Plan for today

- Shadow Prices
  - Example

- Obtaining the Dual Problem

Announcements:
- HW 4 posted
- Survey in Canvas (precursor to Quiz 1)

SHADOW PRICES

… How valuable are the resources?

Tesla Example (from second lecture)

Tesla makes two models C_1 and C_2:

	C_1	C_2	total hours
Plant 1 (prepare Frame I)	1		4
Plant 2 (prepare Frame II)		2	12
Plant 3 (assembly)	3	2	18
Profit	3$	5$	

Question 1: What is the optimal product mix that maximizes the total profit?

Question 2: What is the minimum price that you should offer to buy the entire operation?

Question 3: What is the price that the company should be willing to pay for an extra working hour of plant 1, or 2, or 3?

Question 1: we know how to answer it

1. Formulate an LP

$$\max\ Z = 3x_1 + 5x_2$$
Plant 1: $x_1 \leq 4$
Plant 2: $2x_2 \leq 12$
Plant 3: $3x_1 + 2x_2 \leq 18$
$$x_1 \geq 0, x_2 \geq 0$$

2. Introduce slack variables to form the augmented LP

$$\max\ Z = 3x_1 + 5x_2$$
Plant 1: $x_1 + s_1 = 4$
Plant 2: $x_2 + s_2 = 6$
Plant 3: $3x_1 + 2x_2 + s_3 = 18$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

3. Apply SIMPLEX method to find the optimal solution

$$Z = 36 - \frac{3}{2}s_2 - s_3$$
Basic variables: $x_1^* = 2, x_2^* = 6, s_1^* = 2$
Non-basic variables: $s_2^* = s_3^* = 0$

What about questions 2 and 3: do we know the answer now?

At the optimum: $Z = 36 - \frac{3}{2}s_2 - s_3$

Basic variables: $x_1^* = 2, x_2^* = 6, s_1^* = 2$
Non-basic variables: $s_2^* = s_3^* = 0$

Constraints:

Plant 1: $x_1 \leq 4 \Rightarrow$ at optimum $x_1^* < 4$

Plant 2: $2x_2 \leq 12 \Rightarrow$ at optimum $2x_2^* = 12$

Plant 3: $3x_1 + 2x_2 \leq 18 \Rightarrow$ at optimum $3x_1^* + 2x_2^* = 18$

From the above information, can we tell:

Question 2: What is the minimum price that you should offer for buying the entire operation?

i.e., how much is the company making by not selling the operation?

Ans: 36

Question 3: What is the price that the company would be willing to pay for an additional working-hour at plant 1?

i.e., how much additional profit can be made with this additional hour?

Additional working hour at plant 1 is not of any value, because even the current optimum does not use plant 1's full capacity

How about plants 2 and 3?

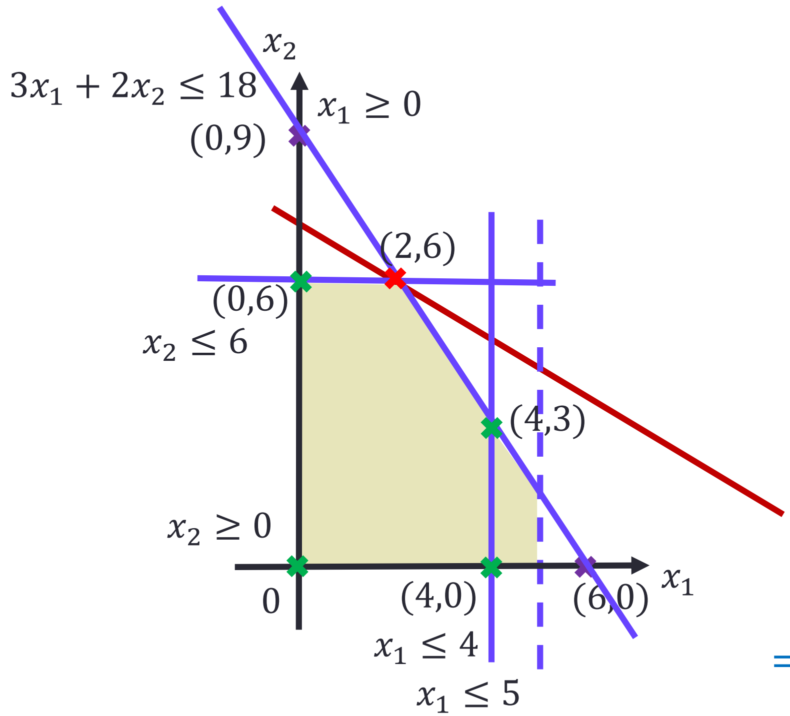
Profit increase if plant 1 has an extra working hour

