Plan for today

- Simplex Method to solve LPs
  - What is a CFP?
    - Augmented LP and Slack Variables
  - How to obtain adjacent CFPs?
  - Which adjacent CFP to move to?
  - How to compute the adjacent CFP?

  - Complete Algorithm

Announcements:

- HW 2 posted in Gradescope
- Instructor office hours: Zoom
- TA office hours: In-person

SIMPLEX METHOD

… where we generalize this solution procedure for larger number of variables

Basic idea behind the Simplex Method

- Focus on CFPs
- Iterative: move from one CFP to another, repeat the process
- Initialization: start from a CFP (typically the origin)
- Compare objective only with adjacent CFPs (neighbors)
- Move to an adjacent CFP that yields the largest improvement in the objective value
  - Each move leads to an improvement, i.e., getting closer to the optimum
  - So the process will stop and the optimum (if it exists) will be found

1. What is a CFP?
2. How do we obtain adjacent CFPs?
3. Which adjacent CFP should we move to?
4. How to determine the adjacent CFP and its objective value?

These questions are trivial if you have a two-dimensional figure to look at, but not so obvious if all you have are symbols and inequalities

Linear Programming Formulation: Standard Form

Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

- Objective should be maximization
- Constraints should be less than or equal to
  - With the linear function of variables appearing on the LHS and
  - Constant appearing on the RHS
- All variables have non-negativity constraints

1. WHAT IS A CFP?

Setting up for Simplex: Augment LP

- Start with LP in standard form

LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ \text{subject to:}\quad x_1 &\le 4 \\ x_2 &\le 6 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

- Divide constraints into two categories: non-negativity and others
- For each of the other constraints, add a *non-negative variable* to the LHS and set it to equality

$$x_1 \le 4 \longrightarrow x_1 + x_3 = 4, x_3 \ge 0 \qquad x_2 \le 6 \longrightarrow x_2 + x_4 = 6, x_4 \ge 0 \qquad 3x_1 + 2x_2 \le 18 \longrightarrow 3x_1 + 2x_2 + x_5 = 18, x_5 \ge 0$$

  - The newly introduced variables, x_3, x_4, x_5 in this case, are referred to as **slack variables**
- Do not forget: Add non-negativity constraints for the slack variables

augmented LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ \text{subject to:}\quad x_1 + x_3 &= 4 \\ x_2 + x_4 &= 6 \\ 3x_1 + 2x_2 + x_5 &= 18 \\ x_1 \ge 0, x_2 \ge 0, x_3 &\ge 0, x_4 \ge 0, x_5 \ge 0 \end{aligned}$$

- Constraints are either non-negativity constraints or equalities
  - The resulting LP is referred to as **augmented LP**

Augmented LP solves original LP

Introducing slack variables and augmenting the LP does not change the problem

- If $(x_1, x_2, x_3, x_4, x_5)$ is feasible for the augmented LP, then (x_1, x_2) is feasible for the original problem
- If (x_1, x_2) is feasible for the original LP, then $(x_1, x_2, 4 - x_1, 6 - x_2, 18 - 3x_1 - 2x_2)$ is feasible for the augmented LP

original LP

$$\begin{aligned} \max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:} \quad x_1 \le 4 \\ x_2 \le 6 \\ 3x_1 + 2x_2 \le 18 \\ x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

augmented LP

$$\begin{aligned} \max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:} \quad x_1 + x_3 = 4 \\ x_2 + x_4 = 6 \\ 3x_1 + 2x_2 + x_5 = 18 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0 \end{aligned}$$

