Plan for today

- Duality Theory
  - Motivations
  - Duality Theorem
  - Weak Duality Theorem
  - Strong Duality Theorem
  - Complementary Slackness Conditions
  - Complementary Slackness Property

- How to be a good friend?

Obtaining the dual problem (general case)

Primal LP

$$\boxed{\begin{gathered}\text{Max } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0\end{gathered}}
\begin{gathered} \\ \longrightarrow y_1 \\ \longrightarrow y_2 \\ \\ \longrightarrow y_m \\ \\ \end{gathered}$$

n variables, *m* constraints

maximization problem

Dual LP

$$\boxed{\begin{gathered}\text{Min } b_1y_1 + b_2y_2 + \cdots + b_my_m \\ a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2 \\ \vdots \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n \\ y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0\end{gathered}}
\begin{gathered} \\ \longrightarrow x_1 \\ \longrightarrow x_2 \\ \\ \longrightarrow x_n \\ \\ \end{gathered}$$

m variables, *n* constraints,

minimization problem

Symmetry Property: The dual of the dual LP is the primal LP

Recap

DUALITY THEORY

Motivations:

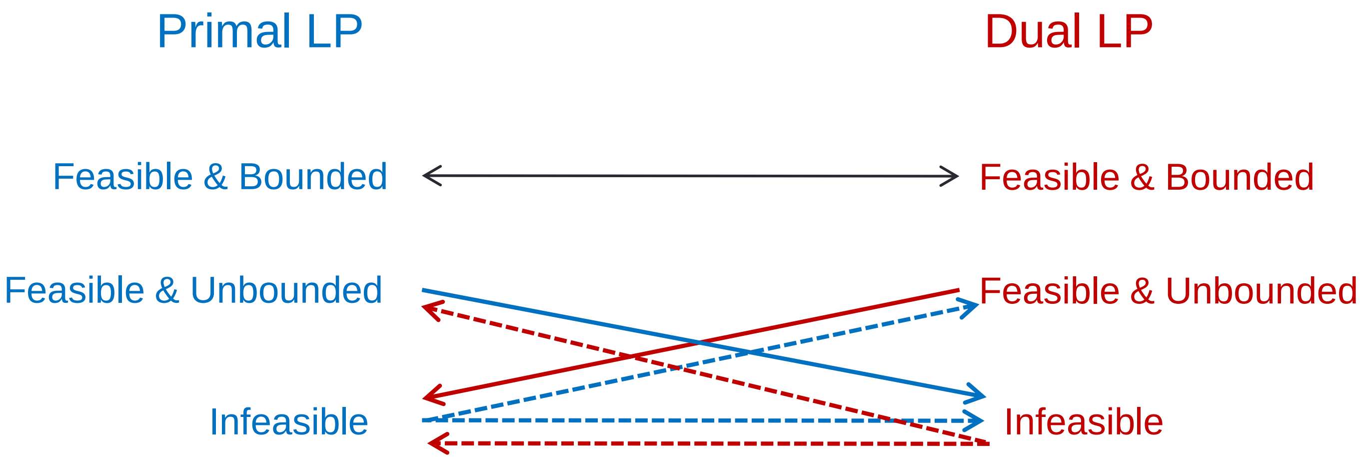
1. Trust thy friend?
2. Convince the st**id boss?

Recap

DUALITY THEORY

1. Duality Theorem
2. Weak Duality Theorem
3. Strong Duality Theorem
4. Complementary Slackness Conditions
5. Complementary Slackness Property

Duality theorem in a figure



Primal LP

Dual LP

Feasible & Bounded

Feasible & Bounded

Feasible & Unbounded

Feasible & Unbounded

Infeasible

Infeasible

Duality Theorem (in words)

The following are the only possible relationships between the primal and the dual problems

Case 1: If primal (dual) problem has a feasible solution and a bounded objective value (and so has an optimal solution), then so does the dual (primal) problem

Case 2: If primal (dual) problem has feasible solutions but unbounded objective value (and so no optimal solution), then the dual (primal) problem has no feasible solution

Case 3: If primal (dual) problem has no feasible solutions, then the dual (primal) problem has either no feasible solutions or unbounded objective value

Weak Duality Theorem

Suppose that the objective function of the primal LP is

$$\max\ c_1x_1 + \cdots + c_nx_n$$

and the objective function of the dual LP is

$$\min\ b_1y_1 + \cdots + b_my_m$$

Let $(\bar{x}_1, .., \bar{x}_n)$ be any **feasible** solution of the primal LP

Let $(\bar{y}_1, \ldots, \bar{y}_m)$ be any **feasible** solution of the dual LP

Then $c_1\bar{x}_1 + \cdots + c_n\bar{x}_n \leq b_1\bar{y}_1 + \cdots + b_m\bar{y}_m$

Recall Example:

Primal Obj: $\max\ 3x_1 + 5x_2$

Dual Obj: $\min\ 4y_1 + 12y_2 + 18y_3$

For any primal feasible $(\bar{x}_1, \bar{x}_2)$
and any dual feasible$(\bar{y}_1, \bar{y}_2, \bar{y}_3)$
$3\bar{x}_1 + 5\bar{x}_2 \leq 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3$

Obj value of
feasible solutions for
the primal max LP

Obj value of
feasible solutions for
the dual min LP

For Primal Maximization LP, Dual Minimization LP,
Maximization LP’s obj value ≤ Minimization LP’s obj value

Weak Duality Theorem

Suppose that the objective function of the primal LP is

$$\max\ c_1x_1 + \cdots + c_nx_n$$

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Then $c_1\bar{x}_1 + \cdots + c_n\bar{x}_n \leq b_1\bar{y}_1 + \cdots + b_m\bar{y}_m$

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$3\bar{x}_1 + 5\bar{x}_2 \leq 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3$

For Primal Maximization LP, Dual Minimization LP,
Maximization LP’s obj value ≤ Minimization LP’s obj value

Obj value of
feasible solutions for
the primal max LP

Obj value of
feasible solutions for
the dual min LP

$-\infty$

$c_1\bar{x}_1 + \cdots + c_n\bar{x}_n$

$b_1\bar{y}_1 + \cdots + b_m\bar{y}_m$

Obj
val

$+\infty$

Strong Duality Theorem

Suppose that the objective function of the primal LP is

$$\max\ c_1x_1 + \cdots + c_nx_n$$

and the objective function of the dual LP is

$$\min\ b_1y_1 + \cdots + b_my_m$$

If $(x_1^*, \dots, x_n^*)$ is an **optimal solution** of the primal LP and $(y_1^*, \dots, y_m^*)$ is an **optimal solution** of the dual LP, then

$$c_1x_1^* + \cdots + c_nx_n^* = b_1y_1^* + \cdots + b_my_m^*$$

Recall Example:

Primal Obj: $\max\ 3x_1 + 5x_2$

Dual Obj: $\min\ 4y_1 + 12y_2 + 18y_3$

For primal opt (x_1^*, x_2^*)
and dual opt (y_1^*, y_2^*, y_3^*)

$$3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^*$$

When both primal and dual have a feasible solution, their optimum objective values will be the same