$x_1^* = 2, x_2^* = 6$
$Z^* = 36$

$$\max\ Z = 3x_1 + 5x_2$$
Plant 1: $x_1 \le 4$
Plant 2: $2x_2 \le 12$
Plant 3: $3x_1 + 2x_2 \le 18$
$$x_1 \ge 0, x_2 \ge 0$$

Change in constraint: $x_1 \le 4 \longrightarrow x_1 \le 5$

for ease of understanding, we go back to the graphic method (feasible region and iso-lines)



corner point

$$2x_2 = 12$$
$$3x_1 + 2x_2 = 18$$

optimal solution

$$x_1 = 2, x_2 = 6$$
$$Z = 3x_1 + 5x_2 = 36$$

⇒optimal solution does not change

$x_1^* = 2, x_2^* = 6$
$Z^* = 36$

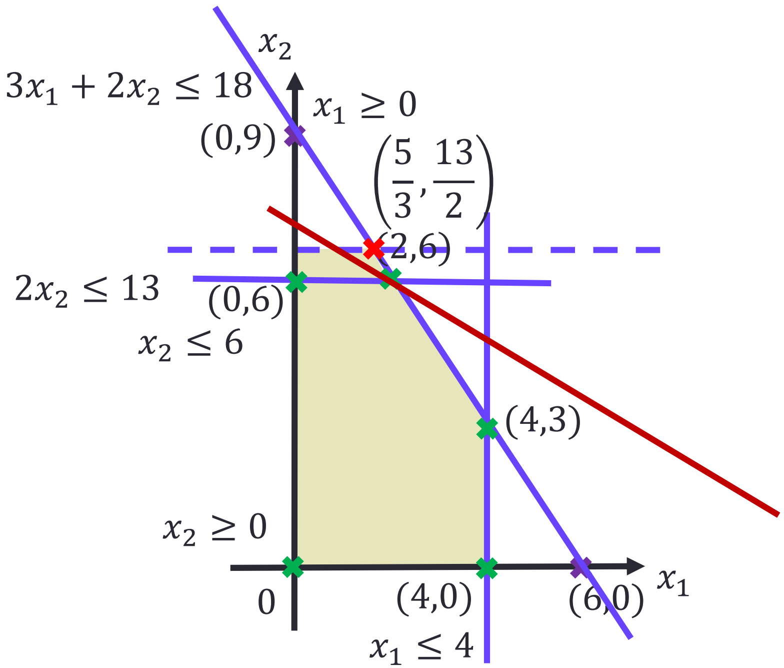
Profit increase if plant 2 has an extra working hour

$x_1^* = 2, x_2^* = 6$
$Z^* = 36$

$$\max\ Z = 3x_1 + 5x_2$$
Plant 1: $x_1 \le 4$
Plant 2: $2x_2 \le 12$
Plant 3: $3x_1 + 2x_2 \le 18$
$$x_1 \ge 0, x_2 \ge 0$$

Change in constraint: $2x_2 \le 12 \longrightarrow 2x_2 \le 13$

for ease of understanding, we go back to the graphic method (feasible region and iso-lines)



New corner point

$$2x_2 = 13$$
$$3x_1 + 2x_2 = 18$$

New optimal solution

$$x_1^* = \frac{5}{3}, x_2^* = \frac{13}{2}$$
$$Z^* = 3x_1^* + 5x_2^* = \frac{75}{2}$$