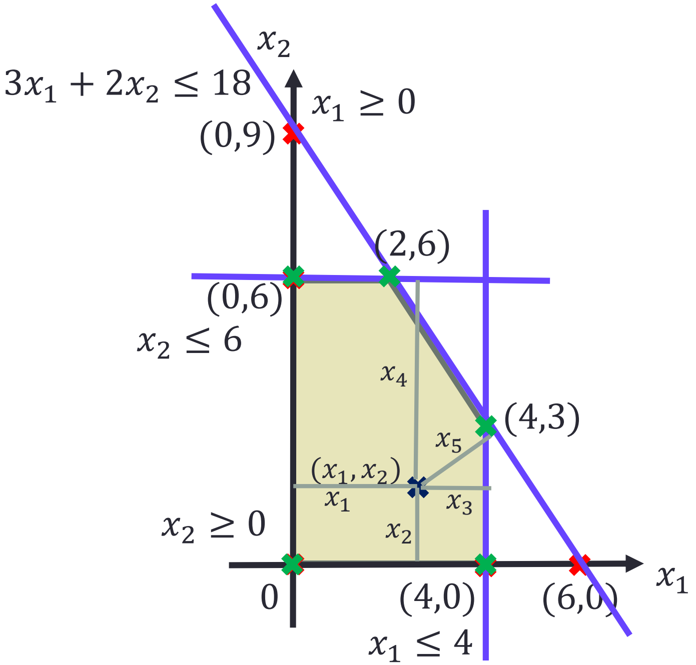
Slack variables (geometric intuition)

original LP

$$\begin{aligned} \max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:} \quad x_1 \le 4 \\ x_2 \le 6 \\ 3x_1 + 2x_2 \le 18 \\ x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

augmented LP

$$\begin{aligned} \max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:} \quad x_1 + x_3 = 4 \\ x_2 + x_4 = 6 \\ 3x_1 + 2x_2 + x_5 = 18 \\ x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0 \end{aligned}$$



- A variable x_i represents the distance from the line $x_i = 0$
- The slack variable for a constraint represents the distance from the line associated with the constraint

Note: This is only an intuition. It does not give the exact Euclidean distance.

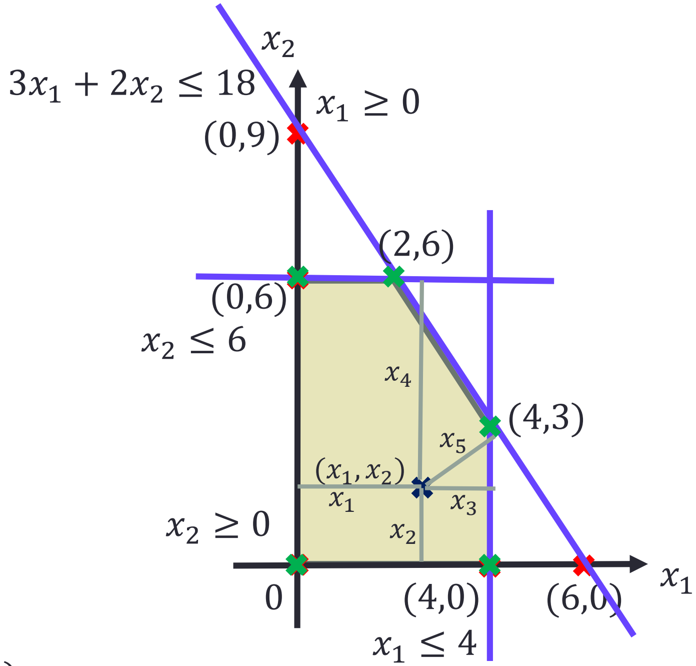
CFPs of original LP from augmented LP

original LP

$$\max\ Z = 3x_1 + 5x_2$$
s. t.:
$$x_1 \leq 4$$
$$x_2 \leq 6$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0, x_2 \geq 0$$

augmented LP

$$\max\ Z = 3x_1 + 5x_2$$
s. t.:
$$x_1 + x_3 = 4$$
$$x_2 + x_4 = 6$$
$$3x_1 + 2x_2 + x_5 = 18$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$



$(x_3, x_5) = (0,0)$ and solve the system $\Rightarrow (x_1, x_2) = (4, 3)$

$(x_4, x_5) = (0,0)$ and solve the system $\Rightarrow (x_1, x_2) = (2, 6)$

Observation: CFPs in the original LP correspond to setting some variables to zero in the augmented LP and solving the system of equations

CFPs of original LP from augmented LP

All CFPs of the original LP can be obtained by setting two variables in the augmented LP to zero and solving the system of equations

