v_i]$ *for any* $A \in \mathcal{A}$ *and any partial evaluation* $(v_i \in D_i)_{i \in I}$, $I \subseteq [n]$ *be at most* t_C. *Suppose there is an* $\epsilon \in (0, 1)$ *and an assignment of reals* $x : \mathcal{A} \to (0, 1)$ *such that for all* $A \in \mathcal{A}$,

$$\mathsf{Pr}(A) \le x'(A)^{1+\epsilon} = \left(x(A) \prod_{B \in \Gamma(A)} (1 - x(B)) \right)^{1+\epsilon}.$$

Then there is a deterministic algorithm that finds a good evaluation in time

$$O\left(t_C \cdot \frac{D M^{3+2/\epsilon} \log^2 M}{\epsilon^2 w_{min}^2} \right),$$

where the parameters D, M, and w_{min} are as defined above.

[3]Throughout this paper log denotes the logarithm to base 2.

We make a few remarks to give a perspective on the magnitudes of the parameters involved in our running time bound. As a guideline to reading the results, it is convenient to think of M as $\tilde{O}(m+n)$ and of w_{min} as $\Omega(1)$.

Indeed, $w_{min} = \Omega(1)$ holds whenever the $x(A)$'s are bounded away from one by a constant. For this setting we also have, without loss of generality,[4] that $x(A) = \Omega(m^{-1})$. Lastly, the factor $(\prod_{B\in\Gamma(A)}(1-x(B)))^{-1}$ is usually small; e.g., in all applications using the symmetric LLL or the simple asymmetric version [14, 15] this factor is a constant. This makes M at most a polynomial in m and n. For most applications of the LLL this also makes M polynomial in the size of the input/output. For all these settings our algorithms are efficient: the running time bound of our sequential algorithm is polynomial in M and that of our parallel algorithm is polylogarithmic in M using at most $M^{O(1)}$ many processors.

Notable exceptions in which M is not polynomial in the input size are the problems in [10]. For these problems M is still $\tilde{O}(m+n)$ but the number of events m is exponential in the number of variables n and the input/output size. For these settings, the problem of checking whether a given evaluation is good is coNP-complete, and obtaining a derandomized algorithm is an open question.

It is illuminating to look at the special case of k-CNF both in the statements of our theorems and in the proofs, as many of the technicalities disappear while retaining the essential ideas. For this reason, we state our results also for k-CNF. The magnitudes of the above parameters in the k-CNF applications are given by $x'(A) > 1/de$, $D = 2$, $M = \tilde{O}(m)$, and $w_{min} \approx k$.

COROLLARY 6. *For any $\epsilon \in (0,1)$ there is a deterministic algorithm that finds a satisfying assignment to any k-CNF formula with m clauses in which each clause shares variables with at most $2^{k/(1+\epsilon)}/e - 1$ other clauses in time $\tilde{O}(m^{3+2/\epsilon})$.*

We also give a parallel deterministic algorithm. This algorithm makes a different complexity assumption about the events, namely, that their decision tree complexity is small. This assumption is quite general and includes almost all applications of the LLL (except, again, for the problems mentioned in [10]). They are an interesting alternative to the assumption that conditional probabilities can be computed efficiently as used in the sequential algorithm.

THEOREM 7. *For a given evaluation, let the time taken by $M^{O(1)}$ processors to check the truth of an event $A \in \mathcal{A}$ be at most t_{eval}. Let t_{MIS} be the time to compute the maximal independent set in an m-vertex graph using $M^{O(1)}$ parallel processors on an EREW PRAM (Exclusive Read Exclusive Write PRAM). Suppose there is an $\epsilon \in (0,1)$ and an assignment of reals $x : \mathcal{A} \to (0,1)$ such that for all $A \in \mathcal{A}$,*

$$\mathsf{Pr}(A) \le x'(A)^{1+\epsilon} = \left(x(A)\prod_{B\in\Gamma(A)}(1-x(B))\right)^{1+\epsilon}.$$

If there exists a constant c such that every event $A \in \mathcal{A}$ has decision tree complexity[5]

[4] With $x(A)$ being bounded away from one and given the ϵ-slack assumed in our theorems, one can always reduce ϵ slightly to obtain a small constant-factor gap between $x'(A)$ and $\mathsf{Pr}(A)$ (similar to [18]) and then increase any extremely small $x(A)$ to at least c/m for some small constant $c > 0$. Increasing $x(A)$ in this way only weakens the LLL condition for the event A itself. Furthermore, the effect on the LLL condition for any event due to the changed $(1-x(B))$ factors of one of its (at most m) neighboring events B accumulates to at most $(1-c/m)^m$, which can be made larger than the produced gap between $x'(A)$ and $\mathsf{Pr}(A)$.

[5] Informally, we say that a function $f(x_1,\ldots,x_n)$ has decision tree complexity at most k if we can determine its value by adaptively querying at most k of the n input variables.

at most $c \min\{-\log x'(A), \log M\}$*, then there is a parallel algorithm that finds a good evaluation in time*

$$O\left(\frac{\log M}{\epsilon w_{min}}(t_{MIS} + t_{eval}) + \gamma \log D\right)$$

using $M^{O((c/\epsilon)\log D)}$ *processors.*

The fastest known algorithm for computing the maximal independent set in an m-vertex graph using $M^{O(1)}$ parallel processors on an EREW PRAM runs in time $t_{MIS} = O(\log^2 m)$ [2, 13]. Using this in the theorem, we get the following corollary for k-CNF.

COROLLARY 8. *For any* $\epsilon \in (0,1)$ *there is a deterministic parallel algorithm that uses* $m^{O(1/\epsilon)}$ *processors on an EREW PRAM and finds a satisfying assignment to any* k*-CNF formula with* m *clauses in which each clause shares variables with at most* $2^{k/(1+\epsilon)}/e$ *other clauses in time* $O(\log^3 m/\epsilon)$.

4. Techniques. In this section, we informally describe the main ideas of our approach in the special context of k-CNF formulas and indicate how they generalize. Reading this section is not essential but provides intuition behind the techniques used for developing deterministic algorithms for the general LLL. For the sake of exposition in this section, we omit numerical constants in some mathematical expressions. Familiarity with the Moser–Tardos paper [18] is useful but not necessary for this section.

4.1. The Moser–Tardos derandomization. Let F be a k-CNF formula with m clauses. We note immediately that if $k > 1 + \log m$, then the probability that a random assignment does not satisfy a clause is $2^{-k} \leq 1/(2m)$. Thus the probability that on a random assignment F has an unsatisfied clause is at most $1/2$, and hence a satisfying assignment can be found in polynomial time using the method of conditional probabilities (see, e.g., [14]). Henceforth, we assume that $k \leq 1 + \log m$. We also assume that each clause in F shares variables with at most $2^k/e - 1$ other clauses; thus the LLL guarantees the existence of a satisfying assignment.

To explain our techniques we first need to outline the deterministic algorithms of Moser and Moser and Tardos which work in polynomial time, albeit only for $k = O(1)$. Consider a table T of values: for each variable in $\mathcal{P}$ the table has a sequence of values, each picked at random according to its distribution. We can run Algorithm 1 using such a table: instead of randomly sampling afresh each time a new evaluation for a variable is needed, we pick its next unused value from T. The fact that the randomized algorithm terminates quickly in expectation (Theorem 4) implies that there exist small tables (i.e., small lists for each variable) on which the algorithm terminates with a satisfying assignment. The deterministic algorithm finds one such table.

The constraints to be satisfied by such a table can be described in terms of *witness trees*: for a run of the randomized algorithm, whenever an event is resampled, a witness tree "records" the sequence of resamplings that led to the current resampling. We will not define witness trees formally here; see [18] or section 5 for a formal definition. We say that a witness (we will often just use "witness" instead of "witness tree") is *consistent* with a table if this witness arises when the table is used to run Algorithm 1. If the algorithm using a table T does not terminate after a small number of resamplings, then it has a large consistent witness certifying this fact. Thus if we use a table which has no large consistent witness, the algorithm should terminate quickly.

