Plan for today

- Simplex Method to solve LPs
  - Complete Algorithm
  - Initialization
  - Iterations
  - Termination
  - Other Issues

Announcements:

- HW 3 posted in Gradescope
- Instructor office hours: Zoom
- TA office hours: In-person

SIMPLEX METHOD

How to make selection easier: **proper format**

$$\max\ Z = 3x_1 + 5x_2$$

s. t.:

$$\begin{aligned} x_1 \quad\quad + x_3 \quad\quad &= 4 \\ x_2 \quad + x_4 \quad &= 6 \\ 3x_1 + 2x_2 \quad\quad + x_5 &= 18 \end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

- Initialization: basic variables (x_3, x_4, x_5), non-basic variables $x_1 = 0, x_2 = 0$
- Write objective function in terms of only non-basic variables,

$$Z = 3x_1 + 5x_2$$

… helps us select the entering variable by a simple comparison of coefficients

Entering variable: x_1 or x_2

3 vs 5

Choose x_2

- Rewrite the equality constraints: Each equality constraint should contain
  - one basic variable on the LHS with coefficient 1,
  - constants and non-basic variables on the RHS and
  - the RHS constant should be non-negative

$$x_3 = 4 - x_1$$
$$x_4 = 6 - x_2$$
$$x_5 = 18 - 3x_1 - 2x_2$$

… helps us select the leaving variable by a simple comparison of the ratios between the RHS constant and the entering variable's coefficient in each equation

Leaving variable: x_3 or x_4 or x_5

– vs $\frac{6}{1}$ vs $\frac{18}{2}$

Choose x_4

- This **proper format** of objective and equality constraints should be maintained in each step to facilitate the selection of entering and leaving variables

After determining which variables to swap…

$$Z = 3x_1 + 5x_2$$
$$x_3 = 4 - x_1$$
$$x_4 = 6 - x_2$$
$$x_5 = 18 - 3x_1 - 2x_2$$

Non-basic variables: x_1, x_2
Basic variables:
$x_3 = 4, x_4 = 6, x_5 = 18$

- x_2 is the entering variable, x_4 is the leaving variable
- Step 1: express the entering basic variable as a function of non-basic variables and the leaving variable (use the equality constraint for the leaving basic variable to do this)

$$x_2 = 6 - x_4$$

- Step 2: use the expression to substitute the entering basic variable in the objective function

$$Z = 3x_1 + 5x_2 = 30 + 3x_1 - 5x_4$$

- Step 3: substitute the entering basic variable in the remaining constraints

$$x_3 = 4 - x_1$$
$$x_5 = 18 - 3x_1 - 2x_2 = 6 - 3x_1 + 2x_4$$

- Now (x_2, x_3, x_5) are basic variables and (x_1, x_4) are non-basic variables
  - The objective function contains only non-basic variables
  - Each equality constraint contains:
    - one basic variable on the LHS with coefficient 1
    - constants and non-basic variables on the RHS and
    - the RHS constant is non-negative

$$Z = 30 + 3x_1 - 5x_4$$
$$x_3 = 4 - x_1$$
$$x_2 = 6 - x_4$$
$$x_5 = 6 - 3x_1 + 2x_4$$

Corresponds to the solution:
$x_1 = 0, x_2 = 6, x_3 = 4,$
$x_4 = 0, x_5 = 6,$
$Z = 30$
which is the corner point $x_1 = 0, x_2 = 6$

Next iteration……

$$Z = 30 + 3x_1 - 5x_4$$

$$x_3 = 4 - x_1$$
$$x_2 = 6 - x_4$$
$$x_5 = 6 - 3x_1 + 2x_4$$

Non-basic variables: x_1, x_4
Basic variables:
$x_3 = 4, x_2 = 6, x_5 = 6$

- which variable should be the entering variable, which one should be the leaving variable?
  x_1 enters
  x_5 leaves
- expression for the entering basic variable
  $$x_1 = 2 + \frac{2}{3}x_4 - \frac{1}{3}x_5$$
- Substitute in the objective function
  $$Z = 30 + 3x_1 - 5x_4 = 36 - 3x_4 - x_5$$
- Substitute in all equality constraints
  $$x_1 = 2 + \frac{2}{3}x_4 - \frac{1}{3}x_5$$
  $$x_3 = 4 - x_1 = 2 - \frac{2}{3}x_4 + \frac{1}{3}x_5$$
  $$x_2 = 6 - x_4$$

