Plan for today

- Obtaining the Dual Problem
- Duality theory: intro
  - Primal Dual Connections for the example
- Duality Theory
  - Motivations
  - Duality Theorem

SHADOW PRICES

… How valuable are the resources?

SHADOW PRICES

Shadow price of a constraint
(in a maximization problem)
… is the rate of objective increase when the constraint is relaxed

SHADOW PRICES

Shadow price of a constraint
(in a minimization problem)
… is the rate of objective decrease when the constraint is relaxed

Summary and a Connection

$$\max\ Z = 3x_1 + 5x_2$$
Plant 1: $x_1 \le 4$
Plant 2: $2x_2 \le 12$
Plant 3: $3x_1 + 2x_2 \le 18$
$$x_1 \ge 0, x_2 \ge 0$$

Optimum

$$Z = 36 - \frac{3}{2}s_2 - s_3$$

Basic variables: $x_1^* = 2, x_2^* = 6, s_1^* = 2$
Non-basic variables: $s_2^* = s_3^* = 0$

Question 2: What is the minimum price that you should offer to buy the entire operation?

Answer: \$36

Question 3: How much would the company be willing to pay for an extra working hour at each plant?

Answer: \$0 for plant 1, \$$\frac{3}{2}$ for plant 2, \$1 for plant 3

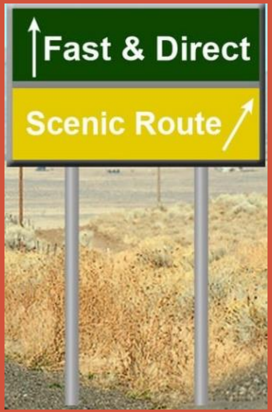
$\left(0, \frac{3}{2}, 1\right)$ are the **shadow prices** of the three plants

What is the objective function in proper format at the last step of the SIMPLEX Method?

$$Z = 36 - 0s_1 - \frac{3}{2}s_2 - s_3$$

See a connection?

Shadow price of each plant is appearing as the coefficient of the corresponding slack variable in the objective (with a negative sign)



DIRECT ANSWERS TO Q2 AND Q3

… where we formulate an LP whose solution gives the shadow prices

Direct answers to Q2 and Q3

Tesla makes two models C_1 and C_2:

	C_1	C_2	total hours
Plant 1 (prepare Frame I)	1		4
Plant 2 (prepare Frame II)		2	12
Plant 3 (assembly)	3	2	18
Profit	3$	5$	

Question 2: What is the minimum price that you should offer to buy the entire operation?

Question 3: What is the price/worth of each working hour at plant 1, or 2, or 3?

Decision variables:

y_j be the price/worth of *each* working hour at plant *j for* $j = 1,2,3$

Objective: minimize the total price paid/worth

Worth of entire capacity at Plant 1 $= 4y_1$

Worth of entire capacity at Plant 2 $= 12y_2$

Worth of entire capacity at Plant 3 $= 18y_3$

Total worth of entire operation is $4y_1 + 12y_2 + 18y_3$

$$\min 4y_1 + 12\, y_2 + 18y_3$$

Constraints: (Combination worth has to be at least the profit)

The total worth of one hour at plant 1 and three hours at plant 3 is at least 3$

The total worth of two hours at plant 2 and two hours at plant 3 is at least 5$

$$y_1 + 3y_3 \geq 3$$

$$2y_2 + 2y_3 \geq 5$$

$$y_1, y_2, y_3 \geq 0$$

Compare the two LPs, what do you observe

Q1: what is the optimal product mix that maximizes the total profit?

Primal LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 \ge 0&, x_2 \ge 0 \end{aligned}$$

LP1

Q2: what is the minimum price that you should offer to buy these three plants?

Dual LP

$$\begin{aligned} \min\ 4y_1 + 12\, y_2 &+ 18y_3 \\ y_1 + 3y_3 &\ge 3 \\ 2y_2 + 2y_3 &\ge 5 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

LP2

- number of variables in LP1 versus number of constraints (excluding non-negativity constraints) in LP2
- constants on the RHS of constraints in LP1 versus coefficients in the objective function of LP2
- col coefficients in the LHS of constraints of LP1 versus row coefficients in the LHS of constraints of LP2

$$\begin{bmatrix} x_1 & x_2 \\ 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} y_1 & y_2 & y_3 \\ 1 & 0 & 3 \\ 0 & 2 & 2 \end{bmatrix}$$