The deterministic algorithms of Moser and Moser and Tardos compute a list L of witness trees satisfying the following properties:

1. Consider an arbitrary but fixed table T. If no witness in L is consistent with T, then there is no large witness tree consistent with T.
2. The expected number of witnesses in L consistent with a random table is less than 1. This property is needed in order to apply the method of conditional probabilities to find a small table with which no tree in L is consistent.
3. The list L is of polynomial size. This property is necessary for the method of conditional probabilities to be efficient.

We now describe how these properties arise naturally while using Algorithm 1 and how to find the list L. In the context of k-CNF formulas with m clauses satisfying the degree bound, Moser (and also Moser and Tardos when their general algorithm is interpreted for k-CNF) prove two lemmas that they use for derandomization. The *expectation lemma* states that the expected number of large (size at least $\log m$) consistent witness trees (among all possible witness trees) is less than $1/2$ (here randomness is over the choice of the table). At this point we could try to use the method of conditional probabilities to find a table such that there are no large witness trees consistent with it. However, there are infinitely many witness trees, and so it is not clear how to proceed by this method.

This difficulty is resolved by the *range lemma* which states that if, for some u, no witness tree with size in the range $[u, ku]$ is consistent with a table, then no witness tree of size at least u is consistent with the table. Thus, the list L is the set of witness trees of size in the range $[u, ku]$. Now one can find the required table by using the method of conditional probabilities to exclude all tables with a consistent witness in L. The number of witnesses in L is $m^{\Omega(k^2)}$. To proceed by the method of conditional probabilities we need to explicitly maintain L and find values for the entries in the table so that none of the witnesses in L remains consistent with it.

Thus, the algorithm of Moser (and respectively Moser and Tardos) works in polynomial time only for constant k. Clearly, it is the size of L that is the bottleneck toward achieving polynomial running time for $k = \omega(1)$. One possible way to deal with the large size of L would be to maintain L in an implicit manner, thereby using a small amount of space. We do not know how to achieve this. We solve this problem in a different way by working with a new (though closely related) notion of witness trees, which we explain next.

4.2. Partial witness trees. For a run of the Moser–Tardos randomized algorithm using a table T, for each resampling of an event, we get one witness tree consistent with T. Given a consistent witness tree of size $ku + 1$, removing the root gives rise to up to k new consistent witnesses, whose union is the original witness minus the root. Clearly one of these new subtrees has size at least u. This proves their range lemma. The range lemma is optimal for the witness trees. That is, for a given u it is not possible to reduce the multiplicative factor of k between the two endpoints of the range $[u, ku]$.

We overcome this limitation by introducing *partial witness trees*, which have properties similar to those of witness trees but have the additional advantage of allowing a tighter range lemma. The only difference between witness trees and partial witness trees is that the root, instead of being labeled by a clause C (as is the case for witness trees), is labeled by a *subset* of variables from C. Now, instead of removing the root to construct new witness trees as in the proof of the Moser–Tardos range lemma, each subset of the set labeling the root gives a new consistent partial witness tree.

This flexibility allows us to prove the range lemma for the smaller range $[u, 2u]$. The number of partial witness trees is larger than the number of witness trees because there are $2^k m$ choices for the label of the root (as opposed to m choices in the case of witness trees) since the root may be labeled by any subset of variables in a clause. But $2^k \leq 2m$, as explained at the beginning of section 4.1. Thus for each witness tree there are at most $2^k \leq 2m$ partial witnesses, and the expectation lemma holds with similar parameters for partial witnesses as well. The method of conditional probabilities now needs to handle partial witness trees of size in the range $[\log m, 2 \log m]$, which is the new L. The number of partial witnesses in this range is $m^{\Omega(k)}$, which is still too large. The next ingredient reduces this number to a manageable size.

4.3. ϵ-slack. By introducing an ϵ-slack, that is, by making the slightly stronger assumption that each clause intersects at most $2^{(1-\epsilon)k}/e$ other clauses, we can prove a stronger expectation lemma: the expected number of partial witnesses of size more than $(4 \log m)/\epsilon k$ is less than $1/2$. Indeed, the number of labeled trees of size u and degree at most d is less than $(ed)^u \leq 2^{(1-\epsilon)ku}$ (see [11]). Thus the number of partial witnesses of size u is less than $2^k m 2^{(1-\epsilon)ku}$, where the factor $2^k m$ ($\leq 2m^2$) accounts for the number of possible labels for the root. Moreover, the probability that a given partial witness tree of size u is consistent with a random table is $2^{-k(u-1)}$ (as opposed to 2^{-ku} in the case of a witness tree). This is proved in a manner similar to that for witness trees. Thus the expected number of partial witnesses of size at least $\gamma = 4 \log m/\epsilon k$ consistent with a random table is at most

$$\sum_{u \geq \gamma} 2^k m 2^{(1-\epsilon)ku} \cdot 2^{-k(u-1)} \leq \sum_{u \geq \gamma} 2^{2k} m 2^{-\epsilon ku} \leq \sum_{u \geq \gamma} 4m^3 2^{-\epsilon ku} \leq 1/2.$$

Now, by the new expectation and range lemmas it is sufficient to consider partial witnesses of size in the range $[(4 \log m)/\epsilon k, (8 \log m)/\epsilon k]$. The number of partial witnesses of size in this range is polynomial in m; thus the list L of trees that the method of conditional probabilities needs to maintain is polynomial in size.

4.4. General version. More effort is needed to obtain a deterministic algorithm for the general version of the LLL. Here, the events are allowed to have significantly varying probabilities of occurrence and unrestricted structure.

One issue is that an event could possibly depend on all n variables. In that case, taking all variable subsets of a label for the root of a partial witness would give up to 2^n different possible labels for the roots. However, for the range lemma to hold true, we do not need to consider all possible variable subsets for the root; instead, for each root event A it is sufficient to have a preselected choice of $2\text{vbl}(A)$ labels. This preselected choice of labels $\mathbb{B}_A$ is fixed for each event A in the beginning.

The major difficulty in derandomizing the general LLL is in finding a list L satisfying the three properties mentioned earlier for applying the method of conditional probabilities. The range lemma can still be applied. However, the existence of low probability events with (potentially) many neighbors may lead to as many as $O(m^u)$ partial witnesses of size in the range $[u, 2u]$. Indeed, it can be shown that there are instances in which there is no setting of u such that the list L containing all witnesses of size in the range $[u, 2u]$ satisfies properties 2 and 3 mentioned in section 4.1.

The most important ingredient for working around this in the general setting is the notion of *weight of a witness tree*. The weight of a tree is the sum of the weights of individual vertices; more weight is given to those vertices whose corresponding bad events have smaller probability of occurrence. Our deterministic algorithm for

the general version finds a list L that consists of partial witnesses with weight (as opposed to size) in the range $[\gamma, 2\gamma]$, where γ is a number depending on the problem. It is easy to prove a similar range lemma for weight-based partial witnesses which guarantees property 1 for this list. Further, the value of γ can be chosen so that the expectation lemma of Moser and Tardos can be adjusted to lead to property 2 for L. Unfortunately one cannot prove property 3 by counting the number of partial witnesses using combinatorial enumeration methods as in [17]. This is due to the possibility of up to $O(m)$ neighbors for each event A in the dependency graph. However, the strong coupling between weight and probability of occurrence of bad events can be used to obtain property 3 directly from the expectation lemma.