→

$$Z = 36 - 3x_4 - x_5$$
$$x_1 = 2 + \frac{2}{3}x_4 - \frac{1}{3}x_5$$
$$x_3 = 2 - \frac{2}{3}x_4 + \frac{1}{3}x_5$$
$$x_2 = 6 - x_4$$

- Next iteration? STOP since no entering variable to improve obj value
- So the optimal solution is

$$x_1^* = 2, x_2^* = 6, x_3^* = 2$$
$$Z = 36$$

Non-basic variables: x_4, x_5
Basic variables: x_1, x_2, x_3
Corresponds to the solution:
$x_1 = 2, x_2 = 6, x_3 = 2, x_4 = 0,$
$x_5 = 0$

which is the corner point $x_1 = 2, x_2 = 6$

Simplex Method: the complete algorithm

- **Initialization**
  - transform the original LP into the augmented LP, determine basic and non-basic variables
  - Rewrite constraints in proper format:
    - one basic variable on the LHS with coefficient 1,
    - constants and non-basic variables on the RHS and
    - RHS constant should be non-negative
  - Rewrite objective function in proper format: contains only non-basic variables

- **Iteration**
  1. select the entering variable: the variable with the largest positive coefficient in the obj. function
  2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
  3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
  4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

- **Termination: Optimality Test**
  - if every variable in the objective function is with a negative coefficient, then no entering variable can be found ⇒ **the optimal solution has been found**
  - if the entering variable can be increased to infinity without driving any other basic variable to below zero, then no leaving variable can be found ⇒ **the problem is unbounded**

SIMPLEX METHOD

initialization

To begin simplex, which variables should be set to basic variables?

Simplex Method: initialization

(easy case, example)

Constraints in proper format:

- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

Obj function in proper format:

- contains only non-basic variables

original LP

$$\begin{aligned} &\max Z = 7x_1 + 6x_2 \\ &9x_1 - 2x_2 \leq 12 \\ &3x_1 + x_2 \leq 30 \\ &x_1, x_2 \geq 0 \end{aligned}$$

augmented LP

$$\begin{aligned} &\max Z = 7x_1 + 6x_2 \\ &9x_1 - 2x_2 + x_3 = 12 \\ &3x_1 + x_2 + x_4 = 30 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

If every equality has a non-negative RHS

- all slack variables are initially basic variables
- all other (original) variables are non-basic variables and are moved to the right

Initialization: (x_3, x_4 basic variables, x_1, x_2 non-basic variables)

$$\begin{aligned} &Z = 7x_1 + 6x_2 \\ &x_3 = 12 - 9x_1 + 2x_2 \\ &x_4 = 30 - 3x_1 - x_2 \end{aligned}$$

initial solution:
Non-basic variables: $x_1 = 0, x_2 = 0$,
Basic variables: $x_3 = 12, x_4 = 30$

Simplex Method: initialization

complicated case example: step 1

Constraints in proper format:

- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

Obj function in proper format:

- contains only non-basic variables

original LP

$$\begin{aligned} \max Z &= 7x_1 + 6x_2 \\ 9x_1 - 2x_2 &\le -12 \\ 3x_1 + x_2 &\le 30 \\ x_1, x_2 &\ge 0 \end{aligned}$$

augmented LP

$$\begin{aligned} \max Z &= 7x_1 + 6x_2 \\ 9x_1 - 2x_2 + x_3 \qquad &= -12 \\ 3x_1 + x_2 \qquad + x_4 &= 30 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

$$\begin{aligned} Z &= 7x_1 + 6x_2 \\ x_3 &= -12 - 9x_1 + 2x_2 \\ x_4 &= 30 - 3x_1 - x_2 \end{aligned}$$

If the RHS of an equation is negative

Not proper format!