Strong Duality Theorem implies Weak Duality Theorem

Suppose that the objective of the primal LP is

$$\max\ c_1x_1 + \cdots + c_nx_n$$

and the objective of the dual LP is

$$\min\ b_1y_1 + \cdots + b_my_m$$

Let $(\bar{x}_1, .., \bar{x}_n)$ be any **feasible** solution of the primal LP

Let $(x_1^*, \ldots, x_n^*)$ be an **optimal solution** of the primal LP

$$c_1\bar{x}_1 + \cdots + c_n\bar{x}_n \le c_1x_1^* + \cdots + c_nx_n^*$$

Let $(\bar{y}_1, \ldots, \bar{y}_m)$ be any **feasible** solution of the dual LP

Let $(y_1^*, \ldots, y_m^*)$ be an **optimal solution** of the dual LP

$$b_1y_1^* + \cdots + b_my_m^* \le b_1\bar{y}_1 + \cdots + b_m\bar{y}_m$$

By strong duality property $c_1x_1^* + \cdots + c_nx_n^* = b_1y_1^* + \cdots + b_my_m^*$

So it must be the case that $c_1\bar{x}_1 + \cdots + c_n\bar{x}_n \le b_1\bar{y}_1 + \cdots + b_m\bar{y}_m$

Recall Example:

Primal Obj: $\max\ 3x_1 + 5x_2$

Dual Obj: $\min\ 4y_1 + 12y_2 + 18y_3$

For primal opt (x_1^*, x_2^*)
and dual opt (y_1^*, y_2^*, y_3^*)

$$3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^*$$

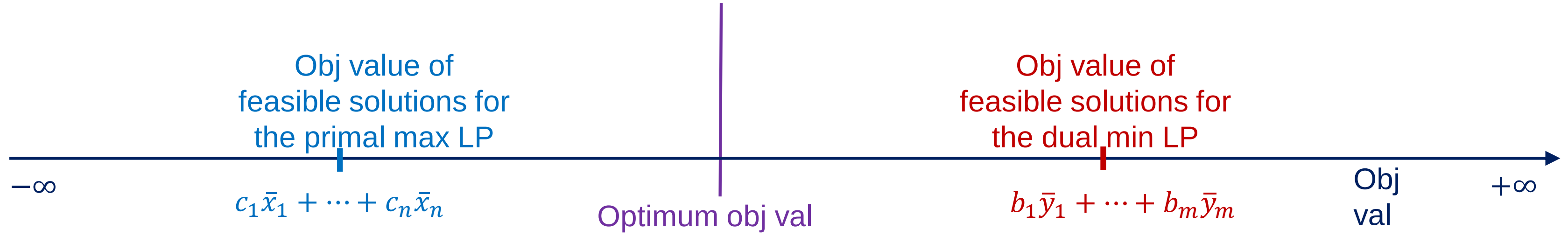
For any primal feasible $(\bar{x}_1, \bar{x}_2)$

$$3\bar{x}_1 + 5\bar{x}_2 \le 3x_1^* + 5x_2^*$$

For any dual feasible$(\bar{y}_1, \bar{y}_2, \bar{y}_3)$

$$4y_1^* + 12y_2^* + 18y_3^* \le 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3$$

$$\Rightarrow 3\bar{x}_1 + 5\bar{x}_2 \le 5\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3$$



Obj value of
feasible solutions for
the primal max LP

Obj value of
feasible solutions for
the dual min LP

$-\infty$

$c_1\bar{x}_1 + \cdots + c_n\bar{x}_n$

Optimum obj val

$b_1\bar{y}_1 + \cdots + b_m\bar{y}_m$

Obj
val

$+\infty$

Strong Duality Theorem is useful to verify optimality

Primal LP

$$\begin{aligned} \max\ Z = 3x_1 + 5x_2 \\ x_1 \leq 4 \\ 2x_2 \leq 12 \\ 3x_1 + 2x_2 \leq 18 \\ x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

y_1

y_2

y_3

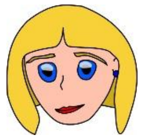
From friend

Dual LP

$$\begin{aligned} \min 4y_1 + 12\ y_2 + 18y_3 \\ y_1 + 3y_3 \geq 3 \\ 2y_2 + 2y_3 \geq 5 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

x_1

x_2



Your guess

Suppose we have primal feasible solution $(\bar{x}_1, \bar{x}_2)$ and dual feasible $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ with the same obj value. Then

$$3\bar{x}_1 + 5\bar{x}_2 \leq 3x_1^* + 5x_n^* = 4y_1^* + 12y_2^* + 18y_m^* \leq 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3 = 3\bar{x}_1 + 5\bar{x}_2$$

by strong duality

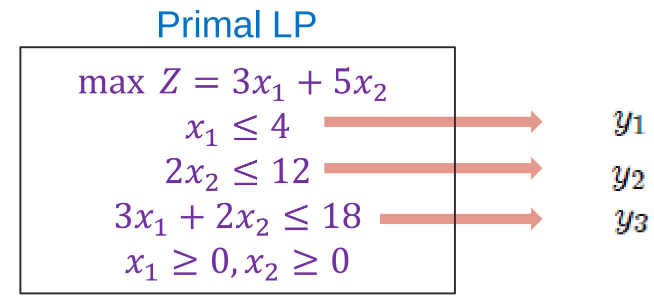
since obj values are the same

So it must be the case that all inequalities in the sequence are equations

So $(\bar{x}_1, \bar{x}_2)$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ are optimal feasible solutions for primal and dual LPs respectively

Primal and Dual feasible solutions with the same objective values are optimal solutions to primal and dual problems respectively

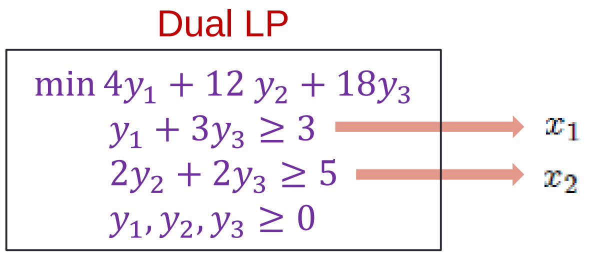
Complementary Slackness Conditions



Primal LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ x_1 &\le 4 \longrightarrow y_1 \\ 2x_2 &\le 12 \longrightarrow y_2 \\ 3x_1 + 2x_2 &\le 18 \longrightarrow y_3 \\ x_1 \ge 0, x_2 &\ge 0 \end{aligned}$$

(x_1^*, x_2^*): optimal solution of the primal LP



Dual LP

$$\begin{aligned} \min 4y_1 + 12\, y_2 &+ 18y_3 \\ y_1 + 3y_3 &\ge 3 \longrightarrow x_1 \\ 2y_2 + 2y_3 &\ge 5 \longrightarrow x_2 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