E.g.,

Original LP: CFP: $(x_1, x_2) = (4, 0)$

Set $x_2 = 0, x_3 = 0$ and solve:

$$x_1 = 4$$
$$x_4 = 6$$
$$3x_1 + x_5 = 18$$

Solving gives $(x_1, x_4, x_5) = (4, 6, 6)$

Original LP: CFP: $(x_1, x_2) = (4, 3)$

Set $x_3 = 0, x_5 = 0$ and solve:

$$x_1 = 4$$
$$x_2 + x_4 = 6$$
$$3x_1 + 2x_2 = 18$$

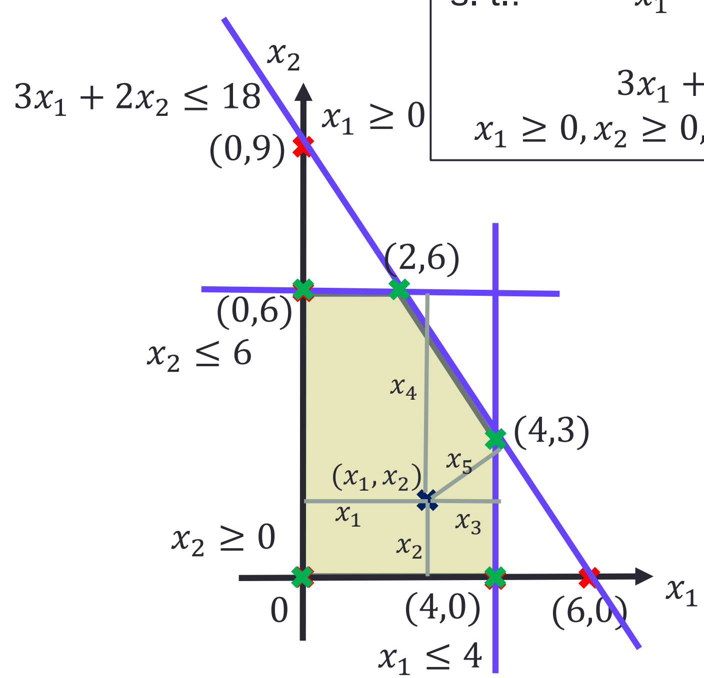
Solving gives $(x_1, x_2, x_4) = (4, 3, 3)$

original LP

$$\begin{aligned} &\max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:}\quad & x_1 \le 4 \\ & x_2 \le 6 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

augmented LP

$$\begin{aligned} &\max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:}\quad & x_1 + x_3 = 4 \\ & x_2 + x_4 = 6 \\ & 3x_1 + 2x_2 + x_5 = 18 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0 \end{aligned}$$



CFPs of original LP from augmented LP (geometry)

All CFPs of the original LP can be obtained by setting two variables in the augmented LP to zero and solving the system of equations

E.g.,

Original LP: CFP: $(x_1, x_2) = (4, 0)$

Set $x_2 = 0, x_3 = 0$ and solve:

$$x_1 = 4$$
$$x_4 = 6$$
$$3x_1 + x_5 = 18$$

Solving gives $(x_1, x_4, x_5) = (4, 6, 6)$

Original LP: CFP: $(x_1, x_2) = (4, 3)$

Set $x_3 = 0, x_5 = 0$ and solve:

$$x_1 = 4$$
$$x_2 + x_4 = 6$$
$$3x_1 + 2x_2 = 18$$

Solving gives $(x_1, x_2, x_4) = (4, 3, 3)$

Original LP: CFP: $(x_1, x_2) = (0, 6)$

Set $x_1 = 0, x_4 = 0$ and solve the rest

Original LP: CFP: $(x_1, x_2) = (2, 6)$

Set $x_4 = 0, x_5 = 0$ and solve the rest

Original LP: CFP: $(x_1, x_2) = (0, 0)$

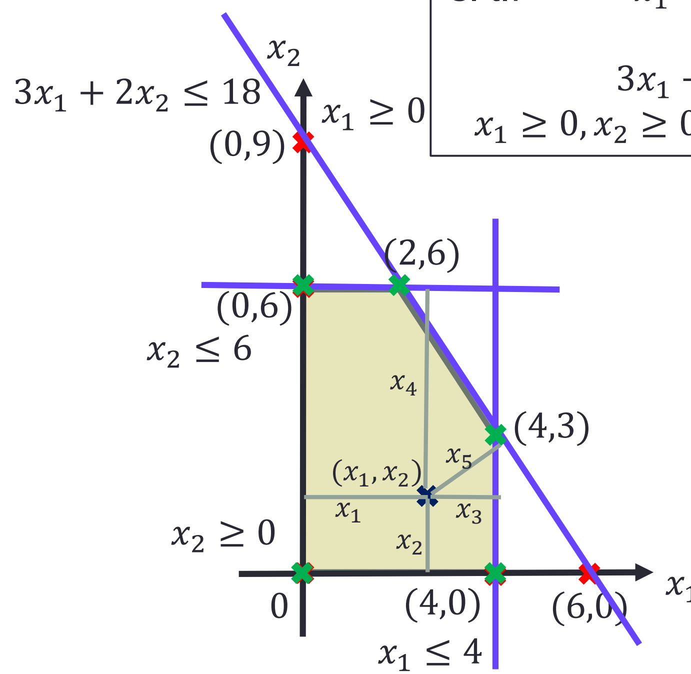
Set $x_1 = 0, x_2 = 0$ and solve the rest

original LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ \text{s. t.:}\quad & x_1 \le 4 \\ & x_2 \le 6 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

augmented LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ \text{s. t.:}\quad & x_1 + x_3 = 4 \\ & x_2 + x_4 = 6 \\ & 3x_1 + 2x_2 + x_5 = 18 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0 \end{aligned}$$



CFPs of original LP from augmented LP

- Consider a LP that has n variables and *m* (excluding non-negativity) constraints

$$\begin{aligned} &\max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:}\quad & x_1 \le 4 \\ & x_2 \le 6 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

original LP

$$\longrightarrow$$

$$\begin{aligned} &\max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:}\quad & x_1 + x_3 = 4 \\ & x_2 + x_4 = 6 \\ & 3x_1 + 2x_2 + x_5 = 18 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0 \end{aligned}$$

augmented LP

- Augmented LP: Each constraint is given a nonnegative slack variable, so we have $m + n$ variables, and m equalities

- CFP: Set n of these variables to 0, and solve for the rest of the *m* variables from the system of m equalities
  - The n variables whose values were set to 0 are referred to as ***non-basic variables***
  - The remaining variables (whose values are obtained by solving the system of equations) are referred to as ***basic variables***
  - If solution assigns non-negative values to all variables, then we have a ***basic feasible solution***

  - Values of original variables (not of those added as slack variables) in the basic feasible solution defines a CFP of the original LP

2. HOW TO OBTAIN ADJACENT CFPS?

How to obtain an adjacent CFP

Basic variables: variables whose values are obtained by solving equations
Non-basic variables: variables whose values are fixed to 0

CFP (4,3) is adjacent to CFP (2,6)

CFP (4,3) Basic variables: x_1, x_2, x_4
Non-basic variables: $x_3 = 0, x_5 = 0$
$x_1 = 4, x_2 + x_4 = 6, 3x_1 + 2x_2 = 18$

CFP (2,6) Basic variables: x_1, x_2, x_3
Non-basic variables: $x_4 = 0, x_5 = 0$
$x_1 + x_3 = 4, x_2 = 6, 3x_1 + 2x_2 = 18$

original LP

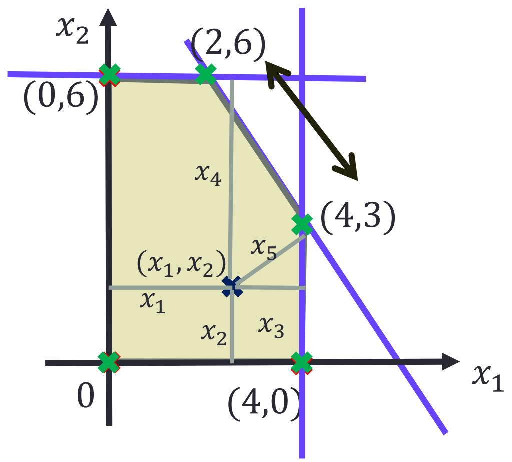
$$\begin{aligned} &\max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:}\quad & x_1 \le 4 \\ & x_2 \le 6 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

augmented LP

$$\begin{aligned} &\max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:}\quad & x_1 + x_3 = 4 \\ & x_2 + x_4 = 6 \\ & 3x_1 + 2x_2 + x_5 = 18 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0 \end{aligned}$$