Cannot initialize slack variables as basic variables and original variables as non-basic variables!

step (i): multiply both sides of such an equation in the augmented LP by -1 to make the RHS positive

augmented LP

$$\begin{aligned} \max Z &= 7x_1 + 6x_2 \\ 9x_1 - 2x_2 + x_3 \qquad &= -12 \\ 3x_1 + x_2 \qquad + x_4 &= 30 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

step (i) →

$$\begin{aligned} \max Z &= 7x_1 + 6x_2 \\ -9x_1 + 2x_2 - x_3 \qquad &= 12 \\ 3x_1 + x_2 \qquad + x_4 &= 30 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

$(P1)$

Simplex Method: initialization

complicated case example: step 2

Constraints in proper format:

- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

Obj function in proper format:

- contains only non-basic variables

step (ii): add another slack variable (say x_5) to the same equation

$$\begin{aligned}
&\max Z = 7x_1 + 6x_2 \\
&-9x_1 + 2x_2 - x_3 = 12 \\
&3x_1 + x_2 + x_4 = 30 \\
&x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

$(P1)$

step (ii) →

$$\begin{aligned}
&\max Z = 7x_1 + 6x_2 \\
&-9x_1 + 2x_2 - x_3 + x_5 = 12 \\
&3x_1 + x_2 + x_4 = 30 \\
&x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

$(P2)$

- but (P2) is a different problem from (P1), since it has a larger feasibility region
  - e.g., $x_1 = 1, x_2 = 6$ is feasible for (P2), in which case $x_3 = 0, x_4 = 21, x_5 = 9$, but not feasible for (P1)

- however, any solution of (P2) given by $(x_1, x_2, x_3, x_4, x_5)$ is feasible for (P1) if $x_5 = 0$

Simplex Method: initialization

complicated case example: step 3

Constraints in proper format:

- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

Obj function in proper format:

- contains only non-basic variables

step (iii): change the objective function to force $x_5 = 0$ in the optimal solution

$$\begin{aligned} &\max Z = 7x_1 + 6x_2 \\ &-9x_1 + 2x_2 - x_3 = 12 \\ &3x_1 + x_2 + x_4 = 30 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned} \quad (P1)$$

$$\begin{aligned} &\max Z = 7x_1 + 6x_2 \\ &-9x_1 + 2x_2 - x_3 + x_5 = 12 \\ &3x_1 + x_2 + x_4 = 30 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned} \quad (P2)$$

step (iii)

$$\begin{aligned} &\max Z = 7x_1 + 6x_2 - Mx_5 \\ &-9x_1 + 2x_2 - x_3 + x_5 = 12 \\ &3x_1 + x_2 + x_4 = 30 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned} \quad (P3)$$

- Choose a very large M, so that an optimal solution for (P3) can never have $x_5 > 0$

- Finding an optimal solution for (P1) is the same as finding an optimal solution for (P3) if we use a very large M

… The Big-M method

Simplex Method: initialization

complicated case example: step 4

Constraints in proper format:

- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

Obj function in proper format:

- contains only non-basic variables

$$\begin{aligned} &\max Z = 7x_1 + 6x_2 \\ &-9x_1 + 2x_2 - x_3 = 12 \\ &3x_1 + x_2 + x_4 = 30 \\ &x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

(P1)

$$\begin{aligned} &\max Z = 7x_1 + 6x_2 - Mx_5 \\ &-9x_1 + 2x_2 - x_3 + x_5 = 12 \\ &3x_1 + x_2 + x_4 = 30 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

(P3)

Initialization of (P3) is easy: we can start with x_4, x_5 as basic variables, and x_1, x_2, x_3 as non-basic variables

$$x_5 = 12 + 9x_1 - 2x_2 + x_3$$
$$x_4 = 30 - 3x_1 - x_2$$

initial solution:
Non-basic variables: $x_1 = 0, x_2 = 0, x_3 = 0$
Basic variables: $x_4 = 30, x_5 = 12$

Are we done with the initialization? Not yet, need to write obj in proper format

step (iv): substitute out the basic variable in the objective function

$$\begin{aligned} Z &= 7x_1 + 6x_2 - Mx_5 \\ &= 7x_1 + 6x_2 - M(12 + 9x_1 - 2x_2 + x_3) \\ &= (7 - 9M)x_1 + (6 + 2M)x_2 - Mx_3 - 12M \end{aligned}$$