4.5. Parallel algorithm. For the parallel algorithm, we use the technique of limited-independence spaces or, more specifically, k-wise δ-dependent probability spaces due to Naor and Naor [19] and its extensions [4, 8]. This is a well-known technique for derandomization. The basic idea here is that instead of using perfectly random bits in the randomized algorithm, we use random bits chosen from a limited-independence probability space. For many algorithms it turns out that their performance does not degrade when using bits from such a probability space; but now the advantage is that these probability spaces are much smaller in size, and so one can enumerate all the sample points in them and choose a good one, thereby obtaining a deterministic algorithm. This tool was applied by Alon [1] to give a deterministic parallel algorithm for k-uniform hypergraph 2-coloring and other applications of the LLL, but with much worse parameters than ours. Our application of this tool is quite different from the way Alon uses it: Alon starts with a random 2-coloring of the hypergraph chosen from a small size limited-independence space; he then shows that at least one of the sample points in this space has the property that the monochromatic hyperedges and almost monochromatic hyperedges form small connected components. For such a coloring, one can alter it locally over vertices in each component to get a valid 2-coloring.

In contrast, our algorithm is very simple (we describe it for k-CNF; the arguments are very similar for hypergraph 2-coloring and for the general LLL): recall that for a random table, the expected number of consistent partial witnesses with size in the range $[(4\log m)/\epsilon k, (8\log m)/\epsilon k]$ is at most $1/2$ (for the case of k-CNF). Each of these partial witnesses uses at most $((8\log m)/\epsilon k)\cdot k = ((8\log m)/\epsilon)$ entries from the table. Now, instead of using a completely random table, we use a table chosen according to an $(8\log m/\epsilon)$-wise independent distribution (i.e., any subset of at most $(8\log m)/\epsilon$ entries has the same joint distribution as in a random table). So any partial witness tree is consistent with the new random table with the same probability as before, and hence the expected number of partial witnesses consistent with the new random table is still at most $1/2$. But now the key point to note is that the number of tables in the new limited-independence distribution is much smaller and we can try each of them in parallel until we succeed with one of the tables. To make the probability space even smaller we use k-wise δ-dependent distributions, but the idea remains the same. Finally, to determine whether a table has no consistent partial witness whose size is at least $(4\log m)/\epsilon k$, we run the parallel algorithm of Moser and Tardos on the table.

In order to apply the above strategy to the general version, we require that the number of variables on which witnesses depend be small, and hence the number of variables on which events depend should also be small. In our general parallel algorithm we relax this to some extent: instead of requiring that each event depend on few variables, we only require that the decision tree complexity of the event be small. The idea behind the proof remains the same.

5. The partial witness structure. In this section we define the partial witness structures and their weight. We then prove the new range lemma using these weights.

5.1. Definitions. For every $A \in \mathcal{A}$ we fix an arbitrary rooted *binary variable splitting* $\mathbb{B}_A$. It is a binary tree in which all vertices have labels which are nonempty subsets of $\mathrm{vbl}(A)$: the root of $\mathbb{B}_A$ is labeled by $\mathrm{vbl}(A)$ itself, the leaves are labeled by distinct singleton subsets of $\mathrm{vbl}(A)$, and every nonleaf vertex in $\mathbb{B}_A$ is labeled by the disjoint union of the labels of its two children. This means that every nonroot nonleaf vertex is labeled by a set $\{v_{i_1}, \ldots, v_{i_k}\}$, $k \geq 2$, while its children are labeled by $\{v_{i_1}, \ldots, v_{i_j}\}$ and $\{v_{i_{j+1}}, \ldots, v_{i_k}\}$ for some $1 \leq j \leq k-1$. Note that $\mathbb{B}_A$ consists of $2|\mathrm{vbl}(A)| - 1$ vertices. We abuse the notation $\mathbb{B}_A$ to also denote the set of labels of the vertices of this binary variable splitting. The binary variable splitting is not to be confused with the (partial) witness tree, which we define next. The elements from $\mathbb{B}_A$ will be used solely to define the possible labels for the roots of partial witness trees. An example of a binary variable splitting $\mathbb{B}_A$ can be found in Figure 1.

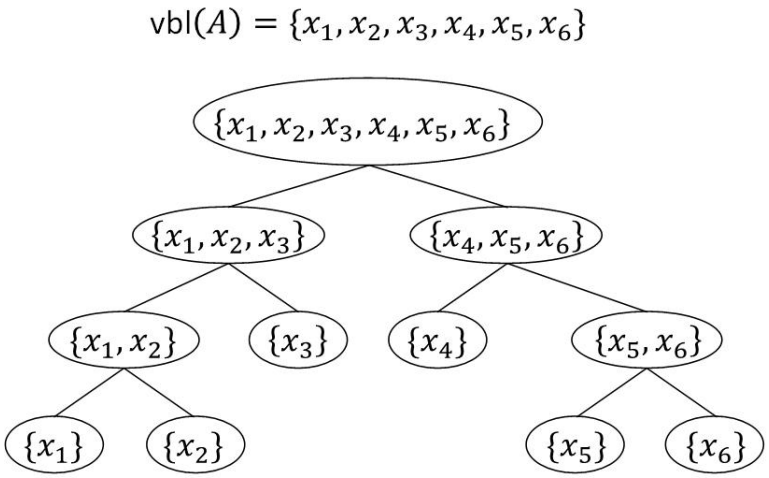


FIG. 1. *A binary variable splitting for an event A that depends on variables $x_1, x_2, x_3, x_4, x_5, x_6$.*

A *partial witness tree* τ_S is a finite rooted tree whose vertices apart from the root are labeled by events from $\mathcal{A}$, while the root is labeled by some subset S of variables with $S \in \mathbb{B}_R$ for some $R \in \mathcal{A}$. Each child of the root must be labeled by an event A that depends on at least one variable in S (thus a neighbor of the corresponding root event); the children of every other vertex that is labeled by an event B must be labeled either by B or by a neighboring event of B, i.e., its label should be from $\Gamma^+(B)$. Define $V(\tau_S)$ to be the set of vertices of τ_S. For notational convenience, we use $\overline{V}(\tau_S) := V(\tau_S) \setminus \{\mathrm{Root}(\tau_S)\}$ and denote the label of a vertex $v \in \overline{V}(\tau_S)$ by $[v]$.

A *full witness tree* is a special case of a partial witness where the root is the complete set $\mathrm{vbl}(A)$ for some $A \in \mathcal{A}$. In such a case, we relabel the root with A instead of $\mathrm{vbl}(A)$. Note that this definition of a full witness tree is the same as that of the witness trees in [18].

Define the weight of an event $A \in \mathcal{A}$ to be $w(A) = -\log x'(A)$. Define the *weight of a partial witness tree* τ_S as the sum of the weights of the labels of the vertices in

$\overline{V}(\tau_S)$, i.e.,

$$w(\tau_S) := \sum_{v \in \overline{V}(\tau_S)} w([v]) = -\log\left(\prod_{v \in \overline{V}(\tau_S)} x'([v])\right).$$

The *depth* of a vertex v in a witness tree is the distance of v from the root in the witness tree. We say that a partial witness tree is *proper* if for every vertex v, all children of v have distinct labels.

Similarly to [16], we will control the randomness used by the algorithm using a *table* of evaluations, denoted by T. It is convenient to think of T as a matrix. This table contains one row for each variable in $\mathcal{P}$. Each row contains evaluations for its variable. Note that the number of columns in the table could possibly be infinite. In order to use such a table in the algorithm, we maintain a pointer t_i for each variable $P_i \in \mathcal{P}$ indicating the column containing its current value used in the evaluation of the events. We denote the value of P_i at t_i by $T(i, t_i)$. If we want to resample an evaluation for P_i, we increment the pointer t_i by one and use the value at the new location.

We call a table T a *random table* if, for all variables $P_i \in \mathcal{P}$ and all positions j, the entry $T(i, j)$ is picked independently at random according to the distribution of P_i. It is clear that running Algorithm 1 is equivalent to using a random table to run Algorithm 2 below.

