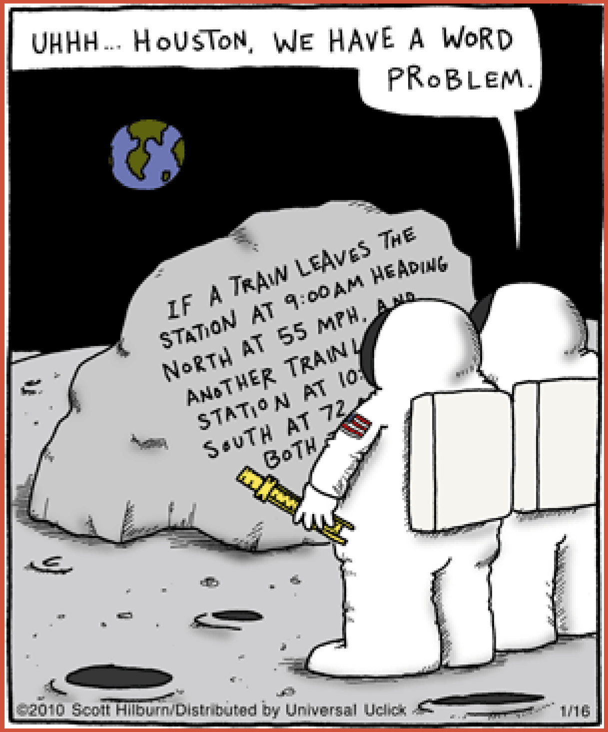
Plan for today

- How to solve LPs?
  - Graphic method
    - Plotting constraints
    - Plotting objective (iso-line)
    - Identifying optimum
  - Scenarios for an LP
    - Infeasible
    - Degenerate
    - Unbounded
  - Computing Corner Feasible Points
  - Optimality Test
- Simplex Method to solve LPs
  - What is a CFP?
    - Augmented LP and Slack Variables

<u>Announcements:</u>

- HW 2 posted in Gradescope after lecture
- Instructor office hours: Zoom
- TA office hours: In-person



FORMULATIONS

… where we model problems mathematically

- So that we can use standard solution techniques

Linear Programming Formulation

- A Linear Programming Formulation is an optimization problem formulation in which
    - the objective function and
    - all constraints

  are linear functions of decision variables, which are real variables

- More specifically, in a linear programming, the objective function and all constraints satisfy the following four conditions
    - Proportionality
    - Additivity
    - Divisibility
    - Certainty

LINEAR PROGRAMMING FORMULATION EXAMPLES

… where we see more examples of problems that can be formulated as LPs

Linear Programming Formulation: Standard Form

Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \\
x_1 \ge 0, x_2 \ge 0, \dots, x_n &\ge 0
\end{aligned}
$$

- Objective should be maximization
- Constraints should be less than or equal to
  - With the linear function of variables appearing on the LHS and
  - Constant appearing on the RHS
- All variables have non-negativity constraints

LINEARIZATION TECHNIQUES

… where we see how to massage certain non-linear formulations into linear formulations

LINEARIZATION TECHNIQUES

1. Absolute values

LINEARIZATION TECHNIQUES

2. Ratio of variables

LINEARIZATION TECHNIQUES

3. Maximizing the minimum

HOW TO SOLVE A LINEAR PROGRAMMING PROBLEM

Method 1: Graphic

Recap

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

GRAPHIC METHOD

… where we

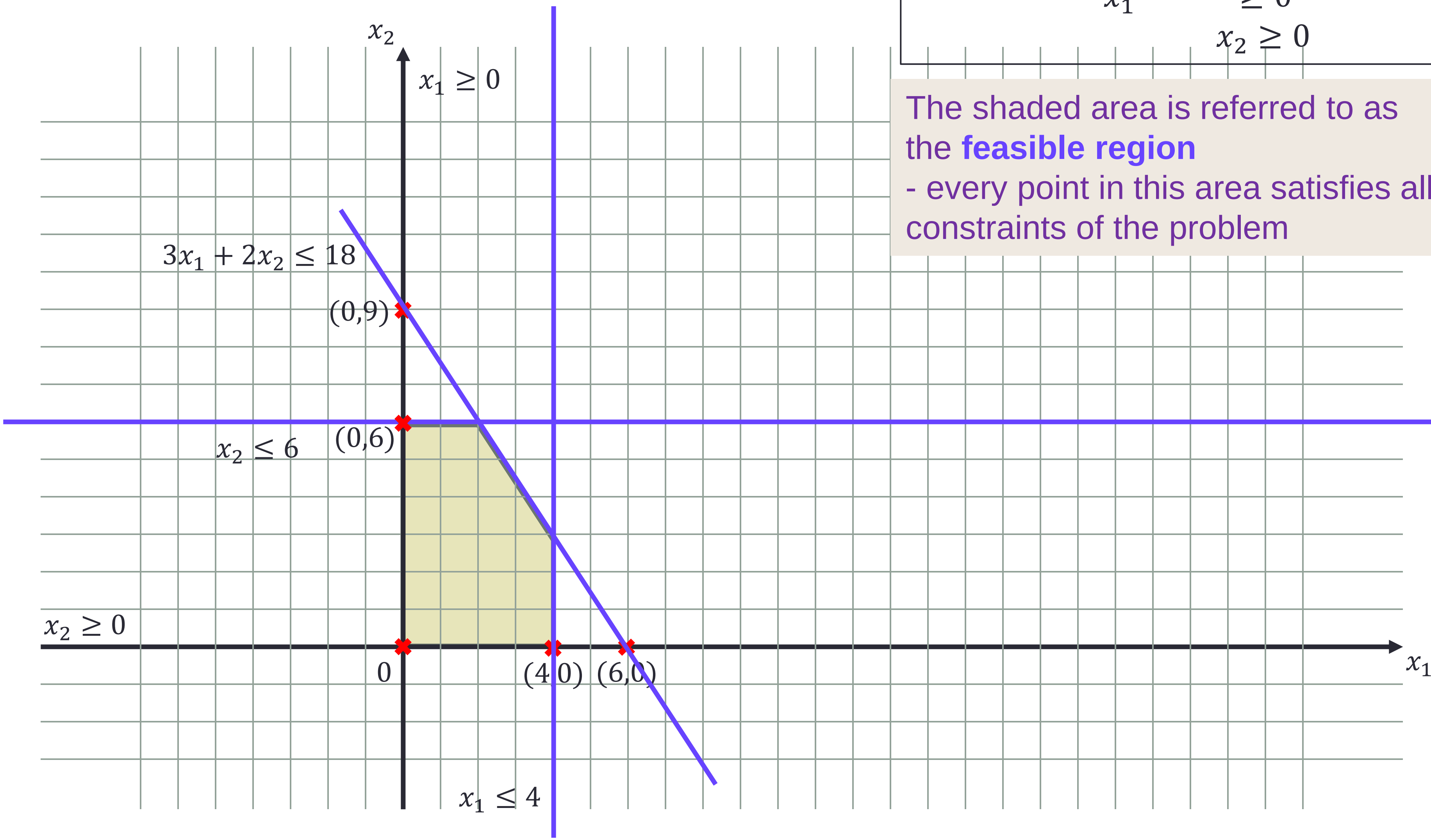
- see how to solve an LP using the Graphic Method and
- understand the scenarios that can happen with an LP