To obtain a CFP adjacent to a given CFP:
1. choose one basic variable to fix its value to 0 (i.e., make it non-basic)
2. choose one non-basic variable to solve for its value (i.e., make it basic)

i.e., swap a basic variable with a non-basic variable



3. WHICH ADJACENT CFP TO MOVE TO?

Which adjacent CFP to move to?

- To obtain an adjacent CFP, we exchange a basic variable for a non-basic variable (in the augmented LP)

- Obvious way: try every pair and choose the one that
  - is feasible and
  - yields the maximum improvement of the objective function

  - Inefficient (too many pairs):
    - Suppose that the augmented LP has m equalities, and $m + n$ variables,
    - Then we have m basic variables and n non-basic variables
    - The total number of possible swaps is $m \times n$
    - If $m = 10000, n = 5000 \Rightarrow 50$ million evaluations at each step

- The Simplex Method
  - Select a non-basic variable to make it a basic variable based on
    - The rate at which we can improve the objective value by making that non-basic variable strictly positive (recall: it is currently non-basic, so its current value is zero)

  - Select a basic variable to make it a non-basic variable based on
    - how much improvement can we achieve without violating any constraints

Specifically…..

$$\max\ Z = 3x_1 + 5x_2$$

s. t.:

$$\begin{aligned} x_1 \quad\quad + x_3 \quad\quad &= 4 \\ x_2 \quad\quad + x_4 \quad &= 6 \\ 3x_1 + 2x_2 \quad\quad + x_5 &= 18 \end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

Initial solution: basic variables: $x_3 = 4, x_4 = 6, x_5 = 18$, non-basic variables: $x_1 = x_2 = 0$

Select the entering basic variable:

Take a look at the objective function

Making x_2 positive gives the maximum rate of improvement in the objective,
so make x_2 basic

Select the leaving basic variable:

Which basic variable among x_3, x_4, x_5 restricts x_2 the most?

The first variable that decreases to zero when we increase x_2 is x_4,
so it determines the amount by which we can increase x_2 and so make it non-basic

How to make selection easier: **proper format**

$$\max\ Z = 3x_1 + 5x_2$$

s. t.:

$$\begin{aligned} x_1 \quad\quad + x_3 \quad\quad &= 4 \\ x_2 \quad + x_4 \quad &= 6 \\ 3x_1 + 2x_2 \quad\quad + x_5 &= 18 \end{aligned}$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

- Initialization: basic variables (x_3, x_4, x_5), non-basic variables $x_1 = 0, x_2 = 0$
- Write objective function in terms of only non-basic variables,

$$Z = 3x_1 + 5x_2$$

… helps us select the entering variable by a simple comparison of coefficients

Entering variable: x_1 or x_2

3 vs 5

Choose x_2

- Rewrite the equality constraints: Each equality constraint should contain
  - one basic variable on the LHS with coefficient 1,
  - constants and non-basic variables on the RHS and
  - the RHS constant should be non-negative

$$x_3 = 4 - x_1$$
$$x_4 = 6 - x_2$$
$$x_5 = 18 - 3x_1 - 2x_2$$

… helps us select the leaving variable by a simple comparison of the ratios between the RHS constant and the entering variable's coefficient in each equation

Leaving variable: x_3 or x_4 or x_5

$-$ vs $\frac{6}{1}$ vs $\frac{18}{2}$

Choose x_4

- This **proper format** of objective and equality constraints should be maintained in each step to facilitate the selection of entering and leaving variables

4. COMPUTING THE ADJACENT CFP

… after determining which variables to swap

- Could re-solve the system for the basic variables by setting the non-basic variables to zero
- There is a simpler way…

After determining which variables to swap…

$$Z = 3x_1 + 5x_2$$
$$x_3 = 4 - x_1$$
$$x_4 = 6 - x_2$$
$$x_5 = 18 - 3x_1 - 2x_2$$