(y_1^*, y_2^*, y_3^*): optimal solution of the dual LP

Observation 1:

Since (x_1^*, x_2^*) has to satisfy all constraints in the primal LP and (y_1^*, y_2^*, y_3^*) has to satisfy all constraints in the dual LP, we have

$$\begin{aligned} y_1^*(4 - x_1^*) &\ge 0 \\ y_2^*(12 - 2x_2^*) &\ge 0 \\ y_3^*(18 - 3x_1^* - 2x_2^*) &\ge 0 \end{aligned}$$

$$\begin{aligned} x_1^*(y_1^* + 3y_3^* - 3) &\ge 0 \\ x_2^*(2y_2^* + 2y_3^* - 5) &\ge 0 \end{aligned}$$

Complementary Slackness Conditions

Primal LP

$$\begin{array}{rl} \max\ Z = 3x_1 + 5x_2 & \\ x_1 \le 4 & \longrightarrow y_1 \\ 2x_2 \le 12 & \longrightarrow y_2 \\ 3x_1 + 2x_2 \le 18 & \longrightarrow y_3 \\ x_1 \ge 0, x_2 \ge 0 & \end{array}$$

(x_1^*, x_2^*): optimal solution of the primal LP

Dual LP

$$\begin{array}{rl} \min 4y_1 + 12\ y_2 + 18y_3 & \\ y_1 + 3y_3 \ge 3 & \longrightarrow x_1 \\ 2y_2 + 2y_3 \ge 5 & \longrightarrow x_2 \\ y_1, y_2, y_3 \ge 0 & \end{array}$$

(y_1^*, y_2^*, y_3^*): optimal solution of the dual LP

Observation 2:

$$\begin{array}{r} y_1^*(4 - x_1^*) \\ +\ y_2^*(12 - 2x_2^*) \\ +\ y_3^*(18 - 3x_1^* - 2x_2^*) \\ \hline \end{array} \qquad \begin{array}{r} x_1^*(y_1^* + 3y_3^* - 3) \\ +\ x_2^*(2y_2^* + 2y_3^* - 5) \\ \hline \end{array}$$

$$\underbrace{4y_1^* + 12y_2^* + 18y_3^* - y_1^*x_1^* - 2x_2^*y_2^* - 3x_1^*y_3^* - 2x_2^*y_3^* \ + \ y_1^*x_1^* + 2x_2^*y_2^* + 3x_1^*y_3^* + 2x_2^*y_3^* - 3x_1^* - 5x_2^*}_{4y_1^* + 12y_2^* + 18y_3^* - 3x_1^* - 5x_2^* \qquad = 0}$$

Recall that by the strong duality theorem

$$4y_1^* + 12y_2^* + 18y_3^* = 3x_1^* + 5x_2^*$$

Complementary Slackness Conditions

Primal LP

$$\boxed{\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ x_1 &\le 4 \longrightarrow y_1 \\ 2x_2 &\le 12 \longrightarrow y_2 \\ 3x_1 + 2x_2 &\le 18 \longrightarrow y_3 \\ x_1 \ge 0&, x_2 \ge 0 \end{aligned}}$$

(x_1^*, x_2^*): optimal solution of the primal LP

Dual LP

$$\boxed{\begin{aligned} \min 4y_1 + 12\, y_2 &+ 18y_3 \\ y_1 + 3y_3 &\ge 3 \longrightarrow x_1 \\ 2y_2 + 2y_3 &\ge 5 \longrightarrow x_2 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}}$$

(y_1^*, y_2^*, y_3^*): optimal solution of the dual LP

Observation 1

$$\begin{aligned} y_1^*(4 - x_1^*) &\ge 0 \\ y_2^*(12 - 2x_2^*) &\ge 0 \\ y_3^*(18 - 3x_1^* - 2x_2^*) &\ge 0 \\ x_1^*(y_1^* + 3y_3^* - 3) &\ge 0 \\ x_2^*(2y_2^* + 2y_3^* - 5) &\ge 0 \end{aligned}$$

Observation 2

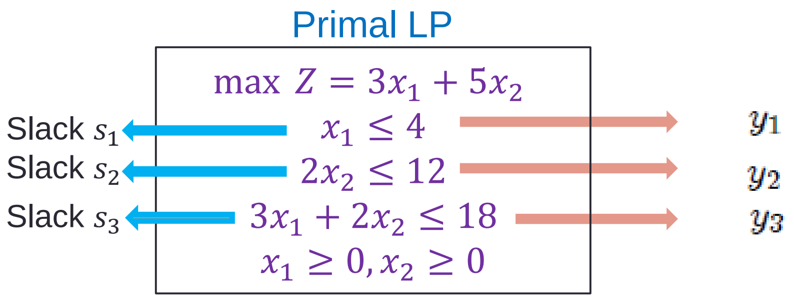
$$\begin{array}{rl} & y_1^*(4 - x_1^*) \\ + & y_2^*(12 - 2x_2^*) \\ + & y_3^*(18 - 3x_1^* - 2x_2^*) \\ + & x_1^*(y_1^* + 3y_3^* - 3) \\ + & x_2^*(2y_2^* + 2y_3^* - 5) \\ \hline & \mathbf{0} \end{array}$$

Observation 3

$$\left.\begin{aligned} y_1^*(4 - x_1^*) &= 0 \\ y_2^*(12 - 2x_2^*) &= 0 \\ y_3^*(18 - 3x_1^* - 2x_2^*) &= 0 \\ x_1^*(y_1^* + 3y_3^* - 3) &= 0 \\ x_2^*(2y_2^* + 2y_3^* - 5) &= 0 \end{aligned}\right\}$$

Complementary Slackness Conditions

Complementary Slackness Conditions



Primal LP

$$\max\ Z = 3x_1 + 5x_2$$

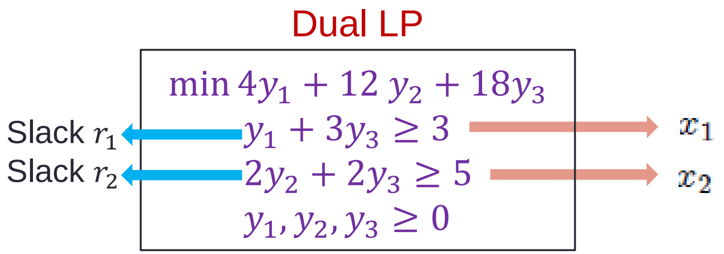
Slack s_1 ← $x_1 \le 4$ → y_1

Slack s_2 ← $2x_2 \le 12$ → y_2

Slack s_3 ← $3x_1 + 2x_2 \le 18$ → y_3

$$x_1 \ge 0, x_2 \ge 0$$

(x_1^*, x_2^*): optimal solution of the primal LP



Dual LP

$$\min 4y_1 + 12\, y_2 + 18y_3$$

Slack r_1 ← $y_1 + 3y_3 \ge 3$ → x_1

Slack r_2 ← $2y_2 + 2y_3 \ge 5$ → x_2

$$y_1, y_2, y_3 \ge 0$$

(y_1^*, y_2^*, y_3^*): optimal solution of the dual LP

Observation 1

$$\begin{aligned} y_1^*(4 - x_1^*) &\ge 0 \\ y_2^*(12 - 2x_2^*) &\ge 0 \\ y_3^*(18 - 3x_1^* - 2x_2^*) &\ge 0 \\ x_1^*(y_1^* + 3y_3^* - 3) &\ge 0 \\ x_2^*(2y_2^* + 2y_3^* - 5) &\ge 0 \end{aligned}$$