Graphic Method: Plot Constraints

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$



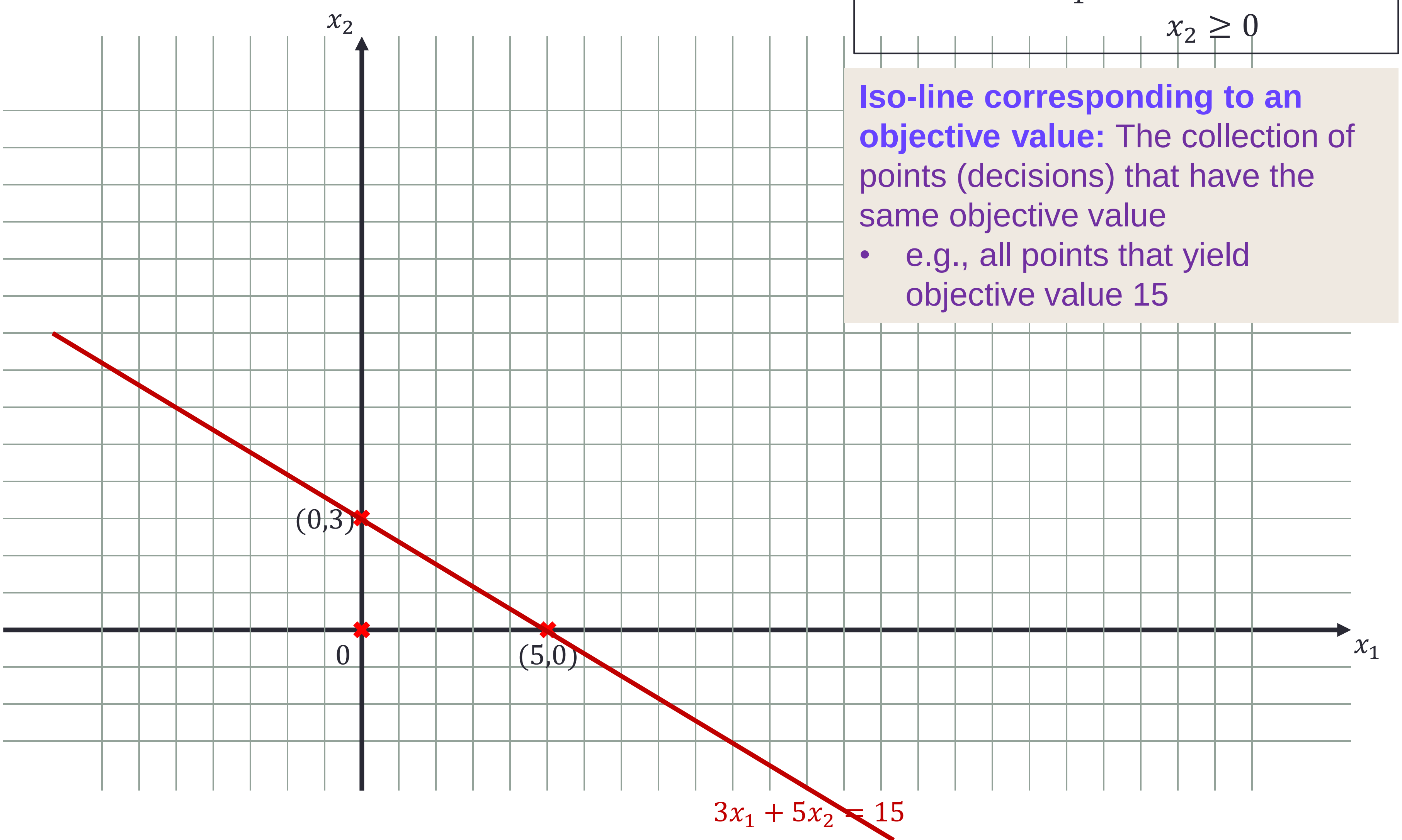
The shaded area is referred to as the **feasible region**
- every point in this area satisfies all constraints of the problem

Graphic Method: Plot Objective

$$
\begin{aligned}
\max\ Z = 3x_1 + 5x_2& \\
\text{subject to:}\quad x_1 &\le 4 \\
2x_2 &\le 12 \\
3x_1 + 2x_2 &\le 18 \\
x_1 &\ge 0 \\
x_2 &\ge 0
\end{aligned}
$$

Iso-line corresponding to an objective value: The collection of points (decisions) that have the same objective value