Non-basic variables: x_1, x_2
Basic variables:
$x_3 = 4, x_4 = 6, x_5 = 18$

- x_2 is the entering variable, x_4 is the leaving variable
- Step 1: express the entering basic variable as a function of non-basic variables and the leaving variable (use the equality constraint for the leaving basic variable to do this)

$$x_2 = 6 - x_4$$

- Step 2: use the expression to substitute the entering basic variable in the objective function

$$Z = 3x_1 + 5x_2 = 30 + 3x_1 - 5x_4$$

- Step 3: substitute the entering basic variable in the remaining constraints

$$x_3 = 4 - x_1$$
$$x_5 = 18 - 3x_1 - 2x_2 = 6 - 3x_1 + 2x_4$$

- Now (x_2, x_3, x_5) are basic variables and (x_1, x_4) are non-basic variables
  - The objective function contains only non-basic variables
  - Each equality constraint contains:
    - one basic variable on the LHS with coefficient 1
    - constants and non-basic variables on the RHS and
    - the RHS constant is non-negative

$$Z = 30 + 3x_1 - 5x_4$$
$$x_3 = 4 - x_1$$
$$x_2 = 6 - x_4$$
$$x_5 = 6 - 3x_1 + 2x_4$$

Corresponds to the solution:
$x_1 = 0, x_2 = 6, x_3 = 4,$
$x_4 = 0, x_5 = 6,$
$Z = 30$
which is the corner point $x_1 = 0, x_2 = 6$

Next iteration……

$$Z = 30 + 3x_1 - 5x_4$$

$$x_3 = 4 - x_1$$
$$x_2 = 6 - x_4$$
$$x_5 = 6 - 3x_1 + 2x_4$$

Non-basic variables: x_1, x_4
Basic variables:
$x_3 = 4, x_2 = 6, x_5 = 6$

- which variable should be the entering variable, which one should be the leaving variable?

  x_1 enters

  x_5 leaves

- expression for the entering basic variable

$$x_1 = 2 + \frac{2}{3}x_4 - \frac{1}{3}x_5$$

- Substitute in the objective function

$$Z = 30 + 3x_1 - 5x_4 = 36 - 3x_4 - x_5$$

- Substitute in all equality constraints

$$x_1 = 2 + \frac{2}{3}x_4 - \frac{1}{3}x_5$$
$$x_3 = 4 - x_1 = 2 - \frac{2}{3}x_4 + \frac{1}{3}x_5$$
$$x_2 = 6 - x_4$$

$$Z = 36 - 3x_4 - x_5$$
$$x_1 = 2 + \frac{2}{3}x_4 - \frac{1}{3}x_5$$
$$x_3 = 2 - \frac{2}{3}x_4 + \frac{1}{3}x_5$$
$$x_2 = 6 - x_4$$

Non-basic variables: x_4, x_5
Basic variables: x_1, x_2, x_3
Corresponds to the solution:
$x_1 = 2, x_2 = 6, x_3 = 2,\ x_4 = 0,$
$x_5 = 0$

which is the corner point $x_1 = 2, x_2 = 6$

- Next iteration? STOP since no entering variable to improve obj value
- So the optimal solution is

$$x_1^* = 2, x_2^* = 6, x_3^* = 2$$
$$Z = 36$$

Simplex Method: the complete algorithm

- **Initialization**
  - transform the original LP into the augmented LP, determine basic and non-basic variables
  - Rewrite constraints in proper format:
    - one basic variable on the LHS with coefficient 1,
    - constants and non-basic variables on the RHS and
    - RHS constant should be non-negative
  - Rewrite objective function in proper format: contains only non-basic variables

- **Iteration**
  1. select the entering variable: the variable with the largest positive coefficient in the obj. function
  2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
  3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
  4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

- **Termination: Optimality Test**
  - if every variable in the objective function is with a negative coefficient, then no entering variable can be found ⇒ **the optimal solution has been found**
  - if the entering variable can be increased to infinity without driving any other basic variable to below zero, then no leaving variable can be found ⇒ **the problem is unbounded**