Now, we are done with the initialization!

$$Z = (7 - 9M)x_1 + (6 + 2M)x_2 - Mx_3 - 12M$$
$$x_5 = 12 + 9x_1 - 2x_2 + x_3$$
$$x_4 = 30 - 3x_1 - x_2$$

initial solution:
Non-basic variables: $x_1 = 0, x_2 = 0, x_3 = 0$
Basic variables: $x_4 = 30, x_5 = 12$

SIMPLEX METHOD

iterations

$$Z = (7 - 9M)x_1 + (6 + 2M)x_2 - Mx_3 - 12M$$

$$x_5 = 12 + 9x_1 - 2x_2 + x_3$$
$$x_4 = 30 - 3x_1 - x_2$$

initial solution:
Non-basic variables: $x_1 = 0, x_2 = 0, x_3 = 0$
Basic variables: $x_4 = 30, x_5 = 12$

Simplex Method: Iteration steps 1 & 2

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

$$Z = (7 - 9M)x_1 + (6 + 2M)x_2 - Mx_3 - 12M$$

$$x_5 = 12 + 9x_1 - 2x_2 + x_3$$
$$x_4 = 30 - 3x_1 - x_2$$

Non-basic vars: $x_1 = 0, x_2 = 0, x_3 = 0$

Basic vars: $x_4 = 30, x_5 = 12$

- select the entering variable: x_2
- select leaving variable by **min ratio test**:

$$x_5 = 12 + 9x_1 - 2x_2 + x_3$$
$$x_4 = 30 - 3x_1 - x_2$$

$\Rightarrow$ x_5 is the leaving variable $\Leftarrow$

x_5 vs x_4

$\frac{12}{2}$ vs $\frac{30}{1}$

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$x_{n+1} = b_1 + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$x_{n+2} = b_2 + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\vdots$$
$$x_{n+m} = b_m + a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

Min-ratio test (for entering variable x_2):

for all $i = 1, \dots, m$ such that $a_{i2} < 0$, calculate $\frac{b_i}{|a_{i2}|}$

choose the equation i that gives the smallest $\frac{b_i}{|a_{i2}|}$

x_{n+i} is the leaving variable

Simplex Method: Iteration steps 3 & 4

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

$$Z = (7-9M)x_1 + (6+2M)x_2 - Mx_3 - 12M$$

$$x_5 = 12 + 9x_1 - 2x_2 + x_3$$
$$x_4 = 30 - 3x_1 - x_2$$

Non-basic vars: $x_1 = 0, x_2 = 0, x_3 = 0$

Basic vars: $x_4 = 30, x_5 = 12$

- now that x_2 is the entering variable and x_5 is the leaving variable
- express the entering variable as a function of non-basic variables

$$x_2 = 6 - \frac{1}{2}x_5 + \frac{9}{2}x_1 + \frac{1}{2}x_3$$

- substitute the entering basic variable in the objective function

$$\begin{aligned}(6+2M)x_2 &= 36 + 12M - (3+M)x_5 + (27+9M)x_1 + (3+M)x_3 \\ &\quad -12M \qquad\qquad\qquad + (7-9M)x_1 \quad - Mx_3 \\ \hline Z &= 36 - (3+M)x_5 + 34x_1 + 3x_3\end{aligned}$$

- substitute the entering basic variable in other equations

$$\begin{aligned}x_4 &= 30 - 3x_1 - x_2 \\ &= 30 - 3x_1 - \left(6 - \frac{1}{2}x_5 + \frac{9}{2}x_1 + \frac{1}{2}x_3\right) \\ &= 24 + \frac{1}{2}x_5 - \frac{15}{2}x_1 - \frac{1}{2}x_3\end{aligned}$$

Simplex Method: after the first iteration

$$Z = 36 - (3+M)x_5 + 34x_1 + 3x_3$$
$$x_2 = 6 - \frac{1}{2}x_5 + \frac{9}{2}x_1 + \frac{1}{2}x_3$$
$$x_4 = 24 + \frac{1}{2}x_5 - \frac{15}{2}x_1 - \frac{1}{2}x_3$$