- e.g., all points that yield objective value 15

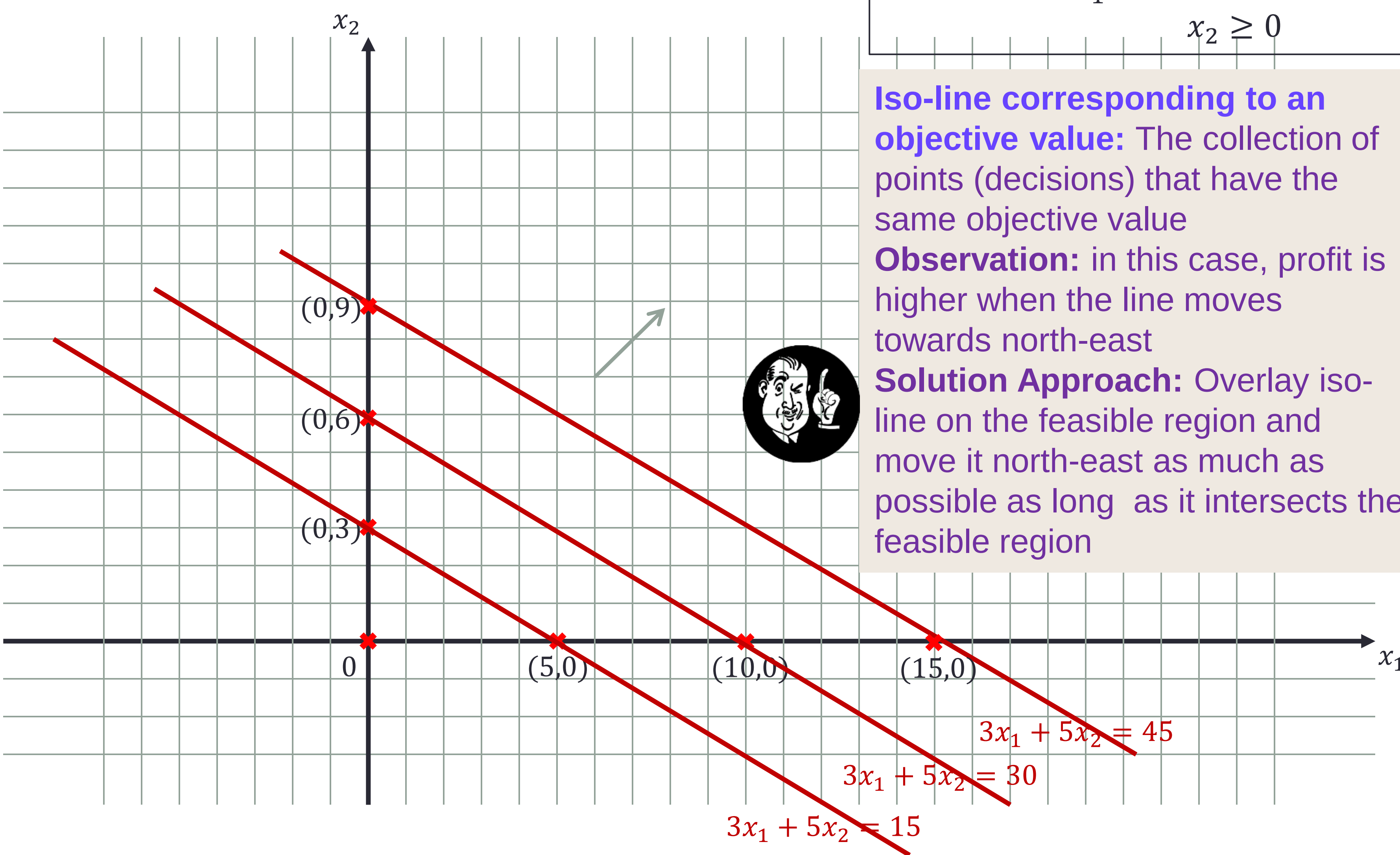


Graphic Method: Plot Objective

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$\begin{aligned} x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$



Iso-line corresponding to an objective value: The collection of points (decisions) that have the same objective value

Observation: in this case, profit is higher when the line moves towards north-east

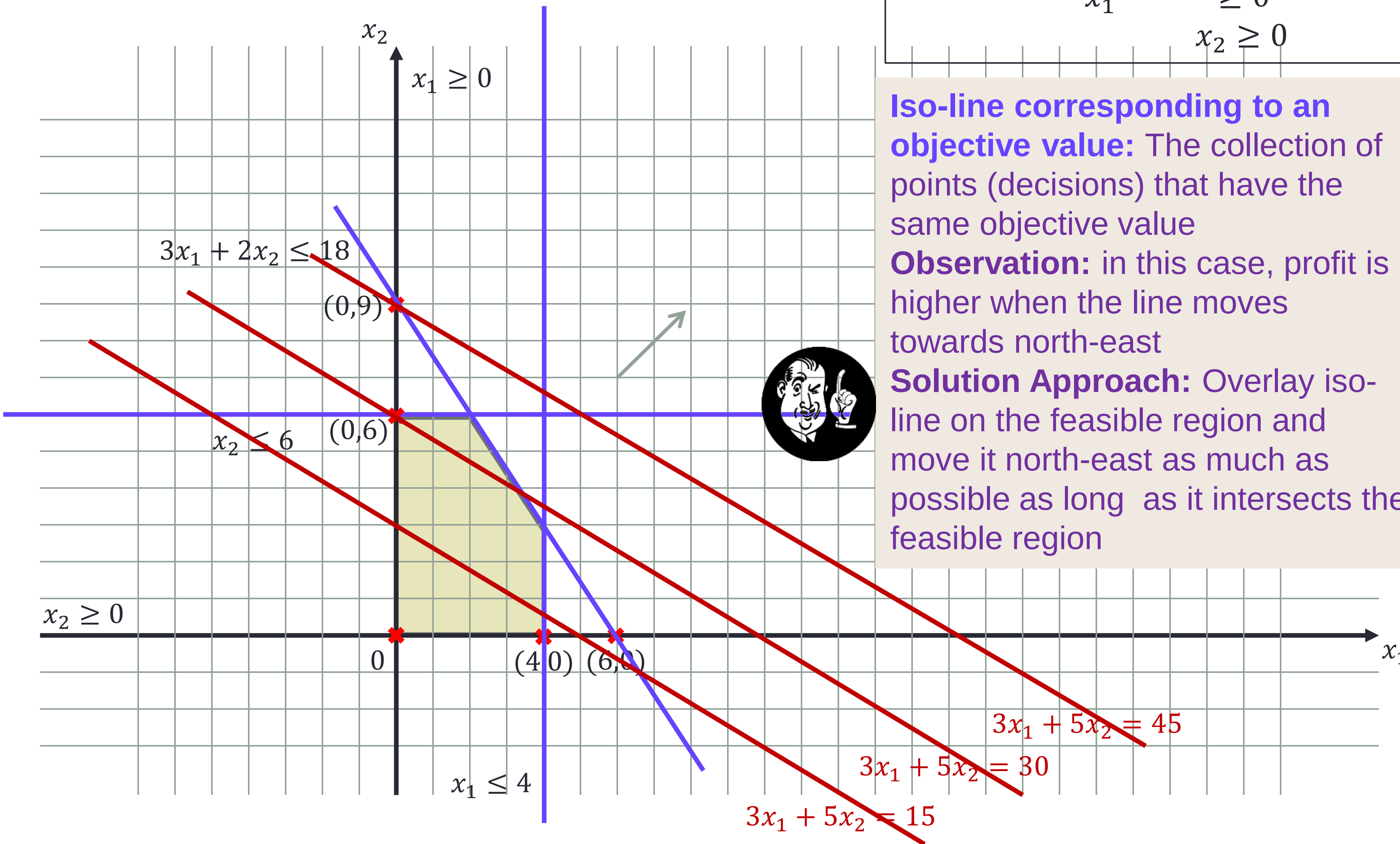
Solution Approach: Overlay iso-line on the feasible region and move it north-east as much as possible as long as it intersects the feasible region