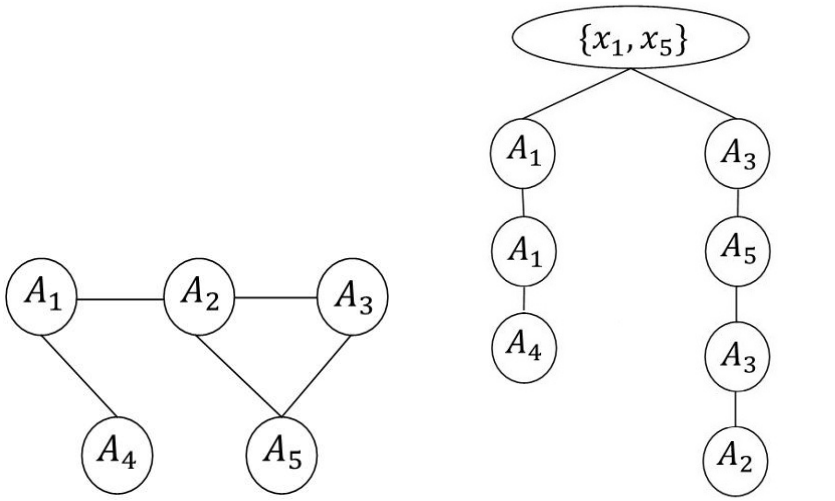
ALGORITHM 2 (Moser–Tardos algorithm with input table).
Input: Table T with values for variables
Output: An assignment of values for variables so that none of the events in $\mathcal{A}$ happens

1. For every variable $P_i \in \mathcal{P}$: Initialize the pointer $t_i = 1$.
2. While $\exists A \in \mathcal{A}$ that happens on the current assignment (i.e., for all $P_i \in \mathcal{P}$: $P_i = T(i, t_i)$) do
   (a) Pick one such A.
   (b) Resample A: For all $P_i \in \mathrm{vbl}(A)$ increment t_i by one.
3. Return for all $P_i \in \mathcal{P} : P_i = T(i, t_i)$.

In the above algorithm, step 2(a) is performed by a fixed arbitrary deterministic procedure. This makes the algorithm well defined.

Let $C : \mathbb{N} \to \mathcal{A}$ be an ordering of the events (with repetitions), which we call the *event-log*. Let the ordering of the events as they have been selected for resampling in the execution of Algorithm 2 using a table T be denoted by an event-log C_T. Observe that C_T is partial if the algorithm terminates after a finite number of resamplings t; i.e., $C_T(i)$ is defined only for $i \in \{1, 2, \ldots, t\}$.

Given an event-log C, associate with each resampling step t and each $S \in \mathbb{B}_{C(t)}$ a partial witness tree $\tau_C(t, S)$ as follows. Define $\tau_C^{(t)}(t, S)$ to be an isolated root vertex labeled S. Going backward through the event-log, for each $i = t-1, t-2, \ldots, 1$: (i) if there is a nonroot vertex $v \in \tau_C^{(i+1)}(t, S)$ such that $C(i) \in \Gamma^+([v])$, then from among all such vertices choose the one whose distance from the root is maximum (break ties arbitrarily) and attach a new child vertex u to v with label $C(i)$, thereby obtaining the tree $\tau_C^{(i)}(t, S)$; (ii) else if $S \cap \mathrm{vbl}(C(i))$ is nonempty, then attach a new child vertex to the root with label $C(i)$ to obtain $\tau_C^{(i)}(t, S)$; (iii) else, set $\tau_C^{(i)}(t, S) = \tau_C^{(i+1)}(t, S)$. Finally, set $\tau_C(t, S) = \tau_C^{(1)}(t, S)$.



Dependency graph Partial witness tree $\tau_C(9, \{x_1, x_5\})$

FIG. 2. *The dependency graph and an example of a partial witness tree constructed from the event-log* $C = A_2, A_3, A_5, A_4, A_1, A_3, A_1, A_5, A_2, \ldots$, *where* $\mathrm{vbl}(A_1) = \{x_1, x_2, x_3\}$, $\mathrm{vbl}(A_2) = \{x_1, x_4, x_5\}$, $\mathrm{vbl}(A_3) = \{x_4, x_5, x_6\}$, $\mathrm{vbl}(A_4) = \{x_3, x_7\}$, $\mathrm{vbl}(A_5) = \{x_4, x_6\}$. *Note that the last occurrence of the event* A_5 *is not added to the witness since it does not share a variable with the variable subset* $\{x_1, x_5\} \subset \mathrm{vbl}(A_2)$ *that was selected as a root.*

Note that if $S = \mathrm{vbl}(A) \in \mathbb{B}_A$, then $\tau_C(t, S)$ is a full witness tree with root A. For such a full witness tree, our construction is the same as the construction of *witness trees associated with the log* in [18].

We say that the partial witness tree τ_S *occurs* in event-log C if there exists $t \in \mathbb{N}$ such that for some $A \in \mathcal{A}$ such that $S \in \mathbb{B}_A$, $C(t) = A$ and $\tau_S = \tau_C(t, S)$. An example illustrating these definitions can be found in Figure 2.

For a table T, a *T-check* on a partial witness tree τ_S uses table T as follows. In decreasing order of depth, visit the nonroot vertices of τ_S, and for a vertex with label A, take the first unused value from T for each $x \in \mathrm{vbl}(A)$ and check whether the resulting evaluation makes A happen. The *T-check passes* if all events corresponding to vertices apart from the root happen when checked. We say that a partial witness tree is *consistent with a table* T if the T-check passes on the partial witness tree.

Most of the above definitions are simple extensions of the ones given in [18].

5.2. Properties. In this section we state and prove two important properties of the partial witness tree which will be useful in obtaining the deterministic sequential and parallel algorithms.

The following lemma proves that, given a witness tree, one can use the T-check procedure to exactly determine which values were used in the resamplings that lead to this witness tree.

LEMMA 9. *For a fixed table* T*, if a partial witness tree* τ_S *occurs in the event-log* C_T*, then*

1. τ_S *is proper.*
2. τ_S *is consistent with* T*.*

Proof. The proof of this lemma is essentially Lemma 2.1 in [18] and is included here for completeness.

Since τ_S occurs in C_T, there exists some time instant t such that for $S \in \mathbb{B}_{C_T(t)}$,

$\tau_S = \tau_{C_T}(t, S)$. For each $v \in \overline{V}(\tau_S)$, let $d(v)$ denote the depth of vertex v and let $q(v)$ denote the largest value q with v contained in $\tau^{(q)}_{C_T}(t)$. We observe that $q(v)$ is the time instant in which v was attached to $\tau_{C_T}(t, S)$ by the procedure constructing $\tau_{C_T}(t, S)$.

If $q(u) < q(v)$ for vertices $u, v \in \overline{V}(\tau_S)$, and vbl($[u]$) and vbl($[v]$) are not disjoint, then $d(u) > d(v)$. Indeed, when adding the vertex u to $\tau^{(q(u)+1)}_{C_T}(t)$ we attach it to v or to another vertex of equal or greater depth. Therefore, for any two vertices $u, v \in \overline{V}(\tau_S)$ at the same depth $d(u) = d(v)$, $[u]$ and $[v]$ do not depend on any common variables; that is, the labels in every level of τ_S form an independent set in G. In particular τ_S must be proper.

Now consider a nonroot vertex v in the partial witness tree τ_S. Let $P_i \in$ vbl($[v]$). Let $\mathcal{D}(i)$ be the set of vertices $w \in \tau_S$ with depth greater than that of v such that $[w]$ depends on variable P_i.

When the T-check considers the vertex v and uses the next unused evaluation of the variable P_i, it uses the evaluation $T(i, |\mathcal{D}(i)|)$. This is because the witness check visits the vertices in order of decreasing depth, and among the vertices with depth equal to that of v, only $[v]$ depends on P_i (as we proved earlier that vertices with equal depth are variable disjoint). So the T-check must have used values for P_i exactly when it was considering the vertices in $\mathcal{D}(i)$.