If we call LP1 as the **Primal** LP, then LP2 is known as the **Dual** of LP1

Obtaining the **dual** problem of an LP

… where the LP is a maximization LP

Step 1. Associate each constraint (excluding non-negativity constraints) with a variable

Primal LP

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \leq 4 \rightarrow y_1$$
$$2x_2 \leq 12 \rightarrow y_2$$
$$3x_1 + 2x_2 \leq 18 \rightarrow y_3$$
$$x_1 \geq 0, x_2 \geq 0$$

Step 2. Use the RHS of constraints to define the objective function, with the goal being min

$$\begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} \Longrightarrow \min 4y_1 + 12y_2 + 18y_3$$

Obtaining the **dual** problem of an LP

… where the LP is a maximization LP

Primal LP

$$\max Z = 3x_1 + 5x_2$$
$$x_1 \le 4 \quad \rightarrow y_1$$
$$2x_2 \le 12 \quad \rightarrow y_2$$
$$3x_1 + 2x_2 \le 18 \quad \rightarrow y_3$$
$$x_1 \ge 0, x_2 \ge 0$$

Step 3. Define constraints (one constraint for each x variable)

1. coefficients of the variable give coefficients of the constraint on LHS
2. coefficient of the variable in the objective function gives the RHS constant
3. Constraint:
   If $x_i \ge 0$ then the constraint is $\ge$
   If $x_i \le 0$ then the constraint is $\le$
   If x_i is unrestricted, then the constraint is $=$

$$x_1: \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \Longrightarrow y_1 + 3y_3 \ge 3$$

$$x_2: \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \Longrightarrow 2y_2 + 2y_3 \ge 5$$

Step 4. Add (non)-negativity conditions

1. If $a_1x_1 + \cdots \le c \rightarrow y_j$ then, $y_j \ge 0$
2. If $a_1x_1 + \cdots \ge c \rightarrow y_j$ then, $y_j \le 0$
3. If $a_1x_1 + \cdots = c \rightarrow y_j$ then, y_j is unrestricted

Dual LP

$$\min 4y_1 + 12\, y_2 + 18y_3$$
$$y_1 + 3y_3 \ge 3$$
$$2y_2 + 2y_3 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

What is the **dual** of the dual problem?

Dual LP

$$\min 4y_1 + 12\, y_2 + 18y_3$$
$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$
$$y_1, y_2, y_3 \geq 0$$

We will call this as the Primal LP and obtain the dual now

Obtaining the **dual** problem of an LP

… where the LP is a minimization LP

Step 1. Associate each constraint (excluding non-negativity constraints) with a variable

Primal LP

$$\min 4y_1 + 12\, y_2 + 18y_3$$
$$y_1 + 3y_3 \geq 3 \;\rightarrow\; x_1$$
$$2y_2 + 2y_3 \geq 5 \;\rightarrow\; x_2$$
$$y_1, y_2, y_3 \geq 0$$

Step 2. Use the RHS of constraints to define the objective function, with the goal being max

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} \Longrightarrow max\; 3x_1 + 5x_2$$

Obtaining the **dual** problem of an LP

… where the LP is a minimization LP

Step 3. Define constraints (one constraint for each y variable)

1. coefficients of the variable give coefficients of the constraint on LHS
2. coefficient of the variable in the objective function gives the RHS constant
3. Constraint:
   If $y_i \geq 0$ then the constraint is $\leq$
   If $y_i \leq 0$ then the constraint is $\geq$
   If y_i is unrestricted, then the constraint is $=$

Step 4. Add (non)-negativity conditions

1. If $a_1 y_1 + \cdots \leq c$ → x_j then, $x_j \leq 0$
2. If $a_1 y_1 + \cdots \geq c$ → x_j then, $x_j \geq 0$
3. If $a_1 y_1 + \cdots = c$ → x_j then, x_j is unrestricted

Primal LP

$$\min 4y_1 + 12\,y_2 + 18y_3$$
$$y_1 + 3y_3 \geq 3 \rightarrow x_1$$
$$2y_2 + 2y_3 \geq 5 \rightarrow x_2$$
$$y_1, y_2, y_3 \geq 0$$

$$y_1: \begin{matrix} x_1 \\ x_2 \end{matrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x_1 \leq 4$$

$$y_2: \begin{matrix} x_1 \\ x_2 \end{matrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow 2x_2 \leq 12$$

$$y_3: \begin{matrix} x_1 \\ x_2 \end{matrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Rightarrow 3x_1 + 2x_2 \leq 18$$