Observation 2

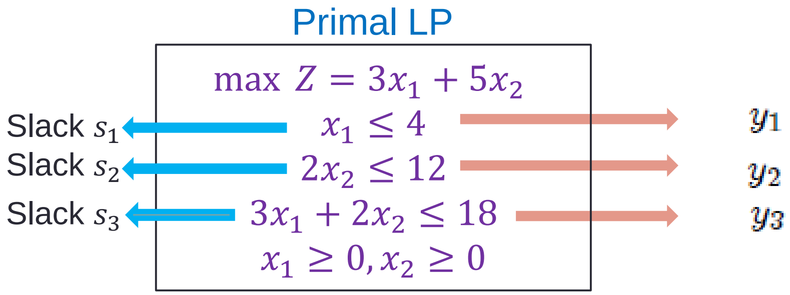
$$\begin{array}{l} y_1^*(4 - x_1^*) \\ +\ y_2^*(12 - 2x_2^*) \\ +\ y_3^*(18 - 3x_1^* - 2x_2^*) \\ +\ x_1^*(y_1^* + 3y_3^* - 3) \\ +\ x_2^*(2y_2^* + 2y_3^* - 5) \\ \hline \mathbf{0} \end{array}$$

Observation 3

$$\left.\begin{aligned} y_1^*(4 - x_1^*) &= 0 \\ y_2^*(12 - 2x_2^*) &= 0 \\ y_3^*(18 - 3x_1^* - 2x_2^*) &= 0 \\ x_1^*(y_1^* + 3y_3^* - 3) &= 0 \\ x_2^*(2y_2^* + 2y_3^* - 5) &= 0 \end{aligned}\right\}$$

Complementary Slackness Conditions

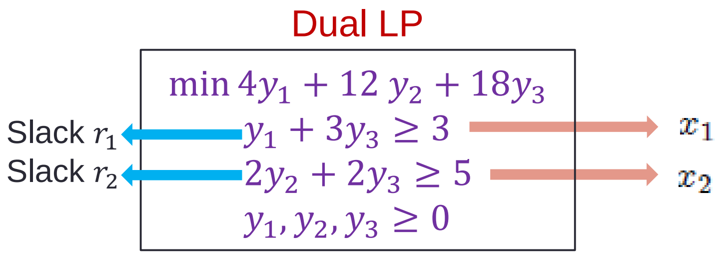
Complementary Slackness Conditions



Primal LP

$$\begin{array}{c} \max\ Z = 3x_1 + 5x_2 \\ x_1 \le 4 \\ 2x_2 \le 12 \\ 3x_1 + 2x_2 \le 18 \\ x_1 \ge 0, x_2 \ge 0 \end{array}$$

(x_1^*, x_2^*): optimal solution of the primal LP



Dual LP

$$\begin{array}{c} \min 4y_1 + 12\,y_2 + 18y_3 \\ y_1 + 3y_3 \ge 3 \\ 2y_2 + 2y_3 \ge 5 \\ y_1, y_2, y_3 \ge 0 \end{array}$$

(y_1^*, y_2^*, y_3^*): optimal solution of the dual LP

Observation 1

$$\begin{aligned} y_1^*(4 - x_1^*) &\ge 0 \\ y_2^*(12 - 2x_2^*) &\ge 0 \\ y_3^*(18 - 3x_1^* - 2x_2^*) &\ge 0 \\ x_1^*(y_1^* + 3y_3^* - 3) &\ge 0 \\ x_2^*(2y_2^* + 2y_3^* - 5) &\ge 0 \end{aligned}$$

Observation 2

$$\begin{array}{l} y_1^*(4 - x_1^*) \\ +\ y_2^*(12 - 2x_2^*) \\ +\ y_3^*(18 - 3x_1^* - 2x_2^*) \\ +\ x_1^*(y_1^* + 3y_3^* - 3) \\ +\ x_2^*(2y_2^* + 2y_3^* - 5) \\ \hline \qquad\qquad 0 \end{array}$$

Observation 3

$$\left.\begin{aligned} y_1^* s_1^* &= 0 \\ y_2^* s_2^* &= 0 \\ y_3^* s_3^* &= 0 \\ x_1^* r_1^* &= 0 \\ x_2^* r_2^* &= 0 \end{aligned}\right\}$$

Complementary Slackness Conditions

Complementary slackness conditions:
the value of a variable in the primal (dual) optimal solution can be non-zero only if the corresponding constraint in its dual (primal) has no slack in a dual (primal) optimal solution

Complementary Slackness Property

Primal LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ x_1 &\le 4 \longrightarrow y_1 \\ 2x_2 &\le 12 \longrightarrow y_2 \\ 3x_1 + 2x_2 &\le 18 \longrightarrow y_3 \\ x_1 \ge 0&, x_2 \ge 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \min 4y_1 + 12\, y_2 &+ 18y_3 \\ y_1 + 3y_3 &\ge 3 \longrightarrow x_1 \\ 2y_2 + 2y_3 &\ge 5 \longrightarrow x_2 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

Suppose we have primal and dual feasible solutions $(\bar{x}_1, \bar{x}_2)$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$respectively
Suppose they satisfy complementary slackness conditions, i.e.,

$$\bar{y}_1(4 - \bar{x}_1) = 0$$
$$\bar{y}_2(12 - 2\bar{x}_2) = 0$$
$$\bar{y}_3(18 - 3\bar{x}_1 - 2\bar{x}_2) = 0$$

$$\bar{x}_1(\bar{y}_1 + 3\bar{y}_3 - 3) = 0$$
$$\bar{x}_2(2\bar{y}_2 + 2\bar{y}_3 - 5) = 0$$

Then

$$\begin{aligned} 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3 &= \bar{y}_1\bar{x}_1 + 2\bar{y}_2\bar{x}_2 + 3\bar{y}_3\bar{x}_1 + 2\bar{y}_3\bar{x}_2 \\ &= \bar{x}_1(\bar{y}_1 + 3\bar{y}_3) + \bar{x}_2(2\bar{y}_2 + 2\bar{y}_3) \\ &= 3\bar{x}_1 + 5\bar{x}_2 \end{aligned}$$

$\Rightarrow$ $(\bar{x}_1, \bar{x}_2)$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ are feasible and they have the same objective value

$\Rightarrow$ $(\bar{x}_1, \bar{x}_2)$ and $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ are optimal