At the time instant of resampling $[v]$, say t_v, Algorithm 2 chooses $[v]$ to be resampled, which implies that $[v]$ happens before this resampling. For $P_i \in$ vbl($[v]$), the value of the variable P_i at t_v is $T(i, |\mathcal{D}(i)|)$. This is because the pointer for P_i was increased for events $[w]$ that were resampled before the current instance, where $w \in \mathcal{D}(i)$. Note that every event which was resampled before t_v and that depends on $[v]$ would be present at depth greater than that of v in τ_S by construction. Hence, $\mathcal{D}(i)$ is the complete set of events which led to resampling of P_i before the instant t_v.

As the T-check uses the same values for the variables in vbl($[v]$) when considering v as the values that led to resampling of $[v]$, it must also find that $[v]$ happens. ∎

Next, we prove a range lemma for partial witnesses, improving the range to a factor of two.

LEMMA 10. *If a partial witness tree of weight at least γ occurs in the event-log C_T and every vertex v in the tree has weight at most γ, then a partial witness tree of weight $\in [\gamma, 2\gamma)$ occurs in the event-log C_T.*

Proof. The proof is by contradiction. Consider a least weight partial witness tree whose weight is at least γ that occurs in the event-log C_T, namely, $\tau_S = \tau_{C_T}(t, S)$ for some t, $S \in \mathbb{B}_A$ where $A = C_T(t)$. A witness tree with weight at least γ exists by assumption, and because there are only finitely many choices for t and S, there exists such a tree with least weight. Suppose, for the sake of contradiction, that $w(\tau_S) \geq 2\gamma$. We may assume that Root(τ_S) has at least one child; otherwise, the weight of the tree is zero. We have two cases.

Case (i): Root(τ_S) has only one child v. Let t' be the largest time instant before t at which $[v]$ was resampled. Note that this resampling of $[v]$ corresponds to the child v of the root of τ_S. Now, consider the partial witness tree $\tau'_S = \tau_{C_T}(t', S' = \text{vbl}([v]))$. Since τ'_S contains one less vertex than τ_S, $w(\tau'_S) < w(\tau_S)$. Also, since the weight of any vertex v in the tree is at most γ, we get that $w(\tau'_S) = w(\tau_S) - w([v]) \geq \gamma$. Finally, by definition of τ'_S, it is clear that τ'_S occurs in the event-log C_T. Thus, τ'_S is a counterexample of smaller weight, contradicting our choice of τ_S.

Case (ii): Root(τ_S) has at least two children. Since the labeling clauses of these children have pairwise disjoint sets of variables and they have to share a variable

with S, we have that S consists of at least two variables. Thus, it also has at least two children in the variable splitting $\mathbb{B}_A$. In $\mathbb{B}_A$, starting from S, we now explore the descendants of S in the following way, looking for the first vertex whose children S_L and S_R reduce the weight of the tree, i.e., $0 < w(\tau_{S_L}), w(\tau_{S_R}) < w(\tau_S)$, where $\tau_{S_L} = \tau_{C_T}(t, S_L)$ and $\tau_{S_R} = \tau_{C_T}(t, S_R)$: if a vertex S_L reduces the weight of the tree without making it zero (i.e., $0 < w(\tau_{S_L}) < w(\tau_S)$), then its variable disjoint sibling S_R must also reduce the weight of the tree; on the other hand, if a vertex S_L reduces the weight of the tree to zero, then its sibling S_R cannot reduce the weight of the tree. Suppose S_L reduces the weight to zero; then we explore S_R to check if its children reduce the weight. It is easy to see that this exploration process stops at the latest when S_L and S_R are leaves in $\mathbb{B}_A$.

By definition, both τ_{S_L} and τ_{S_R} occur in the event-log C_T. Since we pick the first siblings S_L and S_R (in the breadth first search) which reduce the weight, their parent S' is such that $w(\tau_{S'}) \geq w(\tau_S)$, where $\tau_{S'} = \tau_{C_T}(t, S')$. We are considering only those S' such that $S' \subseteq S$. This implies that $w(\tau_{S'}) \leq w(\tau_S)$. Hence, $w(\tau_{S'}) = w(\tau_S)$, and for every vertex that has label A in τ_S, one can find a unique vertex labeled by A in $\tau_{S'}$ and vice versa. Further, S' is the disjoint union of S_L and S_R; therefore, for each vertex with label A in $\tau_{S'}$, one can find a unique vertex labeled by A in either τ_{S_L} or τ_{S_R}.

As a consequence, we have that for every vertex with label A in τ_S, one can find a unique vertex labeled by A in either τ_{S_L} or τ_{S_R}. Hence, $w(\tau_{S_L}) + w(\tau_{S_R}) \geq w(\tau_S)$, and therefore $\max\{w(\tau_{S_L}), w(\tau_{S_R})\} \geq w(\tau_S)/2 \geq \gamma$. So, the witness with larger weight among τ_{S_L} and τ_{S_R} has weight at least γ but less than that of τ_S. This contradicts our choice of τ_S. ∎

6. Deterministic algorithm. In this section we describe our sequential deterministic algorithm and prove Theorem 5.

For the rest of the paper we define a set of *forbidden witnesses* F which contains all partial witness trees with weight between γ and 2γ. We define a table to be a *good table* if no forbidden witness is consistent with it. With these definitions we can state our deterministic algorithm.

ALGORITHM 3 (sequential deterministic algorithm).

1. Enumerate all forbidden witnesses in F.
2. Construct a good table T via the method of conditional probabilities:
   For each variable $p \in \mathcal{P}$, and for each j, $0 \leq j \leq 2\gamma/w_{min}$, do
   - Select a value for $T(p, j)$ that minimizes the expected number of forbidden witnesses that are consistent with T when all entries in the table chosen so far are fixed and the yet to be chosen values are random.
3. Run Algorithm 2 using table T as input.

We next give a short overview of the running time analysis of Algorithm 3 before embarking on the proof of Theorem 5.

The running time of Algorithm 3 depends on the time to construct a good table T by the method of conditional probabilities. To construct such a table efficiently, we prove that the number of forbidden witnesses is small (polynomial in M) using Lemma 12. Further, we need to show that the method of conditional probabilities indeed constructs a good table. We show this by proving in Lemma 11 that the expected number of forbidden witnesses that are consistent with T initially (when all values are random) is smaller than one. This invariant is maintained by the method

of conditional probabilities resulting in a fixed table with less than one (and therefore no) forbidden witnesses consistent with it. By Lemmas 9 and 10, it follows that no witness of weight more than γ occurs when Algorithm 2 is run on the table T. Finally, the maximum number of vertices in a partial witness tree of weight at most γ is small. This suffices to show that the size of table T is small, and thus Algorithm 3 is efficient.

LEMMA 11. *The expected number of forbidden witnesses consistent with a random table T is less than $1/2$.*

Proof. For each event $A \in \mathcal{A}$, let Υ_A and Υ'_A be the set of partial and, respectively, full witness trees in F with root from $\mathbb{B}_A$. With this notation the expectation in question is exactly

$$\sum_{A\in\mathcal{A}} \sum_{\tau\in\Upsilon_A} \mathsf{Pr}\left(\tau \text{ is consistent with } T\right).$$

Note that according to Lemma 9, a partial witness tree is consistent with a table T if and only if it passes the T-check. Clearly, the probability that a witness τ passes the T-check for the random table T is $\prod_{v\in\overline{V}(\tau)} \mathsf{Pr}\left([v]\right)$ (recall that $\overline{V}(\tau)$ denotes the set of nonroot vertices in τ). Using this and the assumption in Theorem 5 that $\mathsf{Pr}\left([v]\right) \le x'([v])^{1+\epsilon}$ we get that the expectation is at most

$$E := \sum_{A\in\mathcal{A}} \sum_{\tau\in\Upsilon_A} \prod_{v\in\overline{V}(\tau)} x'([v])^{1+\epsilon}.$$