Graphic Method: Find Optimum

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$



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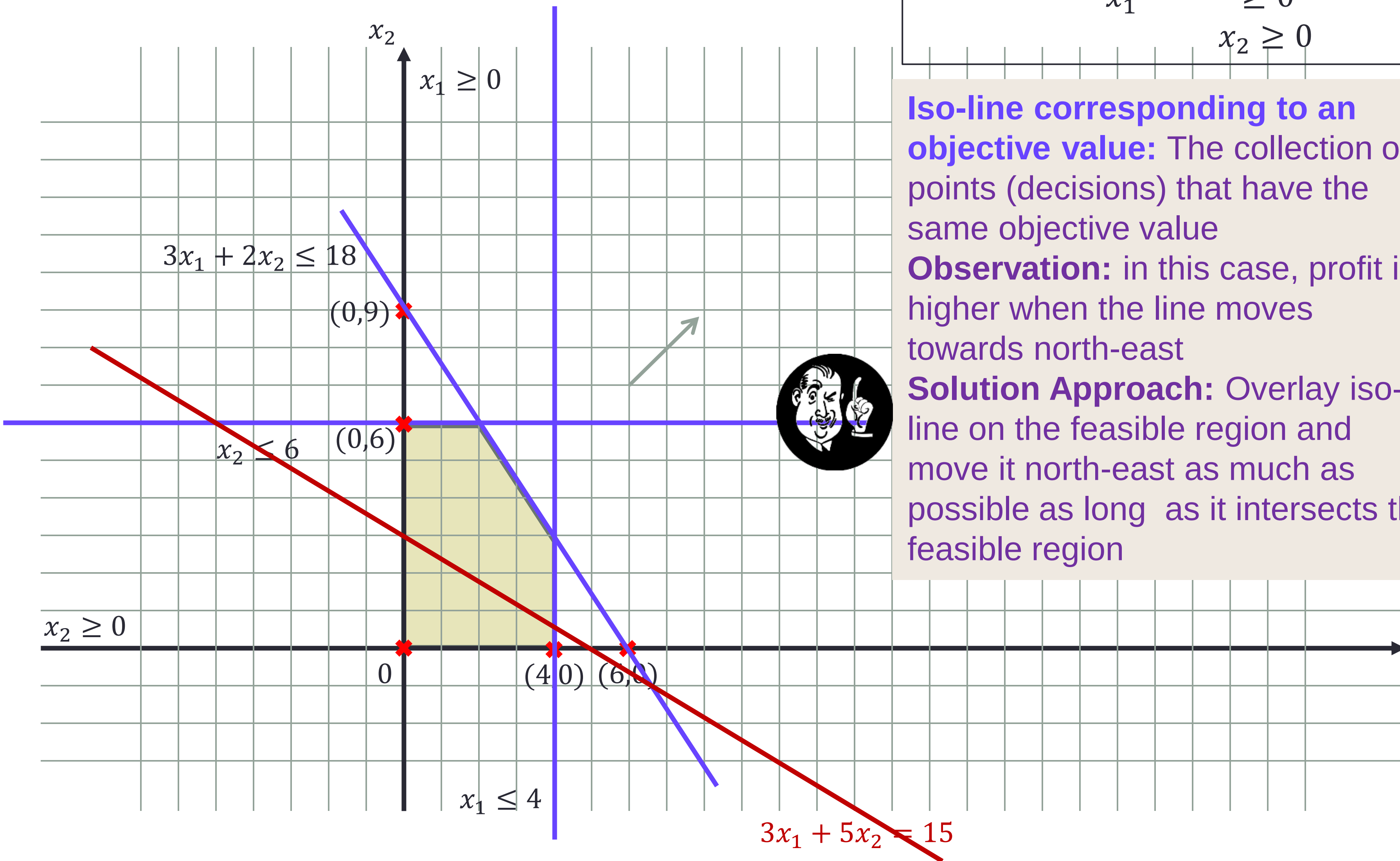

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Solution Approach: Overlay iso-line on the feasible region and move it north-east as much as possible as long as it intersects the feasible region