Complementary Slackness Property: Feasible solutions satisfying complementary slackness conditions are optimal solutions

Complementary Slackness Property

Primal LP

$$\begin{array}{rcl} \text{Max } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n & & \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 & \Longrightarrow & y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 & \Longrightarrow & y_2 \\ \vdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m & \Longrightarrow & y_m \\ x_1 \ge 0, x_2 \ge 0, \dots, x_n \ge 0 & & \end{array}$$

Dual LP

$$\begin{array}{rcl} \text{Min } b_1y_1 + b_2y_2 + \cdots + b_my_m & & \\ a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1 & \Longrightarrow & x_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2 & \Longrightarrow & x_2 \\ \vdots & & \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n & \Longrightarrow & x_n \\ y_1 \ge 0, y_2 \ge 0, \dots, y_m \ge 0 & & \end{array}$$

Suppose we have primal and dual feasible solutions $(\bar{x}_1, \dots, \bar{x}_n)$ and $(\bar{y}_1, \dots, \bar{y}_m)$respectively
Suppose they satisfy complementary slackness conditions, i.e.,

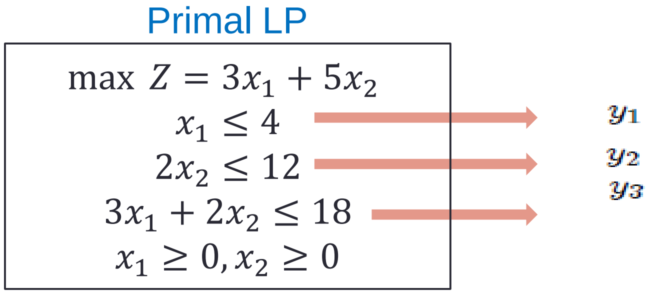
$$\begin{array}{c} \bar{y}_1(b_1 - a_{11}\bar{x}_1 - a_{12}\,\bar{x}_2 - \cdots - a_{1n}\bar{x}_n) = 0 \\ \bar{y}_2(b_2 - a_{21}\bar{x}_1 - a_{22}\bar{x}_2 - \cdots - a_{2n}\bar{x}_n) = 0 \\ \vdots \\ \bar{y}_m(b_m - a_{m1}\bar{x}_1 - a_{m2}\bar{x}_2 - \cdots - a_{mn}\bar{x}_n) = 0 \end{array}$$

$$\begin{array}{c} \bar{x}_1(a_{11}\bar{y}_1 + a_{21}\bar{y}_2 + \cdots + a_{m1}\bar{y}_m - c_1) = 0 \\ \bar{x}_2(a_{12}\bar{y}_1 + a_{22}\bar{y}_2 + \cdots + a_{m2}\bar{y}_m - c_2) = 0 \\ \vdots \\ \bar{x}_n(a_{1n}\bar{y}_1 + a_{2n}\bar{y}_2 + \cdots + a_{mn}\bar{y}_m - c_n) = 0 \end{array}$$

Then $\sum_{j=1}^{m} b_j\bar{y}_j = \sum_{j=1}^{m} \bar{y}_j\left(\sum_{i=1}^{n} a_{ji}\bar{x}_i\right) = \sum_{i=1}^{n} \bar{x}_i\left(\sum_{j=1}^{m} a_{ji}\bar{y}_j\right) = \sum_{i=1}^{n} c_i\bar{x}_i$ $\Rightarrow$ $(\bar{x}_1, \dots, \bar{x}_n)$ and $(\bar{y}_1, \dots, \bar{y}_m)$ are optimal

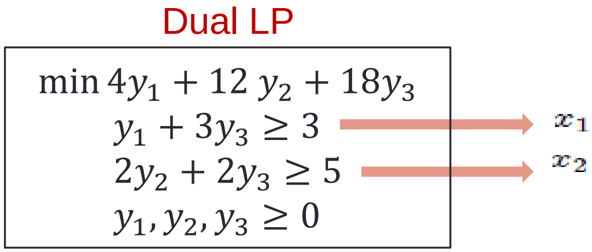
Complementary Slackness Property: Feasible solutions satisfying complementary slackness conditions are optimal solutions

Summary of duality theory



Primal LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ x_1 &\le 4 \quad \rightarrow y_1 \\ 2x_2 &\le 12 \quad \rightarrow y_2 \\ 3x_1 + 2x_2 &\le 18 \quad \rightarrow y_3 \\ x_1 \ge 0, x_2 &\ge 0 \end{aligned}$$



Dual LP

$$\begin{aligned} \min 4y_1 + 12\, y_2 &+ 18y_3 \\ y_1 + 3y_3 &\ge 3 \quad \rightarrow x_1 \\ 2y_2 + 2y_3 &\ge 5 \quad \rightarrow x_2 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

Duality properties for this example: (the primal LP has a finite optimal solution)

1. Duality Theorem: Dual LP also has finite optimal solution
2. Weak duality:

$3\bar{x}_1 + 5\bar{x}_2 \le 4y_1^* + 12y_2^* + 18y_3^* = 36$ if $(\bar{x}_1, \bar{x}_2)$ is feasible for the primal LP

$4\bar{y}_1 + 12\,\bar{y}_2 + 18\bar{y}_3 \ge 3x_1^* + 5x_2^* = 36$ if $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is feasible for the dual LP

3. Strong duality: (x_1^*, x_2^*): optimal for the primal LP
(y_1^*, y_2^*, y_3^*): optimal for the dual LP

$$\Rightarrow 4y_1^* + 12y_2^* + 18y_3^* = 3x_1^* + 5x_2^* = 36$$

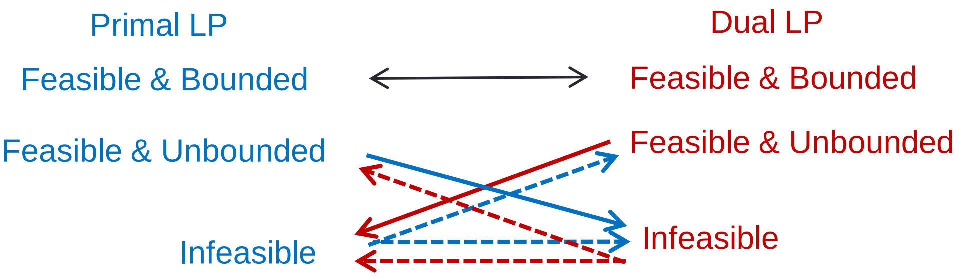
4. Complementary Slackness Conditions:

$$\begin{aligned} y_1^*(4 - x_1^*) &= 0 \\ y_2^*(12 - 2x_2^*) &= 0 \\ y_3^*(18 - 3x_1^* - 2x_2^*) &= 0 \\ x_1^*(y_1^* + 3y_3^* - 3) &= 0 \\ x_2^*(2y_2^* + 2y_3^* - 5) &= 0 \end{aligned}$$