Dual LP

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0, x_2 \geq 0$$

Symmetry Property: The dual of the dual LP is the primal LP

Obtaining the dual problem (general case)

Primal LP

$$\begin{aligned} &\text{Max } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \quad \Rightarrow y_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \quad \Rightarrow y_2 \\ &\vdots \\ &a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \quad \Rightarrow y_m \\ &x_1 \ge 0, x_2 \ge 0, \dots, x_n \ge 0 \end{aligned}$$

n variables, *m* constraints

maximization problem

Dual LP

$$\begin{aligned} &\text{Min } b_1y_1 + b_2y_2 + \cdots + b_my_m \\ &a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1 \quad \Rightarrow x_1 \\ &a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2 \quad \Rightarrow x_2 \\ &\vdots \\ &a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n \quad \Rightarrow x_n \\ &y_1 \ge 0, y_2 \ge 0, \dots, y_m \ge 0 \end{aligned}$$

m variables, *n* constraints,

minimization problem

Symmetry Property: The dual of the dual LP is the primal LP

PRIMAL DUAL CONNECTIONS

Some observations about the example

Primal → Dual Connection

Primal LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 \ge 0&, x_2 \ge 0 \end{aligned}$$

→ profit of the firm

working hour constraint at each plant

At the end of SIMPLEX

$$Z = 36 - \frac{3}{2}s_2 - s_3$$

(s_2, s_3 are slack variables of constraints at plants 2 and 3)

36: the total profit, also the min price to sell the firm

0: (shadow) price of a working hour at plant 1

3/2: (shadow) price of a working hour at plant 2

1 : (shadow) price of a working hour at plant 3

Based on the above, can you tell

- what is the optimal objective value for the dual LP below? Ans: 36
- what are the optimal values of y_1^*, y_2^*, y_3^* for the dual LP below? Ans: $y_1^* = 0, y_2^* = \frac{3}{2}, y_3^* = 1$

Dual LP

$$\begin{aligned} \min 4y_1 + 12\, y_2 &+ 18y_3 \\ y_1 + 3y_3 &\ge 3 \\ 2y_2 + 2y_3 &\ge 5 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

→ min. price to buy the firm

Worth at least profit constraints

Observation: Shadow prices of the primal constraints are the dual optimum values

Recall: y_j (j=1,2,3) is the price of each working hour at plant j

Dual → Primal Connection

Primal LP

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0, x_2 \ge 0$$

$$Z^* = 36$$

$$x_1^* = 2, x_2^* = 6$$

Optimal product mix is
2 batches of doors and 6 batches of windows
and optimal profit is 36

Dual LP

$$\min 4y_1 + 12\, y_2 + 18y_3$$
$$y_1 + 3y_3 \ge 3$$
$$2y_2 + 2y_3 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

Using the above solution, can you tell the rate at which the worth of the company increases with respect to

1) the profit from a batch of Model I? Ans: 2
2) the profit from a batch of Model II? Ans: 6

These are the shadow prices of the constraints in the dual LP

Hint:

Observation: Shadow prices of the primal constraints are the dual optimum values

⇒

(By symmetry): Shadow prices of the dual constraints should be the primal optimum values

Suppose that the SIMPLEX iteration is applied to solve the dual LP below, at the last iteration, the expression for the objective function is of the form

$$obj = \theta + \alpha r_1 + \beta r_2$$

where r_1, r_2 are slack variables for the 1st and 2nd constraints,

$$y_1 + 3y_3 - r_1 = 3$$
$$2y_2 + 2y_3 - r_2 = 5$$

Can you tell the values of these Greek symbols? Ans: $\theta = 36, \alpha = 2, \beta = 6$