To relate this to the full witness trees considered in [18], we associate with every partial witness tree τ (in Υ_A) a full witness tree τ' (in Υ'_A) by replacing the root subset $S \in \mathbb{B}_A$ with the full set $\mathrm{vbl}(A)$. Note that the weights of τ and τ' are the same (as is the quantity $\prod_{v\in\overline{V}(\tau)} x'([v])^{1+\epsilon}$). Note also that every full witness tree has at most $|\mathbb{B}_A|$ partial witness trees associated with it. Hence, we can rewrite the expression to get

$$\begin{aligned} E &\le \sum_{A\in\mathcal{A}} |\mathbb{B}_A| \sum_{\tau\in\Upsilon'_A} \prod_{v\in\overline{V}(\tau)} x'([v])^{1+\epsilon} \\ &\le \sum_{A\in\mathcal{A}} |\mathbb{B}_A| \sum_{\tau\in\Upsilon'_A} \left(\prod_{v\in\overline{V}(\tau)} x'([v]) \right) 2^{-\gamma\epsilon}, \end{aligned}$$

where the last expression follows because, for $\tau \in \Upsilon'_A$, we have

$$\begin{aligned} w(\tau) = -\log \prod_{v\in\overline{V}(\tau)} x'([v]) \ge \gamma \\ \implies \prod_{v\in\overline{V}(\tau)} x'([v]) \le 2^{-\gamma}. \end{aligned}$$

Next we transition from partial to full witness trees by including the root again (and going from $\overline{V}$ to V):

$$E \le \sum_{A\in\mathcal{A}} \frac{|\mathbb{B}_A|}{x'(A)} \left(\sum_{\tau\in\Upsilon'_A} \prod_{v\in V(\tau)} x'([v]) \right) 2^{-\gamma\epsilon}.$$

Now we can use the following result of Moser and Tardos (section 3 in [18]) that bounds the expected number of full witnesses with root A:

$$\sum_{\tau \in \Upsilon'_A} \prod_{v \in V(\tau)} x'([v]) \leq \frac{x(A)}{1 - x(A)}.$$

Their proof makes use of a Galton–Watson process that randomly generates proper witness trees with root A (note that by Lemma 9 all partial witness trees are proper). Using this,

$$\begin{aligned} E &\leq \sum_{A \in \overline{\mathcal{A}}} \frac{|\mathbb{B}_A|}{x'(A)} \cdot \left(\frac{x(A)}{1 - x(A)} \right) 2^{-\gamma\epsilon} \\ &< \frac{M}{2} 2^{-\gamma\epsilon} \leq \frac{1}{2}. \end{aligned}$$

Here the penultimate inequality follows from the fact that $|\mathbb{B}_A| < 2|\mathrm{vbl}(A)|$ and the definition of M, and the last inequality follows from the choice of $\gamma = (\log M)/\epsilon$. ∎

Owing to the definition of forbidden witnesses *via weights*, there is an easy way to count the number of forbidden witnesses using the fact that their expected number is small.

LEMMA 12. *The number of witnesses with weight at most 2γ is at most $O(M^{2(1+1/\epsilon)})$. In particular, the number of forbidden witnesses is less than $M^{2(1+1/\epsilon)}$.*

Proof. Each forbidden witness $\tau \in F$ has weight $w(\tau) \leq 2\gamma$, and thus

$$\begin{aligned} |F|(2^{-2\gamma})^{(1+\epsilon)} &\leq \sum_{\tau \in F} (2^{-w(\tau)})^{(1+\epsilon)} \\ &= \sum_{\tau \in F} \left(\prod_{v \in \overline{V}(\tau)} x'([v]) \right)^{(1+\epsilon)} \\ &= E \leq \frac{M}{2} 2^{-\gamma\epsilon} \leq \frac{1}{2}. \end{aligned}$$

Here, the final line of inequalities comes from the proof of Lemma 11. Therefore the number of forbidden witnesses is at most

$$|F| \leq \left(\frac{M}{2} 2^{-\gamma\epsilon} \right) \cdot 2^{2\gamma(1+\epsilon)} = \left(\frac{M}{2} \right) 2^{\gamma(2+\epsilon)} \leq \frac{1}{2} M^{2(1+1/\epsilon)}.$$

Using the same argument with any γ' instead of γ shows that the number of witnesses with weight in $[\gamma', 2\gamma']$ is at most $(M/2) \cdot 2^{\gamma'(2+\epsilon)}$. Since this is exponential in γ' the total number of witnesses with weight at most 2γ is dominated by a geometric sum which is $O(M^{2(1+1/\epsilon)})$. ∎

We are now ready to prove Theorem 5.

Proof of Theorem 5. We first describe how the set of forbidden witnesses in the first step of the deterministic algorithm (Algorithm 3) is obtained.

Enumeration of witnesses. We enumerate all witnesses of weight at most 2γ and then discard the ones with weight less than γ. According to Lemma 12, there are at most $M^{2(1+1/\epsilon)}$ witnesses of weight at most 2γ and each of them consists of at most $x_{\max} = (2\gamma/w_{min}) + 1 = (2 \log M)/(\epsilon w_{min}) + 1$ vertices. In our discussion so far we

did not need to consider the order of children of a node in our witness trees. However, for the enumeration it will be useful to order the children of each node from left to right. We will build witnesses by attaching nodes level-by-level and from left to right. We fix an order on the events according to their weights, breaking ties arbitrarily, and use the convention that all witnesses are represented so that for any node its children from left to right have increasing weight. We then say a node v is *eligible* to be attached to a witness τ if in the resulting witness τ' the node v is the deepest rightmost leaf in τ'. With this convention the enumeration proceeds as follows.

As a preprocessing step for every event A we sort all the events in $\Gamma^+(A)$ according to their weight in $O(m^2 \log m)$ time. Then, starting with W_1, the set of all possible roots, we incrementally compute all witnesses W_x having $x = 1, \ldots, x_{\max}$ nodes and weight at most 2γ. To obtain W_{x+1} from W_x we take each witness $\tau \in W_x$ and each node $v \in \tau$ and check for all $A \in \Gamma^+([v])$ with weight more than the current children of v, in the order of increasing weight, whether a node v' with $[v'] = A$ is eligible to be attached to τ at v. If it is eligible, and the resulting new witness τ' has weight at most 2γ, then we add τ' to W_{x+1}. It is clear that in this way we enumerate all forbidden witnesses without producing any witness more than once.

We now analyze the time required by the above enumeration procedure. We write down each witness explicitly, taking $O(x_{\max} \log M)$ time and space per witness. For each witness it takes linear (in the number of nodes) time to find the nodes with eligible children. Note that attaching children to a node in the order of increasing weight guarantees that at most one attachment attempt per node fails due to large weight. Thus, the total time to list all forbidden witnesses is at most $O(x_{\max} M^{2(1+1/\epsilon)} \log M)$.

Finding a good table. The running time to find a good table T using the method of conditional probabilities as described in Algorithm 3 can be bounded as follows: for each of the n variables, the table T has $2\gamma/w_{min} = x_{\max}$ entries to be filled in. For each of those entries at most D possible values need to be tested. For each value we compute the conditional expectation of the number of forbidden witnesses that are consistent with the partially filled in table T by computing the conditional probability of each forbidden witness $\tau \in F$ to pass the T-check given the filled in values and summing up these probabilities. This can be done by plugging the fixed values into each of the at most $x_{\max}$ nodes of τ, similarly to the T-check procedure, computing the conditional probability in t_C time and computing the product of these conditional probabilities. Thus, the total time to compute T is at most

$$O(n \cdot x_{\max} \cdot D \cdot |F| \cdot x_{\max} \cdot t_C) = O\left(\frac{D M^{3+2/\epsilon} \log^2 M}{\epsilon^2 w_{min}^2} t_C\right).$$

To complete the proof we show that the running time of the sequential algorithm on a table T obtained by step 2 of the deterministic algorithm is at most $O\left(m^2 \cdot x_{\max} \cdot t_C\right)$.