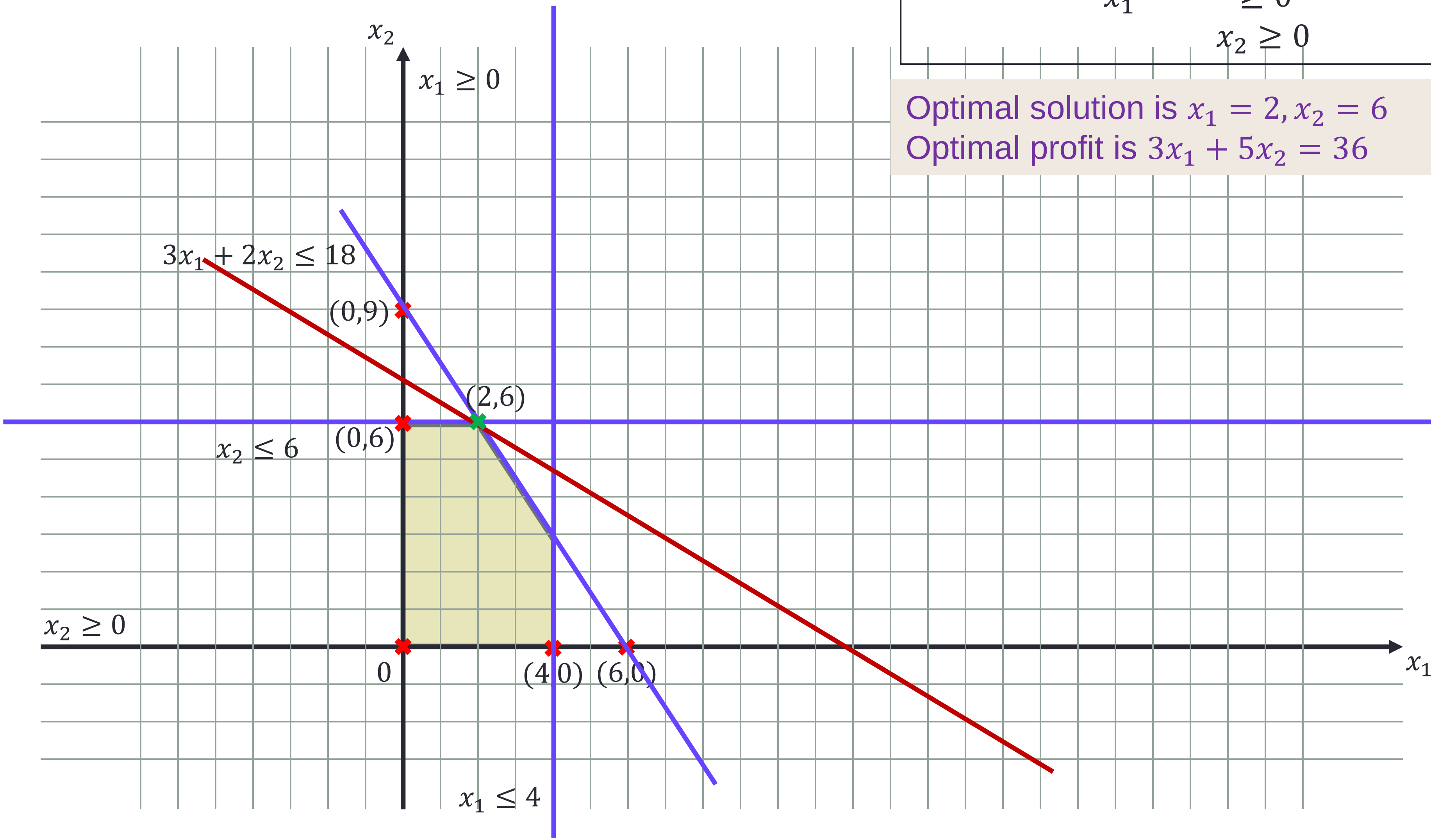
Graphic Method: Find Optimum

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

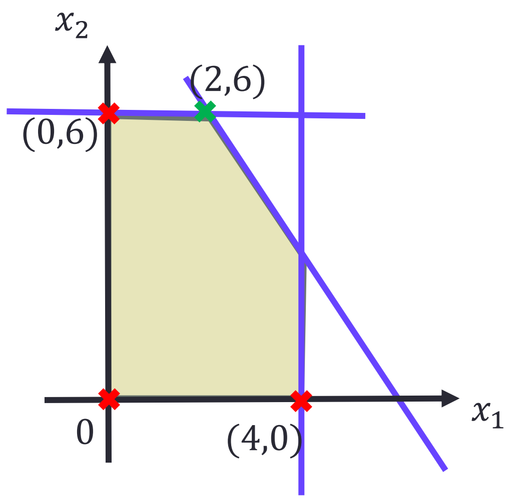
$$\begin{aligned} x_1 &\leq 4 \\ 2x_2 &\leq 12 \\ 3x_1 + 2x_2 &\leq 18 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Optimal solution is $x_1 = 2, x_2 = 6$
Optimal profit is $3x_1 + 5x_2 = 36$



Graphic Method (Overview)

- Decision variables
  - each axis represents a decision variable
  - a point on the plane corresponds to a particular decision
- Constraints
  - each constraint is represented by one side of a straight line
  - the set of points satisfying all constraints corresponds to the feasible region (shaded region) formed by these lines
- Objective
  - a particular objective value is achieved by all points on a line known as the "iso-line corresponding to that objective value"
- Solution Procedure: "push" the iso-line as far as it can go in the direction that improves the objective while still intersecting the feasible region

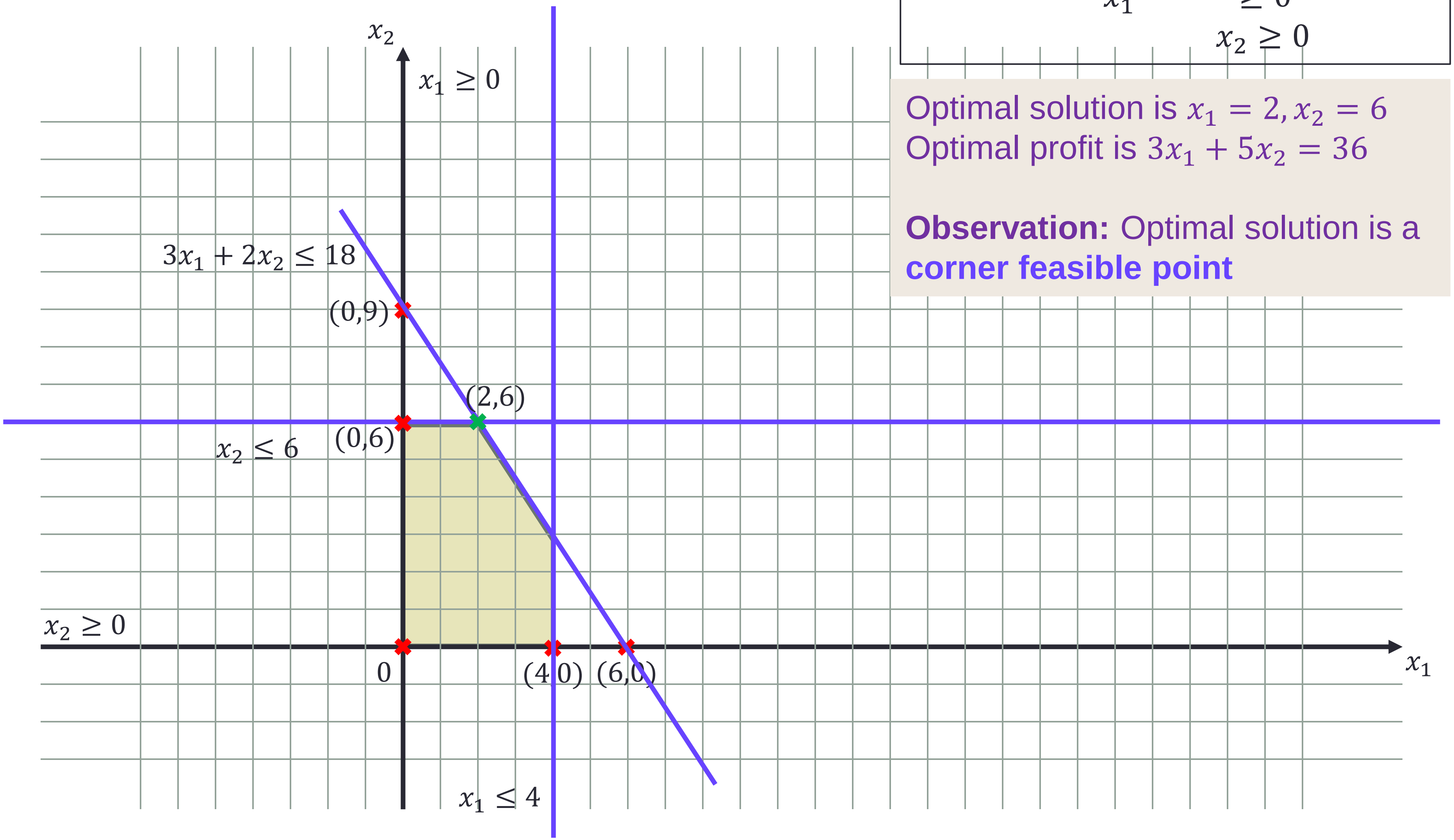


An observation about the solution

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$\begin{aligned} x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$



Optimal solution is $x_1 = 2, x_2 = 6$
Optimal profit is $3x_1 + 5x_2 = 36$

Observation: Optimal solution is a **corner feasible point**

WHAT ELSE COULD HAPPEN WITH AN LP?