Additional capacity at plant 2 improves profit by $\$\frac{3}{2}$

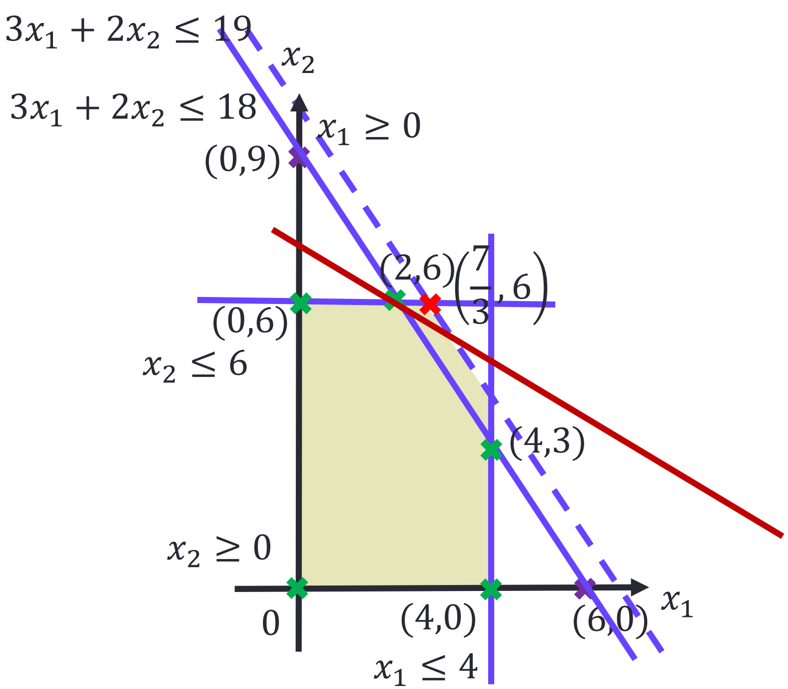
Profit increase if plant 3 has an extra working hour

$x_1^* = 2, x_2^* = 6$
$Z^* = 36$

$$\max\ Z = 3x_1 + 5x_2$$
Plant 1: $x_1 \le 4$
Plant 2: $2x_2 \le 12$
Plant 3: $3x_1 + 2x_2 \le 18$
$$x_1 \ge 0, x_2 \ge 0$$

for ease of understanding, we go back to the graphic method (feasible region and iso-lines)

Change in constraint: $3x_1 + 2x_2 \le 18 \longrightarrow 3x_1 + 2x_2 \le 19$



New corner point

$$2x_2 = 12$$
$$3x_1 + 2x_2 = 19$$

New optimal solution

$$x_1^* = \frac{7}{3}, x_2^* = 6$$
$$Z^* = 3x_1^* + 5x_2^* = 37$$

Additional capacity at plant 3 improves profit by $1

Summary and a Connection

$$\max\ Z = 3x_1 + 5x_2$$
Plant 1: $x_1 \le 4$
Plant 2: $2x_2 \le 12$
Plant 3: $3x_1 + 2x_2 \le 18$
$$x_1 \ge 0, x_2 \ge 0$$

Optimum

$$Z = 36 - \frac{3}{2}s_2 - s_3$$

Basic variables: $x_1^* = 2, x_2^* = 6, s_1^* = 2$
Non-basic variables: $s_2^* = s_3^* = 0$

Question 2: What is the minimum price that you should offer to buy the entire operation?
Answer: \$36

Question 3: How much would the company be willing to pay for an extra working hour at each plant?
Answer: \$0 for plant 1, \$$\frac{3}{2}$ for plant 2, \$1 for plant 3

$\left(0, \frac{3}{2}, 1\right)$ are the **shadow prices** of the three plants

What is the objective function in proper format at the last step of the SIMPLEX Method?

$$Z = 36 - 0s_1 - \frac{3}{2}s_2 - s_3$$

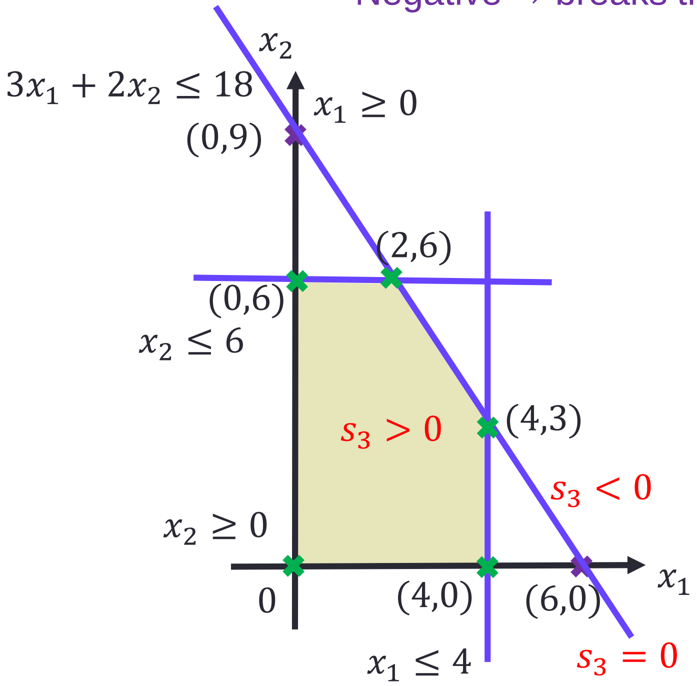
See a connection?