Non-basic variables: $x_1 = x_3 = x_5 = 0$
Basic variables: $x_2 = 6, x_4 = 24$
$Z = 36$

- the problem is still in the same format as the one required for initialization

Constraints in proper format:

- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

Obj function in proper format:

- contains only non-basic variables

- so we can continue to the second iteration in the same manner

Simplex Method: Iteration, steps 1 & 2

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

$$Z = 36 + 34x_1 + 3x_3 - (3+M)x_5$$
$$x_2 = 6 + \frac{9}{2}x_1 + \frac{1}{2}x_3 - \frac{1}{2}x_5$$
$$x_4 = 24 - \frac{15}{2}x_1 - \frac{1}{2}x_3 + \frac{1}{2}x_5$$

Basic variables: $x_2 = 6, x_4 = 24$
Non-basic variables: $x_1 = x_3 = x_5 = 0$
$Z = 36$

- select the entering variable: x_1
- select leaving variable:

x_2 vs x_4
$-$ vs $\frac{24}{15/2}$

$\Rightarrow$ x_4 is the leaving variable

$$\begin{aligned} Z &= c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ x_{n+1} &= b_1 + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_{n+2} &= b_2 + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_{n+m} &= b_m + a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

Min-ratio test (for entering variable x_1):
for all $i = 1, \ldots, m$ such that $a_{i1} < 0$, calculate $\frac{b_i}{|a_{i1}|}$
choose the equation i that gives the smallest $\frac{b_i}{|a_{i1}|}$
x_{n+i} is the leaving variable

Simplex Method: Iteration, steps 3 & 4

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

$$Z = 36 + 34x_1 + 3x_3 - (3+M)x_5$$
$$x_2 = 6 + \frac{9}{2}x_1 + \frac{1}{2}x_3 - \frac{1}{2}x_5$$
$$x_4 = 24 - \frac{15}{2}x_1 - \frac{1}{2}x_3 + \frac{1}{2}x_5$$

Basic variables: $x_2 = 6, x_4 = 24$
Non-basic variables: $x_1 = x_3 = x_5 = 0$
$Z = 36$

- now that x_1 is the entering variable and x_4 is the leaving variable
- express the entering variable as a function of non-basic variables

$$x_1 = \frac{16}{5} - \frac{1}{15}x_3 - \frac{2}{15}x_4 + \frac{1}{15}x_5$$

- substitute the entering basic variable in the objective function

$$\begin{aligned} Z &= 36 + 34x_1 + 3x_3 - (3+M)x_5 \\ &= 36 + 34\left(\frac{16}{5} + \frac{1}{15}x_5 - \frac{1}{15}x_3 - \frac{2}{15}x_4\right) + 3x_3 - (3+M)x_5 \\ &= \frac{724}{5} + \frac{11}{15}x_3 - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5 \end{aligned}$$

- substitute the entering basic variable in other equations

$$\begin{aligned} x_2 = 6 + \frac{9}{2}x_1 + \frac{1}{2}x_3 - \frac{1}{2}x_5 &= 6 + \frac{9}{2}\left(\frac{16}{5} + \frac{1}{15}x_5 - \frac{1}{15}x_3 - \frac{2}{15}x_4\right) + \frac{1}{2}x_3 - \frac{1}{2}x_5 \\ &= \frac{102}{5} + \frac{1}{5}x_3 - \frac{3}{5}x_4 - \frac{1}{5}x_5 \end{aligned}$$

Simplex Method: after the second iteration

$$Z = \frac{724}{5} + \frac{11}{15}x_3 - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5$$
$$x_2 = \frac{102}{5} + \frac{1}{5}x_3 - \frac{3}{5}x_4 - \frac{1}{5}x_5$$
$$x_1 = \frac{16}{5} - \frac{1}{15}x_3 - \frac{2}{15}x_4 + \frac{1}{15}x_5$$