… where we solve more examples with the Graphic method to understand what else could happen to an LP

EXAMPLE 2

Graphic Method

$$\max \; Z = 10x_1 + 5x_2$$

subject to:

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

Feasible region remains the same

Graphic Method

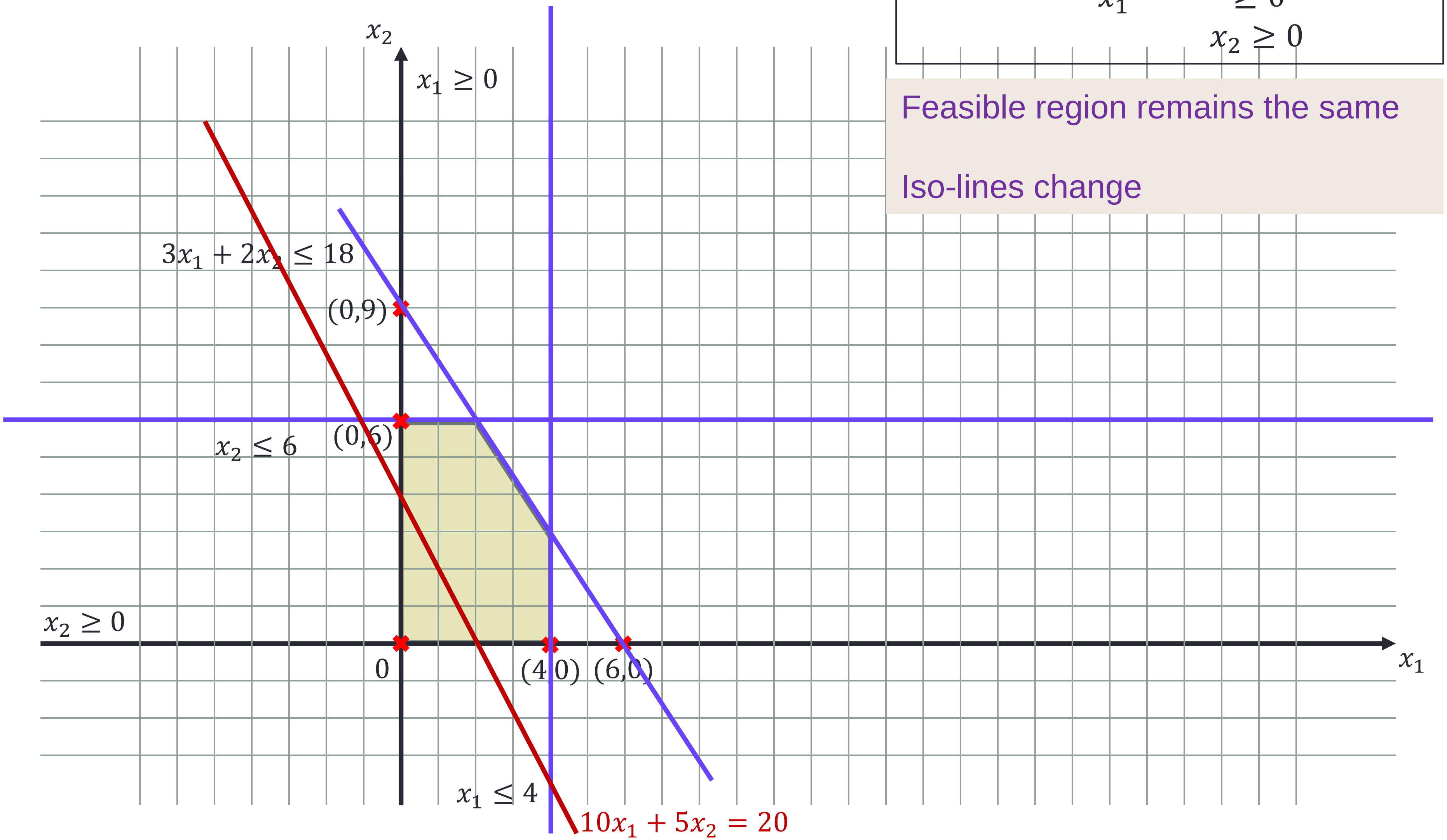
$$\max\ Z = 10x_1 + 5x_2$$

subject to:

$$\begin{aligned} x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

Feasible region remains the same

Iso-lines change



Graphic Method

New solution

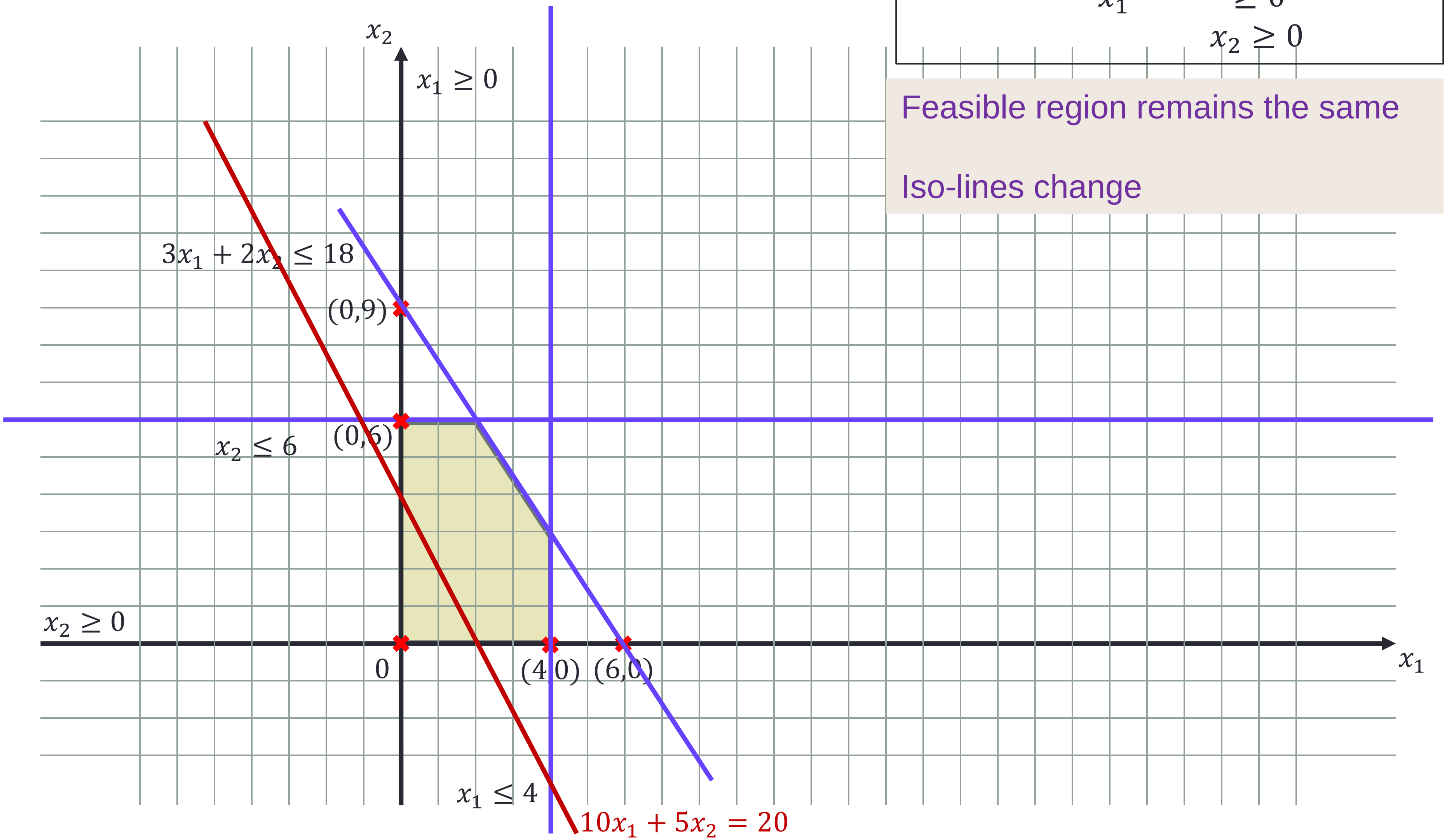
$$\max \ Z = 10x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