Hint: Stare at the primal solution

Primal-Dual Connection: optimal solutions

Primal LP

$$\max\ Z = 3x_1 + 5x_2$$

Slack s_1 ← $x_1 \le 4$ → y_1

Slack s_2 ← $2x_2 \le 12$ → y_2

Slack s_3 ← $3x_1 + 2x_2 \le 18$ → y_3

$$x_1 \ge 0, x_2 \ge 0$$

Dual LP

$$\min 4y_1 + 12\,y_2 + 18y_3$$

Slack r_1 ← $y_1 + 3y_3 \ge 3$ → x_1

Slack r_2 ← $2y_2 + 2y_3 \ge 5$ → x_2

$$y_1, y_2, y_3 \ge 0$$

$$Obj^* = 36$$

$$x_1^* = 2, x_2^* = 6$$

$$Obj = 36 - 0s_1 - \frac{3}{2}s_2 - s_3$$

$$Obj^* = 36$$

$$y_1^* = 0, y_2^* = \frac{3}{2}, y_3^* = 1$$

$$Obj = 36 + 2r_1 + 6r_2$$

Primal-Dual Connection

Primal LP

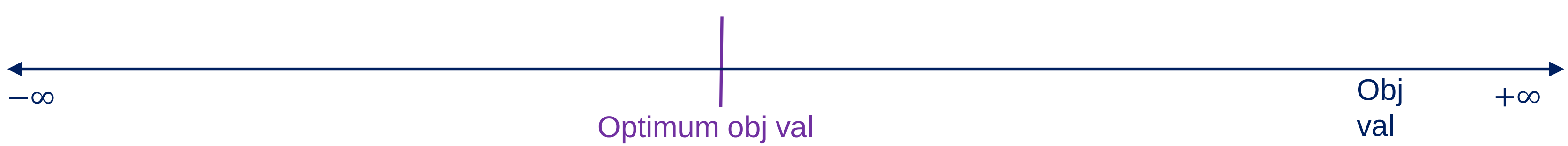
$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \leq 4 \quad \rightarrow \quad y_1$$
$$2x_2 \leq 12 \quad \rightarrow \quad y_2$$
$$3x_1 + 2x_2 \leq 18 \quad \rightarrow \quad y_3$$
$$x_1 \geq 0, x_2 \geq 0$$

Dual LP

$$\min\ 4y_1 + 12\, y_2 + 18y_3$$
$$y_1 + 3y_3 \geq 3 \quad \rightarrow \quad x_1$$
$$2y_2 + 2y_3 \geq 5 \quad \rightarrow \quad x_2$$
$$y_1, y_2, y_3 \geq 0$$

Objective function:

1. If primal LP has an optimal objective value (36 in this case), the dual LP also has the same opt obj value



Primal-Dual Connection

Primal LP

$$\max\ Z = 3x_1 + 5x_2$$
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Dual LP

$$\min 4y_1 + 12\,y_2 + 18y_3$$
$$y_1 + 3y_3 \ge 3 \rightarrow x_1$$
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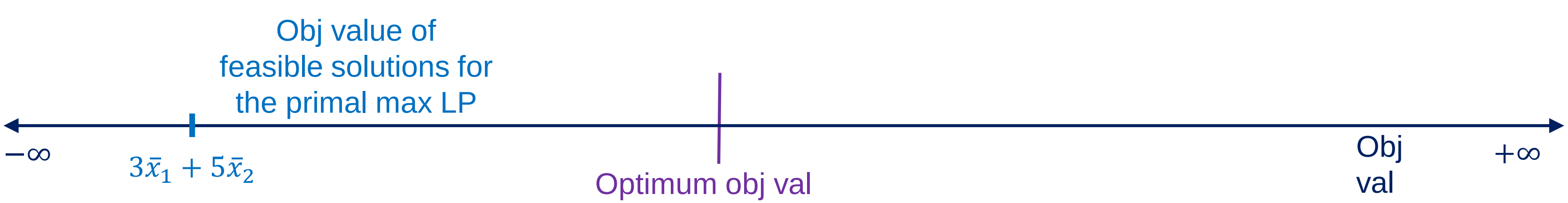
$$y_1, y_2, y_3 \ge 0$$

Objective function:

1. If primal LP has an optimal objective value (36 in this case), the dual LP also has the same opt obj value
2. Objective value of any feasible solution of the maximizing LP is *at most* the optimal objective value of the minimizing LP

If $(\bar{x}_1, \bar{x}_2)$ is feasible for max LP, then $3\bar{x}_1 + 5\bar{x}_2 \le 3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^*$



Primal-Dual Connection

Primal LP

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Dual LP

$$\min 4y_1 + 12\, y_2 + 18y_3$$
$$y_1 + 3y_3 \ge 3 \quad \rightarrow x_1$$
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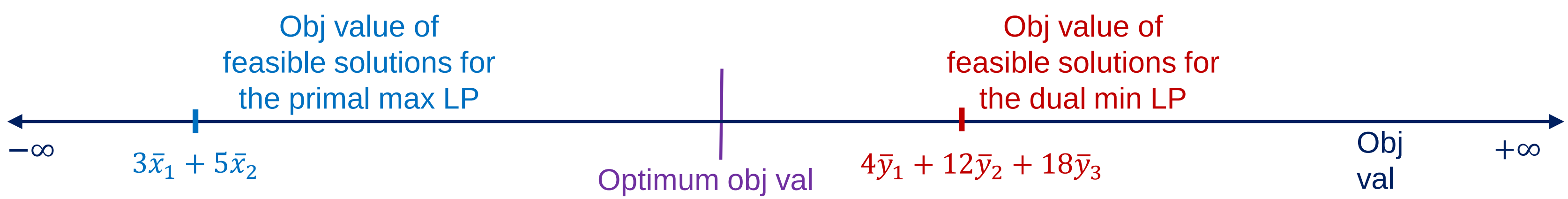
$$y_1, y_2, y_3 \ge 0$$