Basic variables: $x_2 = \frac{102}{5}, x_1 = \frac{16}{5}$

Non-basic variables: $x_3 = x_4 = x_5 = 0$

$$Z = \frac{724}{5}$$

- the constraints and objective are still in proper format

Constraints in proper format:

- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

Obj function in proper format:

- contains only non-basic variables

- so we can continue the third iteration in the same manner

Simplex Method: Iteration, steps 1 & 2

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

$$Z = \frac{724}{5} + \frac{11}{15}x_3 - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5$$
$$x_2 = \frac{102}{5} + \frac{1}{5}x_3 - \frac{3}{5}x_4 - \frac{1}{5}x_5$$
$$x_1 = \frac{16}{5} - \frac{1}{15}x_3 - \frac{2}{15}x_4 + \frac{1}{15}x_5$$

- select the entering variable: x_3
- select leaving variable:

x_2 vs x_1
$-$ vs $\frac{16/5}{1/15}$

$\Rightarrow$ x_1 is the leaving variable

Basic variables: $x_2 = \frac{102}{5}, x_1 = \frac{16}{5}$

Non-basic variables: $x_3 = x_4 = x_5 = 0$

$$Z = \frac{724}{5}$$

$$\begin{aligned} Z &= c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ x_{n+1} &= b_1 + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_{n+2} &= b_2 + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_{n+m} &= b_m + a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

Min-ratio test (for entering variable x_3):

for all $i = 1, \dots, m$ such that $a_{i3} < 0$, calculate $\frac{b_i}{|a_{i3}|}$

choose the equation i that gives the smallest $\frac{b_i}{|a_{i3}|}$

x_{n+i} is the leaving variable

Simplex Method: Iteration, steps 3 & 4

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

$$Z = \frac{724}{5} + \frac{11}{15}x_3 - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5$$
$$x_2 = \frac{102}{5} + \frac{1}{5}x_3 - \frac{3}{5}x_4 - \frac{1}{5}x_5$$
$$x_1 = \frac{16}{5} - \frac{1}{15}x_3 - \frac{2}{15}x_4 + \frac{1}{15}x_5$$

Basic variables: $x_2 = \frac{102}{5}, x_1 = \frac{16}{5}$

Non-basic variables: $x_3 = x_4 = x_5 = 0$

$$Z = \frac{724}{5}$$

- now that x_3 is the entering variable and x_1 is the leaving variable
- express the entering variable as a function of non-basic variables

$$x_3 = 48 - 15x_1 - 2x_4 + x_5$$

- substitute the entering basic variable in the objective function

$$Z = \frac{724}{5} + \frac{11}{15}x_3 - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5$$
$$= \frac{724}{5} + \frac{11}{15}(48 - 15x_1 - 2x_4 + x_5) - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5$$
$$= 180 - 11x_1 - \frac{46}{15}x_4 - Mx_5$$

- substitute the entering basic variable in other equations

$$x_2 = \frac{102}{5} + \frac{1}{5}x_3 - \frac{3}{5}x_4 - \frac{1}{5}x_5 = \frac{102}{5} + \frac{1}{5}(48 - 2x_4 - 15x_1 + x_5) - \frac{3}{5}x_4 - \frac{1}{5}x_5$$
$$= 30 - 3x_1 - x_4$$

Simplex Method: after the third iteration

$$Z = 180 - 11x_1 - \frac{46}{15}x_4 - Mx_5$$
$$x_2 = 30 - 3x_1 - x_4$$
$$x_3 = 48 - 15x_1 - 2x_4 + x_5$$

Basic variables: $x_2 = 30, x_3 = 48$
Non-basic variables: $x_1 = x_4 = x_5 = 0$
$Z = 180$

- the constraints and objective are still in proper format

Constraints in proper format:

- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

Obj function in proper format:

- contains only non-basic variables

- so we can continue the fourth iteration in the same manner
- But no more entering variable, so terminate

Optimum solution is
$x_1^* = 0, x_2^* = 30, x_3^* = 48, x_4^* = 0, x_5^* = 0$
$Z^* = 180$