Feasible region remains the same

Iso-lines change

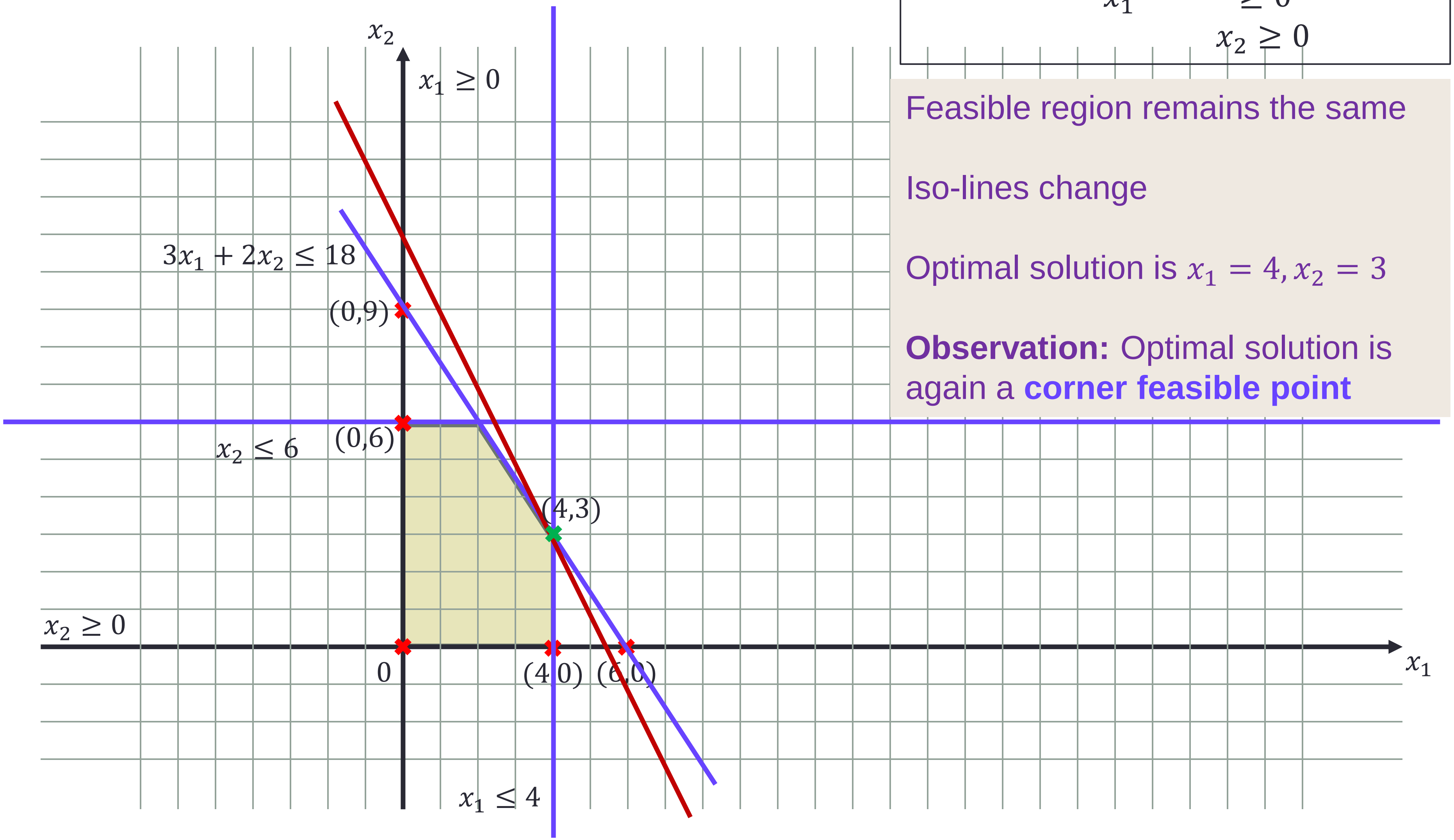


Graphic Method
New solution

$$\max \; Z = 10x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$



Feasible region remains the same

Iso-lines change

Optimal solution is $x_1 = 4, x_2 = 3$

Observation: Optimal solution is again a **corner feasible point**

EXAMPLE 3

Graphic Method

$$\max \ Z = 7.5x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

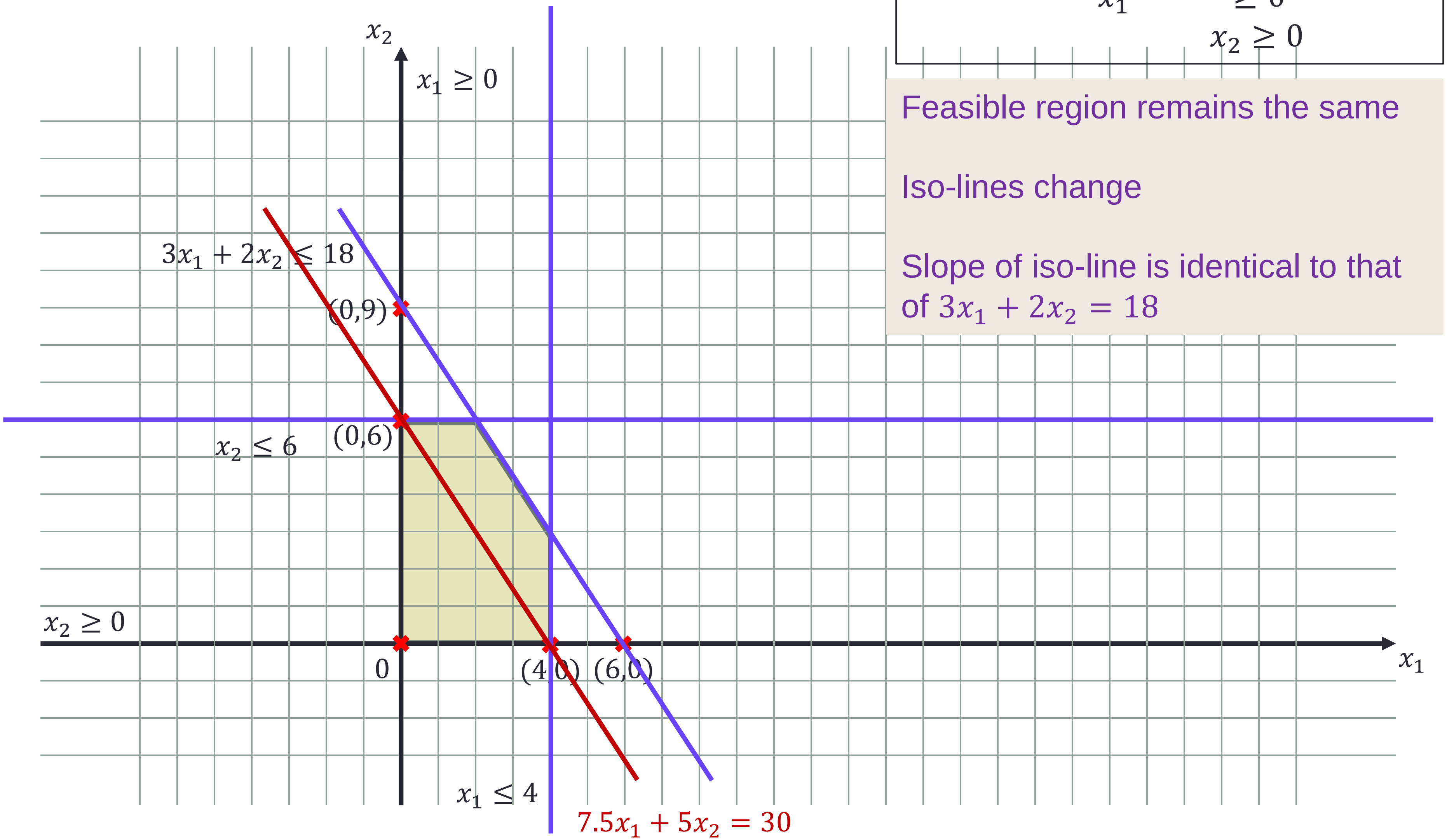