Objective function:

1. If primal LP has an optimal objective value (36 in this case), the dual LP also has the same opt obj value
2. Objective value of any feasible solution of the maximizing LP is *at most* the optimal objective value of the minimizing LP

   If $(\bar{x}_1, \bar{x}_2)$ is feasible for max LP, then $3\bar{x}_1 + 5\bar{x}_2 \le 3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^*$
3. Objective value of any feasible solution of the minimizing LP is *at least* the optimal objective value of the maximizing LP

   If $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is feasible for min LP, then $3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^* \le 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3$



Primal-Dual Connection

Primal LP

$$\begin{aligned} \max\ Z = 3x_1 + 5x_2 & \\ x_1 \le 4 & \quad \rightarrow y_1 \\ 2x_2 \le 12 & \quad \rightarrow y_2 \\ 3x_1 + 2x_2 \le 18 & \quad \rightarrow y_3 \\ x_1 \ge 0, x_2 \ge 0 & \end{aligned}$$

Dual LP

$$\begin{aligned} \min 4y_1 + 12\, y_2 + 18y_3 & \\ y_1 + 3y_3 \ge 3 & \quad \rightarrow x_1 \\ 2y_2 + 2y_3 \ge 5 & \quad \rightarrow x_2 \\ y_1, y_2, y_3 \ge 0 & \end{aligned}$$

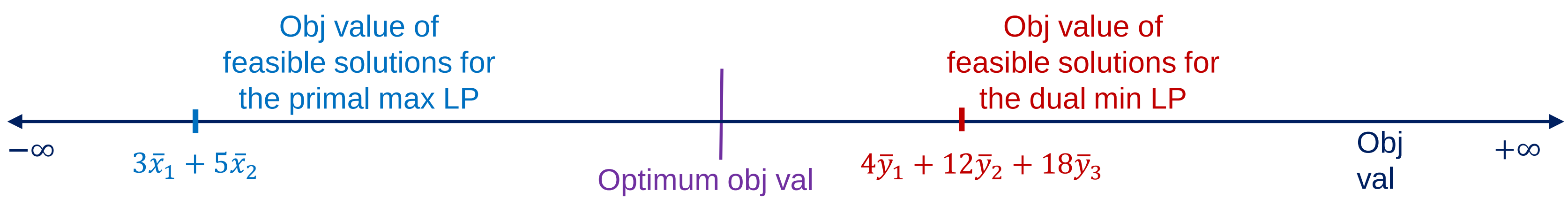
Objective function:

1. If primal LP has an optimal objective value (36 in this case), the dual LP also has the same opt obj value
2. Objective value of any feasible solution of the maximizing LP is *at most* the optimal objective value of the minimizing LP

   If $(\bar{x}_1, \bar{x}_2)$ is feasible for max LP, then $3\bar{x}_1 + 5\bar{x}_2 \le 3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^*$
3. Objective value of any feasible solution of the minimizing LP is *at least* the optimal objective value of the maximizing LP

   If $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is feasible for min LP, then $3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^* \le 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3$
4. Objective value of any feasible solution of the maximizing LP is at most the objective value of any feasible solution of the minimizing LP

   If $(\bar{x}_1, \bar{x}_2)$ is feasible for max LP, $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is feasible for min LP, then
   $3\bar{x}_1 + 5\bar{x}_2 \le 3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^* \le 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3$



Primal-Dual Connection

Primal LP

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$$x_1 \le 4 \quad \rightarrow \quad y_1$$
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$$x_1 \ge 0, x_2 \ge 0$$

Dual LP

$$\min 4y_1 + 12\, y_2 + 18y_3$$
$$y_1 + 3y_3 \ge 3 \quad \rightarrow \quad x_1$$
$$2y_2 + 2y_3 \ge 5 \quad \rightarrow \quad x_2$$
$$y_1, y_2, y_3 \ge 0$$