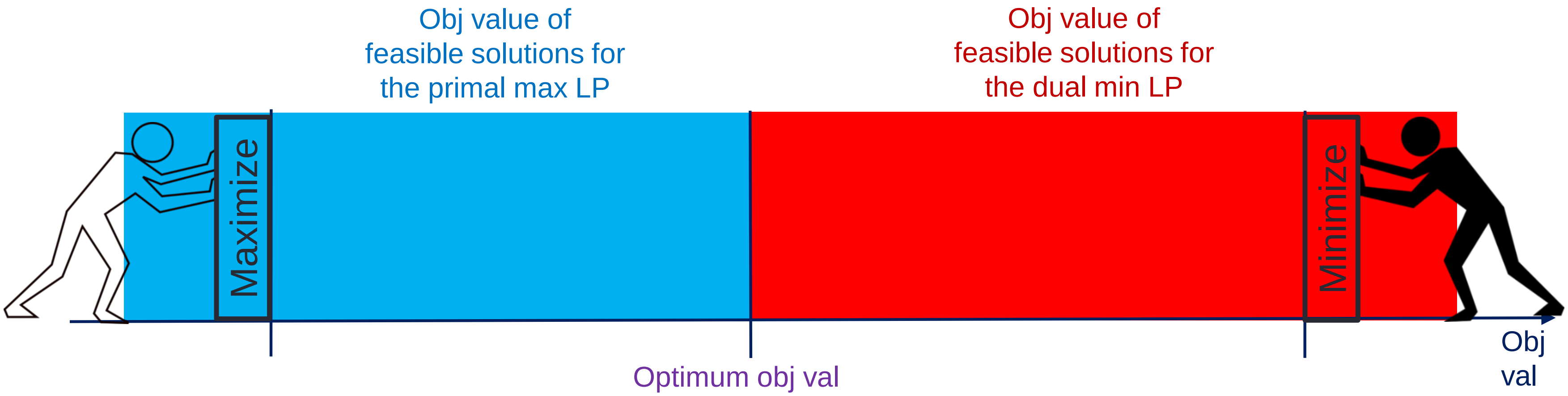
5. Complementary Slackness Property: Feasibility and Complementary Slackness Conditions imply optimality

Summary of duality theory

- Duality Theorem



- Weak and Strong duality properties (For a LP with an optimal solution)



Summary of duality theory

- Complementary Slackness Conditions:

The value of a variable in the primal (dual) optimal solution can be non-zero only if the corresponding constraint in the dual (primal) has no slack in a dual (primal) optimal solution

- Complementary Slackness Property:

Feasible primal and dual solutions that satisfy complementary slackness conditions are optimal primal and optimal dual solutions respectively

To optimize …

Simplex Method (Tabular Form)

- Good friends (good employees) will bring the dual optimal solution that will help check optimality

- How to be a good friend (good employee)?
  - Not only solve the primal LP, but also solve the dual LP
- How to solve the dual LP?

- Well, Simplex method already solves the dual LP!

- Where are the **dual optimal solution values (i.e., the shadow prices)** after termination of the simplex method in tabular form?

SIMPLEX METHOD IN TABULAR FORM

Dual Optimal Values (Shadow Prices): Where are they?

Simplex Method (Tabular Form): Initialization

Original LP

$$\begin{aligned}&\max Z = 6x_1 + 5x_2\\&2x_1 + x_2 \le 4\\&3x_1 + x_2 \le 5\\&x_1, x_2 \ge 0\end{aligned}$$

Augmented LP

$$\begin{aligned}&\max Z = 6x_1 + 5x_2\\&2x_1 + x_2 + s_1 = 4\\&3x_1 + x_2 + s_2 = 5\\&x_1, x_2, x_3, x_4 \ge 0\end{aligned}$$

Initialization

$$\begin{aligned}&Z = 6x_1 + 5x_2\\&s_1 = 4 - 2x_1 - x_2\\&s_2 = 5 - 3x_1 - x_2\end{aligned}$$

Basic variables: $s_1 = 4, s_2 = 5$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

$$\begin{aligned}&Z - 6x_1 - 5x_2 = 0\\&2x_1 + x_2 + s_1 = 4\\&3x_1 + x_2 + s_2 = 5\end{aligned}$$

Slack Variables

Basic Var.	Z	x_1	x_2	s_1	s_2	RHS
Z	1	-6	-5	0	0	0
s_1	0	2	1	1	0	4
s_2	0	3	1	0	1	5

Simplex Method (Tabular Form): Terminating tableau

Slack Variables

Iteration	Basic Var.	Z	x_1	x_2	s_1 y_1^*	s_2 y_2^*	RHS	ratio
3	Z	1	4	0	5	0	20	
	x_2	0	2	1	1	0	4	
	s_2	0	1	0	-1	1	1	

$$\Updownarrow$$

$$\boxed{\begin{aligned} Z + 4x_1 + 5s_1 &= 20 \\ 2x_1 + x_2 + s_1 &= 4 \\ x_1 - s_1 + s_2 &= 1 \end{aligned}} \Longleftrightarrow \boxed{\begin{aligned} Z &= 20 - 4x_1 - 5s_1 \\ x_2 &= 4 - 2x_1 - s_1 \\ s_2 &= 1 - x_1 + s_1 \end{aligned}}$$

Basic variables: $x_2 = 4, s_2 = 1$

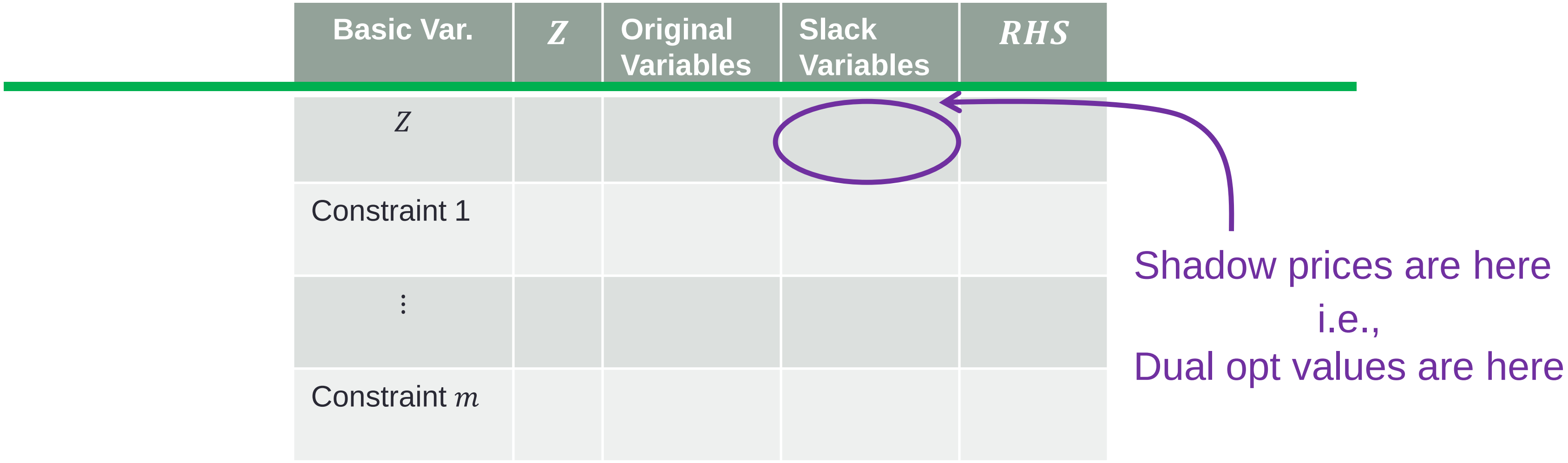
Non-basic variables: $x_1 = s_1 = 0$

$Z = 20$

Shadow price for constraint 1: Coefficient of slack variable s_1 in optimum Z which is 5