Shadow price of each plant is appearing as the coefficient of the corresponding slack variable in the objective (with a negative sign)

Explanation of the Connection

Recall: Slack variable associated with a constraint measures the tightness of that constraint

Slack variable being
Positive ⇒ inside the constraint
Zero ⇒ on the constraint
Negative ⇒ breaks the constraint



$$x_1 + s_1 = 4$$
$$2x_2 + s_2 = 12$$
$$3x_1 + 2x_2 + s_3 = 18$$

Explanation of the Connection (continued)

At the last SIMPLEX iteration, the objective is a decreasing function of every variable

$$Z = 36 - \frac{3}{2}s_2 - s_3$$

- to maximize the objective, these variables are kept at their minimum possible value, i.e., zero

- if that variable happens to be the slack variable,

$$\text{Plant 1: } x_1 + s_1 = 4$$
$$\text{Plant 2: } 2x_2 + s_2 = 12$$
$$\text{Plant 3: } 3x_1 + 2x_2 + s_3 = 18$$

  - increasing its value means moving the corresponding constraint outside (away from the optimum), possibly expanding the feasible region and improving the optimum

$2x_2 \leq 12 \longrightarrow 2x_2 \leq 13$

$2x_2 + s_2 = 12 \qquad 2x_2 + s_2^{new} = 13$

$$s_2^{new} = s_2 + 1 \qquad \Rightarrow s_2 = s_2^{new} - 1$$

$$Z = 36 - \frac{3}{2}s_2 - s_3 = 36 - \frac{3}{2}(s_2^{new} - 1) - s_3 = 36 + \frac{3}{2} - \frac{3}{2}s_2^{new} - s_3$$

- Thus
  - 3/2 is the rate of objective increase if working-hour constraint in plant 2 is relaxed (moved outside)

Explanation of the Connection (continued)

At the last SIMPLEX iteration, the objective is a decreasing function of every variable

$$Z = 36 - \frac{3}{2}s_2 - s_3$$

- to maximize the objective, these variables are kept at their minimum possible value, i.e., zero

- if that variable happens to be the slack variable,

$$\text{Plant 1: } x_1 + s_1 = 4$$
$$\text{Plant 2: } 2x_2 + s_2 = 12$$
$$\text{Plant 3: } 3x_1 + 2x_2 + s_3 = 18$$

  - increasing its value means moving the corresponding constraint outside (away from the optimum), possibly expanding the feasible region and improving the optimum

- Thus
  - 3/2 is the rate of objective increase if working-hour constraint in plant 2 is relaxed (moved outside)
  - 1 is the rate of objective increase if the working-hour constraint in plant 3 is relaxed (moved outside)
  - since s_1 is associated with coefficient 0, we cannot improve the objective by relaxing (moving outside) the working-hour constraint at plant 1

Explanation of the Connection (continued)

At the last SIMPLEX iteration, the objective is a decreasing function of every variable

$$Z = 36 - \frac{3}{2}s_2 - s_3$$

- to maximize the objective, these variables are kept at their minimum possible value, i.e., zero

- if that variable happens to be the slack variable,

$$\text{Plant 1: } x_1 + s_1 = 4$$
$$\text{Plant 2: } 2x_2 + s_2 = 12$$
$$\text{Plant 3: } 3x_1 + 2x_2 + s_3 = 18$$

  - decreasing its value means moving the corresponding constraint inside, possibly shrinking the feasible region and degrading the optimum