Objective function:

5. If one LP is unbounded, then the other LP is …?

Primal-Dual Connection

Primal LP

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \leq 4 \rightarrow y_1$$
$$2x_2 \leq 12 \rightarrow y_2$$
$$3x_1 + 2x_2 \leq 18 \rightarrow y_3$$
$$x_1 \geq 0, x_2 \geq 0$$

Dual LP

$$\min 4y_1 + 12\, y_2 + 18y_3$$
$$y_1 + 3y_3 \geq 3 \rightarrow x_1$$
$$2y_2 + 2y_3 \geq 5 \rightarrow x_2$$
$$y_1, y_2, y_3 \geq 0$$

Objective function:

5. If one LP is unbounded, then the other LP is …?

Primal-Dual Connection

Primal LP

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \le 4 \quad \rightarrow \quad y_1$$
$$2x_2 \le 12 \quad \rightarrow \quad y_2$$
$$3x_1 + 2x_2 \le 18 \quad \rightarrow \quad y_3$$
$$x_1 \ge 0, x_2 \ge 0$$

Dual LP

$$\min 4y_1 + 12\, y_2 + 18y_3$$
$$y_1 + 3y_3 \ge 3 \quad \rightarrow \quad x_1$$
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$$y_1, y_2, y_3 \ge 0$$

Objective function:

1. If primal LP has an optimal objective value (36 in this case), the dual LP also has the same opt obj value
2. Objective value of any feasible solution of the maximizing LP is *at most* the optimal objective value of the minimizing LP

   If $(\bar{x}_1, \bar{x}_2)$ is feasible for max LP, then $3\bar{x}_1 + 5\bar{x}_2 \le 3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^*$

3. Objective value of any feasible solution of the minimizing LP is *at least* the optimal objective value of the maximizing LP

   If $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is feasible for min LP, then $3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^* \le 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3$

4. Objective value of any feasible solution of the maximizing LP is at most the objective value of any feasible solution of the minimizing LP

   If $(\bar{x}_1, \bar{x}_2)$ is feasible for max LP, $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is feasible for min LP, then $3\bar{x}_1 + 5\bar{x}_2 \le 3x_1^* + 5x_2^* = 4y_1^* + 12y_2^* + 18y_3^* \le 4\bar{y}_1 + 12\bar{y}_2 + 18\bar{y}_3$

5. If one LP is unbounded, the other LP is <u>infeasible</u>
6. If one LP is infeasible, the other LP is either infeasible or unbounded