Shadow price for constraint 2: Coefficient of slack variable s_2 in optimum Z which is 0

Simplex Method (Tabular Form): Terminating tableau



Basic Var.	Z	Original Variables	Slack Variables	RHS
Z				
Constraint 1				
$\vdots$				
Constraint m				

Shadow prices are here

i.e.,

Dual opt values are here

Simplex Method (Tabular Form)

- Good friends (good employees) will bring the dual optimal solution that will help check optimality

- How to be a good friend (good employee)?

  Not only solve the primal LP, but also solve the dual LP

- How to solve the dual LP?

- Well, Simplex method already solves the dual LP!

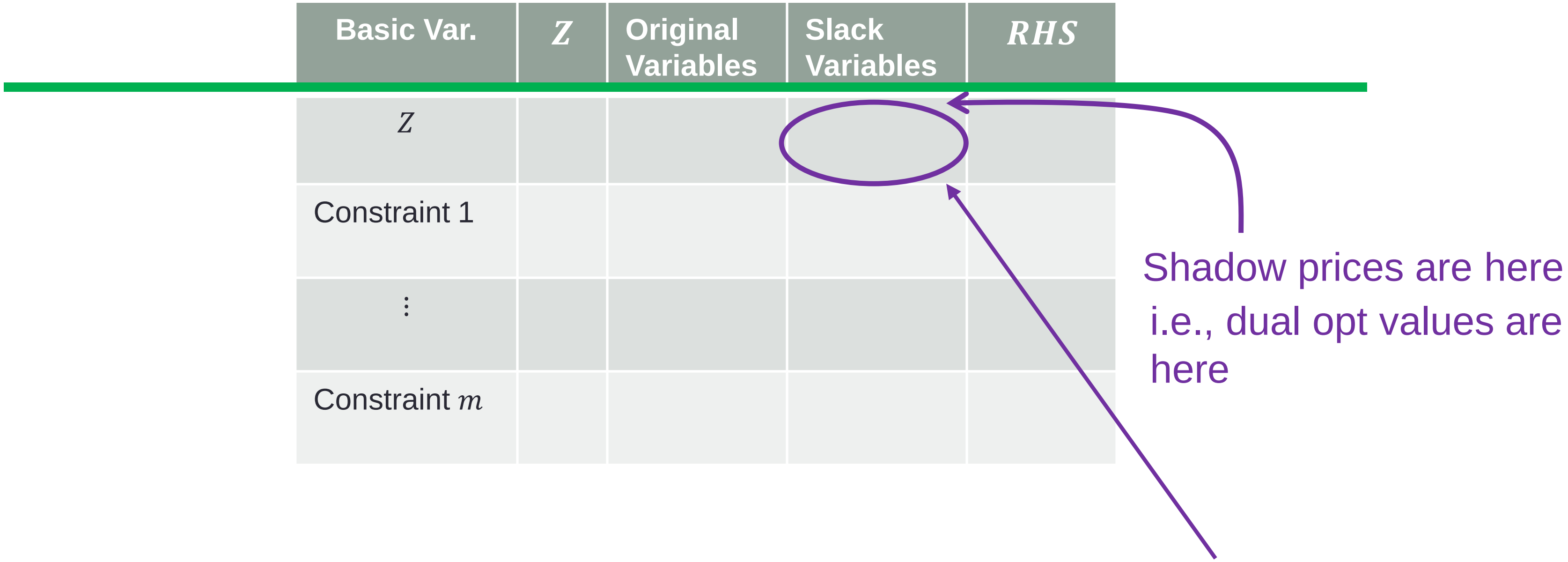
- Where are the **dual optimal solution values (i.e., the shadow prices)** after termination of the simplex method in tabular form?

✓

SIMPLEX METHOD

Complementary Solutions Property

Simplex Method (Tabular Form): Terminating tableau



Basic Var.	Z	Original Variables	Slack Variables	RHS
Z				
Constraint 1				
⋮				
Constraint m				

Shadow prices are here

i.e., dual opt values are here

Qn: What is there in previous iterations before termination?

Simplex Method (Tabular Form): Initialization

Original LP

$$\begin{aligned} &\max Z = 6x_1 + 5x_2 \\ &2x_1 + x_2 \le 4 \\ &3x_1 + x_2 \le 5 \\ &x_1, x_2 \ge 0 \end{aligned}$$

Augmented LP

$$\begin{aligned} &\max Z = 6x_1 + 5x_2 \\ &2x_1 + x_2 + s_1 = 4 \\ &3x_1 + x_2 + s_2 = 5 \\ &x_1, x_2, s_1, s_2 \ge 0 \end{aligned}$$

Initialization

$$\begin{aligned} Z &= 6x_1 + 5x_2 \\ s_1 &= 4 - 2x_1 - x_2 \\ s_2 &= 5 - 3x_1 - x_2 \end{aligned}$$

Basic variables: $s_1 = 4, s_2 = 5$

Non-basic variables: $x_1 = x_2 = 0$

$Z = 0$

$$\begin{aligned} Z - 6x_1 - 5x_2 &= 0 \\ 2x_1 + x_2 + s_1 &= 4 \\ 3x_1 + x_2 + s_2 &= 5 \end{aligned}$$

Basic Var.	Z	x_1	x_2	s_1 y_1	s_2 y_2	RHS
Z	1	-6	-5	0	0	0
s_1	0	2	1	1	0	4
s_2	0	3	1	0	1	5

Simplex Method (Tabular Form): Iterations

Iteration	Basic Var.	Z	x_1	x_2	s_1 y_1	s_2 y_2	RHS	ratio
1	Z	1	0	-3	0	2	10	
	s_1	0	0	$\frac{1}{3}$	1	$-\frac{2}{3}$	$\frac{2}{3}$	2
	x_1	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$	5
2	Z	1	0	0	9	-4	16	
	x_2	0	0	1	3	-2	2	
	x_1	0	1	0	-1	1	1	1
3	Z	1	4	0	5	0	20	
	x_2	0	2	1	1	0	4	
	s_2	0	1	0	-1	1	1	

No pivot col, so optimal solution has been found
Terminate!