SIMPLEX METHOD

termination

Termination: Optimality Test

- no entering variable can be found

  all coefficients (except the constant) in the objective function are negative

- there is an entering variable, but no leaving variable
  - In this case, the LP is **unbounded**

original LP

$$\max Z = x_1 + 2x_2$$
$$3x_1 - 2x_2 \leq 5$$
$$x_1 - x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

augmented LP

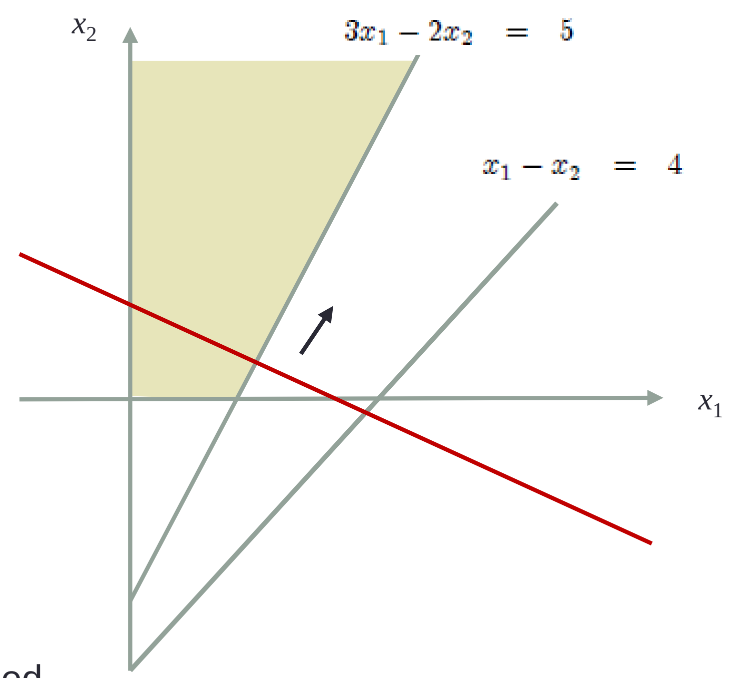
$$\max Z = x_1 + 2x_2$$
$$3x_1 - 2x_2 + x_3 = 5$$
$$x_1 - x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Initialization

$$Z = x_1 + 2x_2$$
$$x_3 = 5 - 3x_1 + 2x_2$$
$$x_4 = 4 - x_1 + x_2$$

Basic variables: $x_3 = 5, x_4 = 4$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

Entering variable: x_2
Leaving variable: None ⇒ Unbounded



OTHER ISSUES

… that could happen while executing the Simplex method

Initialization

Original LP

$$\max Z = -11x_1 + 9x_2$$
$$2x_1 + 8x_2 \leq 5$$
$$-x_1 + x_2 \leq 0$$
$$x_1, x_2 \geq 0$$

Augmented LP

$$\max Z = -11x_1 + 9x_2$$
$$2x_1 + 8x_2 + x_3 = 5$$
$$-x_1 + x_2 + x_4 = 0$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Initialization

$$Z = -11x_1 + 9x_2$$
$$x_3 = 5 - 2x_1 - 8x_2$$
$$x_4 = 0 + x_1 - x_2$$

Basic variables: $x_3 = 5, x_4 = 0$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

Iteration

Iteration 1:

$$Z = -11x_1 + 9x_2$$
$$x_3 = 5 - 2x_1 - 8x_2$$
$$x_4 = 0 + x_1 - x_2$$

Basic variables: $x_3 = 5, x_4 = 0$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

Entering variable: x_2
Leaving variable: x_4

$$x_2 = 0 + x_1 - x_4$$
$$x_3 = 5 - 2x_1 - 8(0 + x_1 - x_4)$$
$$= 5 - 10x_1 + 8x_4$$

$$Z = -11x_1 + 9(0 + x_1 - x_4) = -2x_1 - 9x_4$$

$$Z = -2x_1 - 9x_4$$
$$x_2 = 0 + x_1 - x_4$$
$$x_3 = 5 - 10x_1 + 8x_4$$

Basic variables: $x_2 = 0, x_3 = 5$
Non-basic variables: $x_1 = x_4 = 0$
$Z = 0$

No more entering variable, so terminate

Optimum solution is

$$x_1^* = 0, x_2^* = 0, x_3^* = 5, x_4^* = 0$$
$$Z^* = 0$$

Degeneracy

- If the RHS of the leaving variable is zero, then
  - The entering variable will also take the same RHS value of zero in the new basis
  - The objective value will not increase in the new basis
- We call this situation **degeneracy**
- In simple terms, degeneracy happens when one or more basic variables take a value of 0
- Note that objective stays the same even though we have an entering variable and a leaving variable