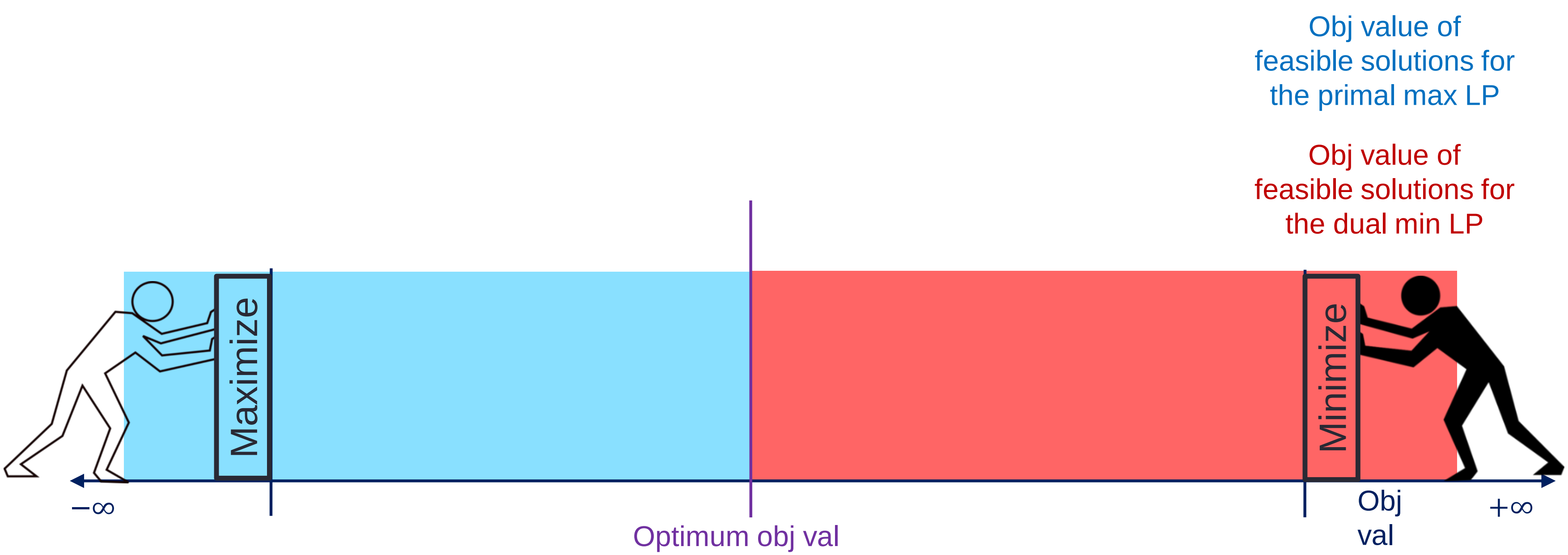
Summary of duality theory

Primal LP

$$
\begin{aligned}
\max\ Z = 3x_1 + 5x_2 & \\
x_1 \le 4 & \quad \rightarrow y_1 \\
2x_2 \le 12 & \quad \rightarrow y_2 \\
3x_1 + 2x_2 \le 18 & \quad \rightarrow y_3 \\
x_1 \ge 0, x_2 \ge 0 &
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
\min 4y_1 + 12\, y_2 + 18y_3 & \\
y_1 + 3y_3 \ge 3 & \quad \rightarrow x_1 \\
2y_2 + 2y_3 \ge 5 & \quad \rightarrow x_2 \\
y_1, y_2, y_3 \ge 0 &
\end{aligned}
$$



DUALITY THEORY

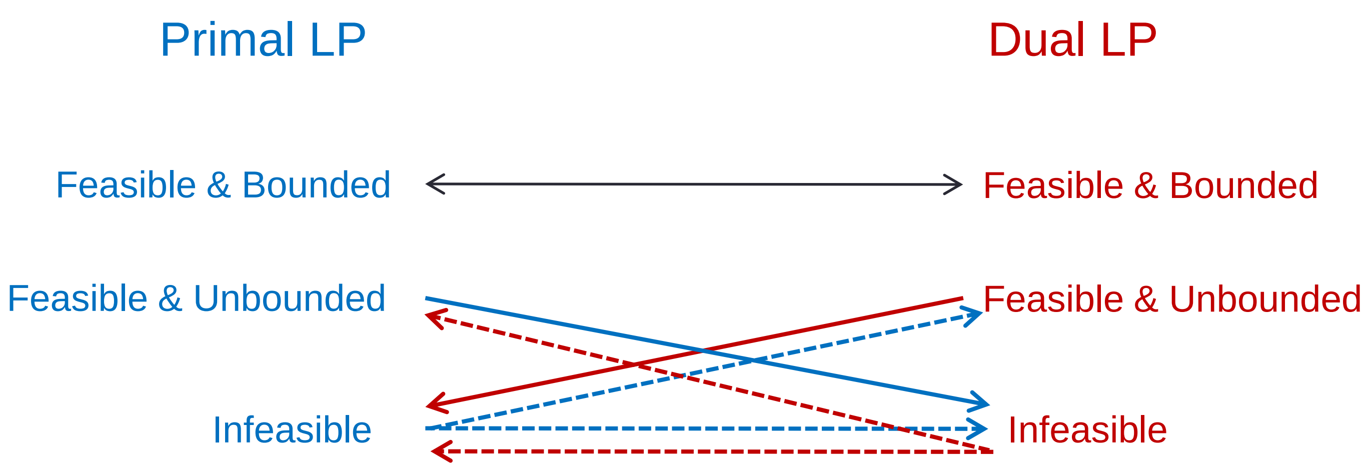
Motivations:

1. Trust thy friend?
2. Convince the st**id boss?

DUALITY THEORY

1. Duality Theorem
2. Weak Duality Theorem
3. Strong Duality Theorem
4. Complementary Slackness Conditions
5. Complementary Slackness Property

Duality theorem in a figure



Recall Example where we have the following relationship:

$Obj^* = 36$ $\quad$ $Obj^* = 36$

$x_1^* = 2, x_2^* = 6$ $\quad$ $y_1^* = 0, y_2^* = \frac{3}{2}, y_3^* = 1$

$Obj = 36 - 0s_1 - \frac{3}{2}s_2 - s_3$ $\quad$ $Obj = 36 + 2r_1 + 6r_2$

which illustrates the first case

Case 2. primal is unbounded and dual is infeasible

Primal LP

$$\max\ 2x_1 - x_2 - 0.5x_3$$
$$x_1 - x_2 \le 2$$
$$x_1 - x_3 \le 3$$
$$x_1, x_2, x_3 \ge 0$$

Unbounded!