First, we note that by running the sequential algorithm using table T, none of the forbidden witnesses can occur in the event-log C_T. This is because the table is obtained by the method of conditional probabilities: in the beginning of the construction of the table, when no value is fixed, the expected number of forbidden witnesses that occur in the event-log is less than $1/2$, as proved in Lemma 11. This invariant is maintained while picking values for variables in the table. Thus, once all values are fixed, the number of witness trees in F that occur in the event-log C_T is still less than $1/2$ and hence zero.

This implies that the sequential algorithm with T as input resamples each event

$A \in \mathcal{A}$ at most $x_{\max}$ times. Indeed, if some event $A \in \mathcal{A}$ is resampled more than $x_{\max}$ times, then A occurs in the event-log C_T at least $x_{\max}$ times. Now, the weight of the partial witness tree associated with the last instance at which A was resampled would be at least $x_{\max} w_{min}$, which is more than 2γ. According to Lemma 10, which is applicable since $\gamma = (\log M)/\epsilon$ is larger than the maximum weight event, there would also be a forbidden witness of weight between γ and 2γ occurring in C_T, a contradiction. Therefore, the number of resamplings done by Algorithm 2 is $O(m \cdot x_{\max})$ and the total running time for Algorithm 2 using table T is $O\left(m^2 \cdot x_{\max} \cdot t_C\right)$: the additional factor $m \cdot t_C$ comes from the time needed to find an event that happens. This running time is smaller than the upper bound for the time needed to find a good table T.

This shows that Algorithm 3 terminates in the stated time bound. Lastly, the correctness of the algorithm follows directly from the fact that the algorithm terminates only if a good assignment is found. ∎

From the general deterministic algorithm it is easy to obtain the corollary regarding k-CNF by using the standard reduction to the symmetric LLL and plugging in the optimal values for the parameters.

Proof of Corollary 6. For a k-CNF formula with clauses $\mathcal{A} = \{A_1, \ldots, A_m\}$, for each clause $A \in \mathcal{A}$ we define an event A and say that the event happens if the clause is unsatisfied. Further, each variable appearing in the formula picks values uniformly at random from $\{0, 1\}$. Then, for every event A, $\mathsf{Pr}(A) = 2^{-k}$. As remarked in section 4.1, we may assume that $k < \log m$; otherwise, the problem becomes simple. If d is the maximum number of clauses with which a clause shares its variables, setting $x(A) = 1/d$ for all $A \in \mathcal{A}$, we obtain that $x'(A) > 1/de$. The condition that $d \leq 2^{k/(1+\epsilon)}/e$ then implies for all events A that $\mathsf{Pr}(A) \leq x'(A)^{1+\epsilon}$, as required by the LLL condition. Therefore, we use parameters $t_C = O(k)$, $w_{min} \approx k$, $D = 2$, $|\mathrm{vbl}(A)| = k$ and obtain $M = O(n+m+mk+d) = O(m \log m)$. With these parameters the corollary follows directly from Theorem 5. ∎

7. Parallel algorithm. In this section we present an efficient parallel algorithm (outlined in section 4.5) and analyze its performance, thereby proving Theorem 7.

In the design of our sequential algorithm, we used Algorithm 2 as a subroutine, which takes an input table and uses it to search for an assignment for which none of the bad events happens. This reduced the problem to finding a good input table. For designing the parallel algorithm, Moser and Tardos already provided the parallel counterpart of Algorithm 2, and thus what remains is to find a good table. Our algorithm relies on the following observation: instead of sampling the values in the table independently at random, if we choose it from a distribution that is a (k, δ)-approximation of the original distribution (for appropriate k and δ), the algorithm behaves as if the values in the table had been chosen independently at random (Proposition 1). The support of a (k, δ)-approximation can be chosen to be small and can be generated fast in parallel, so this gives us a small set of tables which is guaranteed to contain at least one table on which the algorithm terminates quickly (Lemma 13). Our algorithm runs the Moser–Tardos parallel algorithm on each of these tables in parallel and stops as soon as one of the tables leads to a good evaluation.

We begin by describing the two ingredients that we will need.

7.1. Limited-independence probability spaces. We need the notion of (k, δ)-approximate distributions to describe our algorithm.

Definition 1 ((k, δ)-approximations [8]). *Let $\mathcal{S}$ be a product probability distribution on a finite domain $S_1 \times S_2 \times \cdots \times S_s$ given by mutually independent random variables $X_1, \ldots, X_s$, where $X_i \in S_i$. For positive integer k and constant $\delta \in (0, 1)$, a*

probability distribution $\mathcal{Y}$ on $S_1 \times S_2 \times \cdots \times S_s$ is said to be a (k, δ)-approximation of $\mathcal{S}$ if the following holds: for every $I \subseteq [s]$ such that $|I| \leq k$, and every $v \in S_1 \times S_2 \times \cdots \times S_s$, we have

$$|\Pr_{\mathcal{S}}[v_I] - \Pr_{\mathcal{Y}}[v_I]| \leq \delta,$$

where $\Pr_{\mathcal{S}}[v_I]$ denotes the probability that for a random vector $(x_1, \ldots, x_s)$ chosen according to the probability distribution $\mathcal{S}$, we get $x_i = v_i$ for $i \in I$; the definition of $\Pr_{\mathcal{Y}}[v_I]$ is analogous.

The support Y of a (k, δ)-approximation $\mathcal{Y}$ of $\mathcal{S}$ can be constructed efficiently in parallel. We use the construction described in [8] (which in turn uses [19]). This construction builds a (k, δ)-approximation to a product space with t variables with a support size of $|Y| = \text{poly}(2^k, \log t, \delta^{-1})$. The construction can be parallelized to run in time $O(\log t + \log k + \log 1/\delta + \log D)$ using $\text{poly}(2^k/\delta)tD$ processors, where D is again the maximum domain size for a variable.

For our algorithm we want approximately random tables of small size. More formally we will work with tables containing at most $\lceil \gamma / w_{min} \rceil$ columns. So, we set $t = n \cdot \lceil \gamma / w_{min} \rceil$ and $S_1 \times S_2 \times \cdots \times S_s = (D_1 \times D_2 \times \cdots \times D_n)^{\lceil \gamma / w_{min} \rceil}$. We furthermore set $k = 2c\gamma$, $\delta^{-1} = 3M^{2+2/\epsilon} D^{2c\gamma}$, and $\mathcal{S}$ to be the distribution obtained by independently sampling each entry in the table according to its distribution. For these values and recalling that $\gamma = (\log M)/\epsilon$, the support Y of the (k, δ)-approximation $\mathcal{Y}$ obtained by the construction mentioned above has size $\text{poly}(2^{2c\gamma}, \log(n \cdot \gamma / w_{min}), 3M^{2+2/\epsilon} D^{2c\gamma}) = M^{O((c/\epsilon) \log D)}$, and it can be constructed in parallel in time $O(\gamma \log D + \log(1/w_{min}))$ using $M^{O((c/\epsilon) \log D)}$ processors.

7.2. Decision trees. In Theorem 7, our assumption about how the events depend on the variables was in terms of decision tree complexity. In this section we recall the definition of decision trees and show some simple properties needed in what follows.

Let $S = D_1 \times \cdots \times D_n$, and let $f : S \to \{0, 1\}$ be a Boolean function. We denote the elements of S by $(x_1, x_2, \ldots, x_n)$, where $x_i \in D_i$ for $1 \leq i \leq n$. A decision tree for computing $f(x_1, x_2, \ldots, x_n)$ is a rooted tree T, where each internal vertex of the tree is labeled by one of the variables from $\{x_1, \ldots, x_n\}$, and each leaf is labeled by 0 or 1. An internal vertex labeled by x_i, has $|D_i|$ children, with their corresponding edges being labeled by distinct elements from D_i. To compute $f(x_1, x_2, \ldots, x_n)$, the execution of T determines a root-to-leaf path as follows: starting at the root we query the value of the variable labeling a vertex and follow the edge to the child which is labeled by the answer to the query. When we reach the leaf, we output the label of the leaf. The complexity of a decision tree is its depth. The decision tree complexity of a function f is the depth of the shallowest decision tree computing f.