- Similarly
  - 3/2 is the rate of objective decrease if working-hour constraint in plant 2 is violated
  - 1 is the rate of objective decrease if the working-hour constraint in plant 3 is violated
  - since s_1 is associated with coefficient 0, the objective will not decrease if the working-hour constraint at plant 1 is violated

  → *that is why they are referred to as the* ***shadow prices****: price the company is willing to pay to keep the constraints where they are and not bring them inside*

SHADOW PRICES

Shadow price of a constraint
(in a maximization problem)
… is the rate of objective increase when the constraint is relaxed

Another example

	Product 1	Product 2	total available
Resource 1	1	2	6
Resource 2	1	1	5
Profit	\$1	\$5	

Question: find the optimal product mix to maximize profit

$$\max\ Z = x_1 + 5x_2$$
$$\text{Res 1: } x_1 + 2x_2 \leq 6$$
$$\text{Res 2: } x_1 + x_2 \leq 5$$
$$x_1 \geq 0, x_2 \geq 0$$

Original LP

$$\max\ Z = x_1 + 5x_2$$
$$\text{Res 1: } x_1 + 2x_2 + s_1 = 6$$
$$\text{Res 2: } x_1 + x_2 + s_2 = 5$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Augmented LP

$$Z = x_1 + 5x_2$$
$$s_1 = 6 - x_1 - 2x_2$$
$$s_2 = 5 - x_1 - x_2$$

Basic variables: $s_1 = 6, s_2 = 5$
Non-basic variables: $x_1, x_2 = 0$
$Z = 0$

Entering variable: x_2
Leaving variable: s_1

$$Z = 15 - \frac{3}{2}x_1 - \frac{5}{2}s_1$$
$$x_2 = 3 - \frac{1}{2}x_1 - \frac{1}{2}s_1$$
$$s_2 = 2 - \frac{1}{2}x_1 + \frac{1}{2}s_1$$

Optimum

Basic variables: $x_2 = 3, s_2 = 2$
Non-basic variables: $x_1, s_1 = 0$
$Z = 15$

Interpretation of results

$$\max\ Z = x_1 + 5x_2$$
Res 1: $x_1 + 2x_2 \le 6$
Res 2: $x_1 + x_2 \le 5$
$x_1 \ge 0, x_2 \ge 0$

→

$$\max\ Z = x_1 + 5x_2$$
Res 1: $x_1 + 2x_2 + s_1 = 6$
Res 2: $x_1 + x_2 + s_2 = 5$
$x_1, x_2, s_1, s_2 \ge 0$

→

$$Z = 15 - \frac{3}{2}x_1 - \frac{5}{2}s_1$$
$$x_2 = 3 - \frac{1}{2}x_1 - \frac{1}{2}s_1$$
$$s_2 = 2 - \frac{1}{2}x_1 + \frac{1}{2}s_1$$

Optimum

Basic variables: $x_2^* = 3, s_2^* = 2$
Non-basic variables: $x_1^*, s_1^* = 0$
$Z^* = 15$

- It is optimal to build **3** batches of Product 2 but no Product 1 and get a profit of **$15**
- The shadow price of Resource 1 is **$5/2**
  Willing to pay **$5/2** for an additional batch of Part 1
- The shadow price of Resource 2 is **$0**
  The supply (i.e., **5**) is already excessive for the optimal solution, hence not willing to pay for more

Essentially, every resource has a price tag
E.g., **$5/2** is the price tag for which resource?

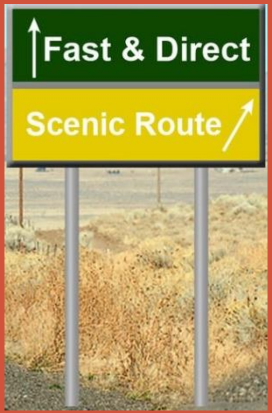
Observation: if the price tag of a resource (constraint) is non-zero, then the corresponding slack variable must be **0** at optimum

SHADOW PRICES

Shadow price of a constraint
(in a maximization problem)
… is the rate of objective increase when the constraint is relaxed

SHADOW PRICES

Shadow price of a constraint
(in a minimization problem)
… is the rate of objective decrease when the constraint is relaxed



DIRECT ANSWERS TO Q2 AND Q3

… where we formulate an LP whose solution gives the shadow prices

Direct answers to Q2 and Q3

Tesla makes two models C_1 and C_2:

	C_1	C_2	total hours
Plant 1 (prepare Frame I)	1		4
Plant 2 (prepare Frame II)		2	12
Plant 3 (assembly)	3	2	18
Profit	3$	5$	

Question 2: What is the minimum price that you should offer to buy the entire operation?