Dual LP

$$\min 2y_1 + 3y_2$$
$$y_1 + y_2 \ge 2$$
$$-y_1 \ge -1 \quad \Longleftrightarrow \quad y_1 \le 1$$
$$-y_2 \ge -0.5 \quad \Longleftrightarrow \quad y_2 \le 0.5$$
$$y_1, y_2 \ge 0$$

Infeasible!

- A pear and an apple can be used to make a plate of Pear-Apple salad that sells for $2
- Initially you have 2 Pears and 3 Apples, but you can always buy more
  - The supply of Pears and Apples is unlimited
- Pear costs $1 and Apple costs $0.5
- Decide how many plates to make (x_1), how many pears (x_2) and apples (x_3) to buy to maximize the profit

Do you see why the primal LP is unbounded?

Do you now see why the dual LP is infeasible?

Case 3. both primal and dual are infeasible

Primal LP

$$\max\ 2x_1 - x_2$$
$$x_1 - x_2 \leq 1$$
$$-x_1 + x_2 \leq -2$$
$$x_1, x_2 \geq 0$$

Dual LP

$$\min y_1 - 2y_2$$
$$y_1 - y_2 \geq 2$$
$$-y_1 + y_2 \geq -1$$
$$y_1, y_2 \geq 0$$

To satisfy all constraints of the primal LP:

$$x_1 - x_2 \leq 1$$
$$x_1 - x_2 \geq 2$$

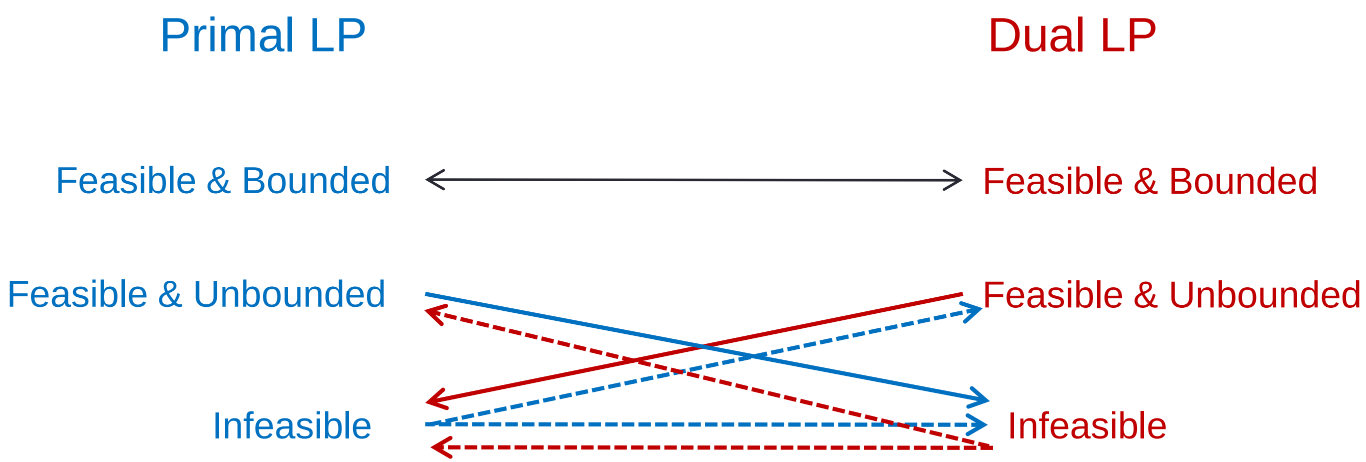
Infeasible!

To satisfy all constraints of the dual LP:

$$y_1 - y_2 \geq 2$$
$$y_1 - y_2 \leq 1$$

Infeasible!

Duality theorem in a figure



Duality Theorem (in words)

The following are the only possible relationships between the primal and the dual problems

Case 1: If primal (dual) problem has a feasible solution and a bounded objective value (and so has an optimal solution), then so does the dual (primal) problem

Case 2: If primal (dual) problem has feasible solutions but unbounded objective value (and so no optimal solution), then the dual (primal) problem has no feasible solution

Case 3: If primal (dual) problem has no feasible solutions, then the dual (primal) problem has either no feasible solutions or unbounded objective value