Example 1

Simplex Method (Tabular Form): Initialization

primal LP

dual LP

Original LP

$$\begin{aligned}\max Z &= 6x_1 + 5x_2\\ 2x_1 + x_2 &\le 4\\ 3x_1 + x_2 &\le 5\\ x_1, x_2 &\ge 0\end{aligned}$$

$$\begin{aligned}\min 4y_1 &+ 5\,y_2\\ 2y_1 + 3y_2 &\ge 6\\ y_1 + y_2 &\ge 5\\ y_1, y_2 &\ge 0\end{aligned}$$

Augmented LP

$$\begin{aligned}\max Z &= 6x_1 + 5x_2\\ 2x_1 + x_2 + s_1 &= 4\\ 3x_1 + x_2 + s_2 &= 5\\ x_1, x_2, s_1, s_2 &\ge 0\end{aligned}$$

Dual obj: $4y_1 + 5y_2$

Comp Slackness Condns

$y_1(4 - 2x_1 - x_2) = 0$

$y_2(5 - 3x_1 - x_2) = 0$

$x_1(2y_1 + 3y_2 - 6) = 0$

$x_2(y_1 + y_2 - 5) = 0$

Basic Var.	Z	x_1	x_2	s_1 y_1	s_2 y_2	RHS
Z	1	-6	-5	0	0	0
s_1	0	2	1	1	0	4
s_2	0	3	1	0	1	5

$4\bar{y}_1 + 5\bar{y}_2 = 0$

$\bar{y}_1(4 - 2\bar{x}_1 - \bar{x}_2) = 0$

$\bar{y}_2(5 - 3\bar{x}_1 - \bar{x}_2) = 0$

$\bar{x}_1(2\bar{y}_1 + 3\bar{y}_2 - 6) = 0$

$\bar{x}_2(\bar{y}_1 + \bar{y}_2 - 5) = 0$

Initialization

$$\begin{aligned}Z &= 6x_1 + 5x_2\\ s_1 &= 4 - 2x_1 - x_2\\ s_2 &= 5 - 3x_1 - x_2\end{aligned}$$

$$\begin{aligned}Z - 6x_1 - 5x_2 &= 0\\ 2x_1 + x_2 + s_1 &= 4\\ 3x_1 + x_2 + s_2 &= 5\end{aligned}$$

Basic variables: $s_1 = 4, s_2 = 5$

Non-basic variables: $x_1 = x_2 = 0$

$Z = 0$

Simplex Method (Tabular Form): Iterations

Iteration	Basic Var.	Z	x_1	x_2	s_1 y_1	s_2 y_2	RHS	ratio
1	Z	1	0	-3	0	2	10	
	s_1	0	0	$\frac{1}{3}$	1	$-\frac{2}{3}$	$\frac{2}{3}$	2
	x_1	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$	5
2	Z	1	0	0	9	-4	16	
	x_2	0	0	1	3	-2	2	
	x_1	0	1	0	-1	1	1	1
3	Z	1	4	0	5	0	20	
	x_2	0	2	1	1	0	4	
	s_2	0	1	0	-1	1	1	

$$4\bar{y}_1 + 5\bar{y}_2 = 10$$
$$\bar{y}_1(4 - 2\bar{x}_1 - \bar{x}_2) = 0$$
$$\bar{y}_2(5 - 3\bar{x}_1 - \bar{x}_2) = 0$$
$$\bar{x}_1(2\bar{y}_1 + 3\bar{y}_2 - 6) = 0$$
$$\bar{x}_2(\bar{y}_1 + \bar{y}_2 - 5) = 0$$

$$4\bar{y}_1 + 5\bar{y}_2 = 16$$
$$\bar{y}_1(4 - 2\bar{x}_1 - \bar{x}_2) = 0$$
$$\bar{y}_2(5 - 3\bar{x}_1 - \bar{x}_2) = 0$$
$$\bar{x}_1(2\bar{y}_1 + 3\bar{y}_2 - 6) = 0$$
$$\bar{x}_2(\bar{y}_1 + \bar{y}_2 - 5) = 0$$

$$4\bar{y}_1 + 5\bar{y}_2 = 20$$
$$\bar{y}_1(4 - 2\bar{x}_1 - \bar{x}_2) = 0$$
$$\bar{y}_2(5 - 3\bar{x}_1 - \bar{x}_2) = 0$$
$$\bar{x}_1(2\bar{y}_1 + 3\bar{y}_2 - 6) = 0$$
$$\bar{x}_2(\bar{y}_1 + \bar{y}_2 - 5) = 0$$

dual LP

$$\min 4y_1 + 5\,y_2$$
$$2y_1 + 3y_2 \geq 6$$
$$y_1 + y_2 \geq 5$$
$$y_1, y_2 \geq 0$$

Dual obj: $4y_1 + 5y_2$

Comp Slackness Condns

$$y_1(4 - 2x_1 - x_2) = 0$$
$$y_2(5 - 3x_1 - x_2) = 0$$
$$x_1(2y_1 + 3y_2 - 6) = 0$$
$$x_2(y_1 + y_2 - 5) = 0$$

No pivot col, so optimal solution has been found Terminate!

Observation: Each iteration of simplex identifies a complementary solution y for the dual LP with same objective value as x, however y is infeasible for the dual LP until termination

Simplex: Complementary Solutions Property

Complementary Solutions Property:

- At each iteration, the simplex algorithm simultaneously maintains a primal feasible solution x and a solution y such that

  x, y satisfy complementary slackness conditions

- If y is feasible to the dual LP, then by complementary slackness property, x and y are optimal primal-dual solutions

Simplex: An alternative viewpoint

- Simplex Algorithm begins at a primal basic feasible solution
- Explores adjacent basic feasible solutions until all Z-row coefficients are non-negative

The Z-row coefficients give the complementary dual solution!

- The complementary dual solution becomes feasible when all Z-row coefficients are non-negative
- At that point, the algorithm has achieved an optimal solution due to the complementary slackness property and so it terminates

Another way to view Simplex

- Begin at a primal basic feasible solution, continue until the complementary dual solution becomes feasible

Can we do the opposite?

- Dual Simplex Method: Begin at a dual basic feasible solution, continue until the complementary primal solution becomes feasible

DUAL SIMPLEX METHOD

… where we see a variant of the simplex method motivated by Duality Theory

Dual Simplex Method

- Starting tableau should have the following properties:
  - All Z-row entries are non-negative
  - Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
  - (Some RHS entries could be negative)

- Iteration:
  1. Determine the leaving basic variable: variable with the most negative RHS (should be strictly negative)
  2. Determine the entering variable (new min-ratio test):
     Choices: non-basic variables with a negative entry in the row of the leaving basic variable
     Selection: pick the one with the smallest absolute value of the ratio between its Z-row entry and its entry in the row of the leaving basic variable
  3. Update tableau similar to simplex method

- Termination: Feasibility test
  - If all RHS values are non-negative, STOP
  - The basic solution in this tableau is primal feasible and hence optimal