Question 3: What is the price/worth of each working hour at plant 1, or 2, or 3?

Decision variables:

y_j be the price/worth of *each* working hour at plant *j for* $j = 1,2,3$

Objective: minimize the total price paid/worth

Worth of entire capacity at Plant 1 $= 4y_1$

Worth of entire capacity at Plant 2 $= 12y_2$

Worth of entire capacity at Plant 3 $= 18y_3$

Total worth of entire operation is $4y_1 + 12y_2 + 18y_3$

$$\min 4y_1 + 12\,y_2 + 18y_3$$

Constraints: (Combination worth has to be at least the profit)

The total worth of one hour at plant 1 and three hours at plant 3 is at least 3$

The total worth of two hours at plant 2 and two hours at plant 3 is at least 5$

$$y_1 + 3y_3 \geq 3$$

$$2y_2 + 2y_3 \geq 5$$

$$y_1, y_2, y_3 \geq 0$$

Compare the two LPs, what do you observe

Q1: what is the optimal product mix that maximizes the total profit?

Primal LP

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0, x_2 \ge 0$$

LP1

Q2: what is the minimum price that you should offer to buy these three plants?

Dual LP

$$\min\ 4y_1 + 12\,y_2 + 18y_3$$
$$y_1 + 3y_3 \ge 3$$
$$2y_2 + 2y_3 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

LP2

- number of variables in LP1 versus number of constraints (excluding non-negativity constraints) in LP2
- constants on the RHS of constraints in LP1 versus coefficients in the objective function of LP2
- col coefficients in the LHS of constraints of LP1 versus row coefficients in the LHS of constraints of LP2

x_1	x_2
1	0
0	2
3	2

y_1	y_2	y_3
1	0	3
0	2	2

If we call LP1 as the **Primal** LP, then LP2 is known as the **Dual** of LP1

Obtaining the **dual** problem of an LP

… where the LP is a maximization LP

Step 1. Associate each constraint (excluding non-negativity constraints) with a variable

Primal LP

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \leq 4 \quad \rightarrow \quad y_1$$
$$2x_2 \leq 12 \quad \rightarrow \quad y_2$$
$$3x_1 + 2x_2 \leq 18 \quad \rightarrow \quad y_3$$
$$x_1 \geq 0, x_2 \geq 0$$

Step 2. Use the RHS of constraints to define the objective function, with the goal being min

$$\begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} \Longrightarrow \min 4y_1 + 12y_2 + 18y_3$$

Obtaining the **dual** problem of an LP

… where the LP is a maximization LP

Step 3. Define constraints (one constraint for each x variable)

1. coefficients of the variable give coefficients of the constraint on LHS
2. coefficient of the variable in the objective function gives the RHS constant
3. Constraint:
   If $x_i \geq 0$ then the constraint is $\geq$
   If $x_i \leq 0$ then the constraint is $\leq$
   If x_i is unrestricted, then the constraint is $=$

Step 4. Add (non)-negativity conditions

1. If $a_1x_1 + \cdots \leq c$ → y_j then, $y_j \geq 0$
2. If $a_1x_1 + \cdots \geq c$ → y_j then, $y_j \leq 0$
3. If $a_1x_1 + \cdots = c$ → y_j then, y_j is unrestricted

Primal LP

$$\max Z = 3x_1 + 5x_2$$
$$x_1 \leq 4 \rightarrow y_1$$
$$2x_2 \leq 12 \rightarrow y_2$$
$$3x_1 + 2x_2 \leq 18 \rightarrow y_3$$
$$x_1 \geq 0, x_2 \geq 0$$

$$x_1: \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \Rightarrow y_1 + 3y_3 \geq 3$$

$$x_2: \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \Rightarrow 2y_2 + 2y_3 \geq 5$$

Dual LP

$$\min 4y_1 + 12\,y_2 + 18y_3$$
$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$
$$y_1, y_2, y_3 \geq 0$$