PROPOSITION 1. *Let $S = D_1 \times \cdots \times D_n$ be a product space of finite domains of size at most $D = \max_i |D_i|$, let $\mathcal{P}$ be an independent product distribution on S, and let $f, f_1, f_2 : S \to \{0, 1\}$ be Boolean functions on S.*

1. *If f_1 and f_2 have decision tree complexity k_1 and k_2, respectively, then the decision tree complexity of $f_1 \wedge f_2$ is at most $k_1 + k_2$.*
2. *If f has decision tree complexity at most k, then every (k, δ)-approximation $\mathcal{Y}$ of $\mathcal{P}$ is $D^k\delta$-indistinguishable from $\mathcal{P}$, i.e.,*

$$|E_{\mathcal{Y}}(f) - E_{\mathcal{P}}(f)| \leq D^k \delta.$$

Proof. For the first claim we recall that a function f having decision tree complexity at most k is equivalent to saying that we can determine $f(x)$ for $x \in S$ by

adaptively querying at most k coordinates of x. If this is true for f_1 and f_2 with decision tree complexity k_1 and k_2, respectively, then we can evaluate $f_1(x) \wedge f_2(x)$ by adaptively querying at most $k_1 + k_2$ components of x. Therefore the conjunction has decision tree complexity at most $k_1 + k_2$.

For the second claim, we fix a decision tree for f with depth at most k. Each one of the leaf-to-root paths in this tree corresponds to a partial assignment of values to at most k components, and this assignment determines the value of f. The expectation of f under any distribution is simply the sum of the probabilities of the paths, resulting in a 1-evaluation at the leaf. Switching from a completely independent distribution to a k-wise independent distribution does not change these probabilities, since the partial assignments involve at most k variables. Similarly switching to a (k, δ)-approximation changes each of these probabilities by at most δ. There are at most D^k paths resulting in a 1-evaluation, which implies that the deviation of the expectation is at most $D^k\delta$. ∎

The following lemma shows that using a (k, δ)-approximate distribution instead of the original one does not change the performance of Algorithm 2 if the events have low decision tree complexity.

LEMMA 13. *Suppose that there exists a constant c such that every event $A \in \mathcal{A}$ has decision tree complexity at most $c \min\{-\log x'(A), \log M\}$. Let $k = 2c\gamma$ and $\delta^{-1} = 3M^{2+2/\epsilon}D^{2c\gamma}$. The expected number of forbidden witnesses consistent with a table T that was created by a (k, δ)-approximation for the distribution of random tables is at most $1/2 + 1/3 < 1$.*

Proof. The event that a partial witness $\tau \in F$ is consistent with T is exactly the conjunction of events $[v]$, $v \in \overline{V}(\tau)$. Using Proposition 1, the decision tree complexity of this event is at most

$$\sum_{v \in \overline{V}(\tau)} c \min\{\log M, -\log x'([v])\} \leq c \sum_{v \in \overline{V}(\tau)} -\log x'([v]) \leq 2c\gamma,$$

where the last inequality follows because, by definition, forbidden witnesses have weight at most 2γ. Lemma 11 shows that, using the original independent distribution $\mathcal{P}$, the expected number of forbidden witnesses occurring is at most $1/2$. The second claim of Proposition 1 proves that switching to a (k, δ)-approximation changes this expectation by at most $D^k\delta = 1/(3M^{2+2/\epsilon})$ for each of the $|F|$ witnesses. To complete the proof, observe that by Lemma 12 we have $|F| \leq M^{2+2/\epsilon}$. ∎

7.3. The parallel algorithm and its analysis. We can now describe our parallel algorithm.

Proof of Theorem 7. We use Algorithm 4 to obtain a good evaluation. We already saw in section 7.1 that the support Y of the (k, δ)-approximation to the random distribution of tables in step 1 can be generated efficiently within the time and the number of processors claimed. We now show that these resources also suffice for the rest of the steps in the algorithm.

Lemma 13 guarantees that there is a table $T \in Y$ for which there is no forbidden witness consistent with it. Steps 2 and 3 are the same as the parallel algorithm in [18]. We will show that on table T this algorithm terminates within at most $\lceil \gamma / w_{min} \rceil$ steps: by using Lemma 4.1 of [18], if the algorithm runs for i iterations, then there exists a consistent witness of height i. Such a witness has weight at least iw_{min}. We know from Lemmas 10 and 9 that no witness of weight more than γ can occur, since otherwise a forbidden witness would be consistent with T. Hence we have $i \leq \gamma / w_{min}$. This means that the thread for table T does not attempt to increment

ALGORITHM 4 (parallel deterministic algorithm).

1. Construct a small set of tables Y which form the support of a (k, δ)-approximate independent distribution $\mathcal{Y}$ using the construction mentioned in section 7.1.
2. For each table $T \in Y$ do in parallel:
   (a) For every variable $P_i \in \mathcal{P}$: initialize the pointer $t_i = 1$.
   (b) While $\exists A \in \mathcal{A}$ that happens when for all $P_i \in \mathcal{P} : P_i = T(i, t_i)$, do
   - Compute, in parallel, a maximal independent set I in the subgraph of $G_{\mathcal{A}}$ induced by the events that happen on the current assignment.
   - Resample all $A \in I$ in parallel: For all $P_i \in \bigcup_{A \in I} \mathrm{vbl}(A)$, increment t_i by one.
   - If $t_i = \lceil \gamma / w_{min} \rceil + 1$ (one more than the total number of samples for P_i in a good table), then halt this thread of computation.
3. Once a valid assignment is found using one of the tables, output it and terminate.

the pointer t_i beyond γ/w_i on table T, and so this thread terminates with a good evaluation. Each of these i iterations takes time t_{eval} to evaluate all m events and time t_{MIS} to compute the independent set on the induced dependency subgraph of size at most m. This proves that after creating the probability space $\mathcal{Y}$, the algorithm terminates in $O((t_{MIS} + t_{eval})\gamma/w_{min})$ time and the termination criterion guarantees correctness. Adding this to $O(\gamma \log D + \log(1/w_{min}))$, the time to construct Y, we get that the total time the algorithm takes is $O((t_{MIS} + t_{eval})\gamma/w_{min} + \gamma \log D)$. The number of processors needed for the loop is bounded by $M^{O(1)}$ for each of the $|Y|$ parallel computations and thus $M^{O((c/\epsilon)\log D)}$ in total. ∎

Again it is easy to obtain the k-CNF result as a corollary of the general algorithm.

Proof of Corollary 8. We apply the LLL in the same way to k-CNF as in the proof of Corollary 6. Again we assume without loss of generality that $k = O(\log n)$ and again get $M = O(mk)$ and $w_{min} \approx k$. Since each clause depends only on k variables, a decision tree complexity of $O(k)$ is obvious. Finally, using Theorem 7 and an algorithm of Alon, Babai, and Itai [2] or Luby [13] to compute the maximal independent set in time $t_{MIS} = O(\log^2 m)$ leads to the claimed running time. ∎

8. Conclusion. Moser and Tardos [18] raised the open question for a deterministic LLL algorithm. We address this question and give a deterministic parallel algorithm that works under nearly the same conditions as its randomized versions.

All known deterministic or (randomized) parallel algorithms need a slack in the LLL conditions (see Table 1). It remains open to remove those ϵ-slacks. Obtaining deterministic constructions for the problems in [10] is another interesting open question.

Acknowledgments. We thank the anonymous reviewers who caught an error in an earlier version and whose comments greatly improved the presentation of this paper. We also thank Aravind Srinivasan for inspiration and Salil Vadhan and David Karger for many helpful comments.

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