$$Z = -11x_1 + 9x_2$$
$$x_3 = 5 - 2x_1 - 8x_2$$
$$x_4 = 0 + x_1 - x_2$$

Basic variables: $x_3 = 5, x_4 = 0$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

$$Z = -2x_1 - 9x_4$$
$$x_2 = 0 + x_1 - x_4$$
$$x_3 = 5 - 10x_1 + 8x_4$$

Basic variables: $x_2 = 0, x_3 = 5$
Non-basic variables: $x_1 = x_4 = 0$
$Z = 0$

Degeneracy is bad

It can cause the simplex method to cycle around the same set of basic feasible solutions without improving the objective value and without terminating

Bland

Bland's rule for entering and leaving variables

- Entering variable:
  - Among the choices (non-basic variables with positive coefficient in the objective function Z expressed in proper format)
    - pick the one with the smallest index
      i.e., x_j with smallest index j
- Leaving variable:
  - Among the choices (basic variables whose equation has a negative coefficient for the entering variable)
    - Pick the one with with the least ratio between RHS constant and absolute value of the coefficient of the entering variable
    - In case of tie, pick the one with the smallest index,
      i.e., x_j with smallest index j

With this choice of entering and leaving variable, it can be provably shown that the Simplex method does not cycle and hence, will terminate in finite number of iterations

OTHER ISSUES

… that could happen while executing the Simplex method

Iteration step maintains positive RHS

after each iteration, the constants on the RHS must be positive

Let us consider an example:

Situation:

$$
\begin{aligned}
x_3 &= 1 - 6x_1 - 4x_2 + 8x_5 - 7x_6 \\
x_4 &= 12 - 9x_1 - 3x_2 + 8x_6 \\
x_7 &= 100 - 3x_2 - 2x_5 \\
x_8 &= 19 - 7x_1 - 6x_5 + 8x_6
\end{aligned}
$$

x_3, x_4, x_7, x_8 are basic variables
x_1, x_2, x_5, x_6 are non-basic variables
Suppose x_5 is the entering variable

- use the min ratio test to select the leaving variable

RHS	Coefficient of x_5	$\frac{RHS}{\lvert coefficient \rvert}$
1	8	$-$
12	0	$-$
100	-2	$\frac{100}{2} = 50$
19	-6	$\frac{19}{6}$

Iteration step maintains positive RHS

If not, then error in calculations

$$x_3 = 1 - 6x_1 - 4x_2 + 8x_5 - 7x_6$$
$$x_4 = 12 - 9x_1 - 3x_2 + 8x_6$$
$$x_7 = 100 - 3x_2 - 2x_5$$
$$x_8 = 19 - 7x_1 - 6x_5 + 8x_6$$

Situation: x_3, x_4, x_7, x_8 are basic variables
x_1, x_2, x_5, x_6 are non-basic variables
Suppose x_5 is the entering variable

RHS	Coefficient of x_5	$\frac{RHS}{\lvert coefficient \rvert}$
1	8	–
12	0	–
100	-2	$\frac{100}{2} = 50$
19	-6	$\frac{19}{6}$

- Entering variable: x_5, Leaving variable: x_8
- Substitute x_5 in other equalities by using $x_5 = \frac{19}{6} - \frac{7}{6}x_1 + \frac{8}{6}x_6 - \frac{1}{6}x_8$
  - New constant in the equalities

$$1 + 8 \times \frac{19}{6}$$
$$12 + 0$$
$$100 - 2 \times \frac{19}{6}$$

  - min ratio test ensures the constant in the third equality stays positive:

$$\frac{100}{2} > \frac{19}{6}, \text{which ensures that } 100 - 2 \times \frac{19}{6} > 0$$