Feasible region remains the same

Graphic Method

$$\max\ Z = 7.5x_1 + 5x_2$$

subject to:

$$\begin{aligned} x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$



Feasible region remains the same

Iso-lines change

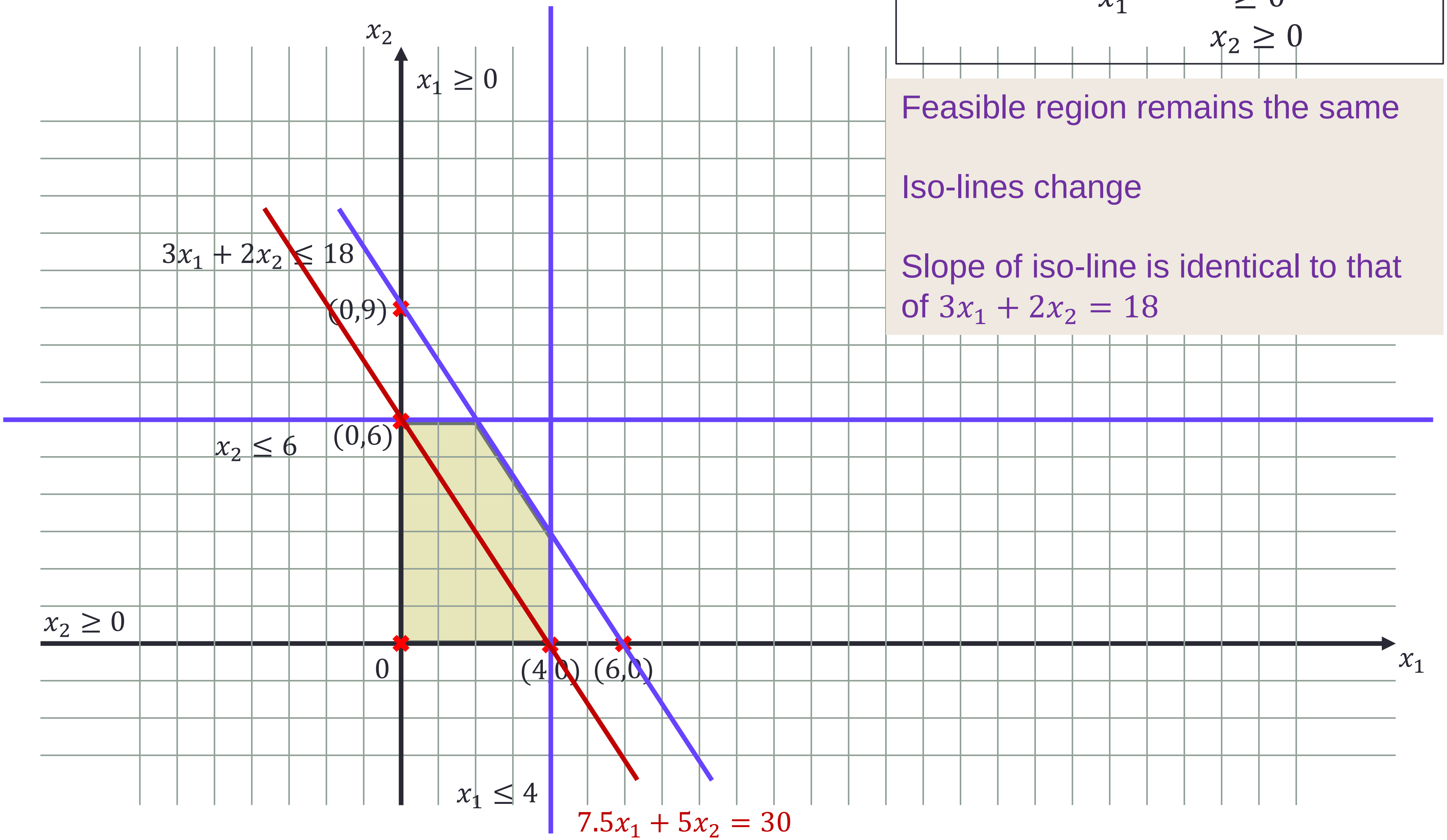
Slope of iso-line is identical to that of $3x_1 + 2x_2 = 18$

Graphic Method
New solution

$$\max\ Z = 7.5x_1 + 5x_2$$

subject to:

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$



Feasible region remains the same

Iso-lines change

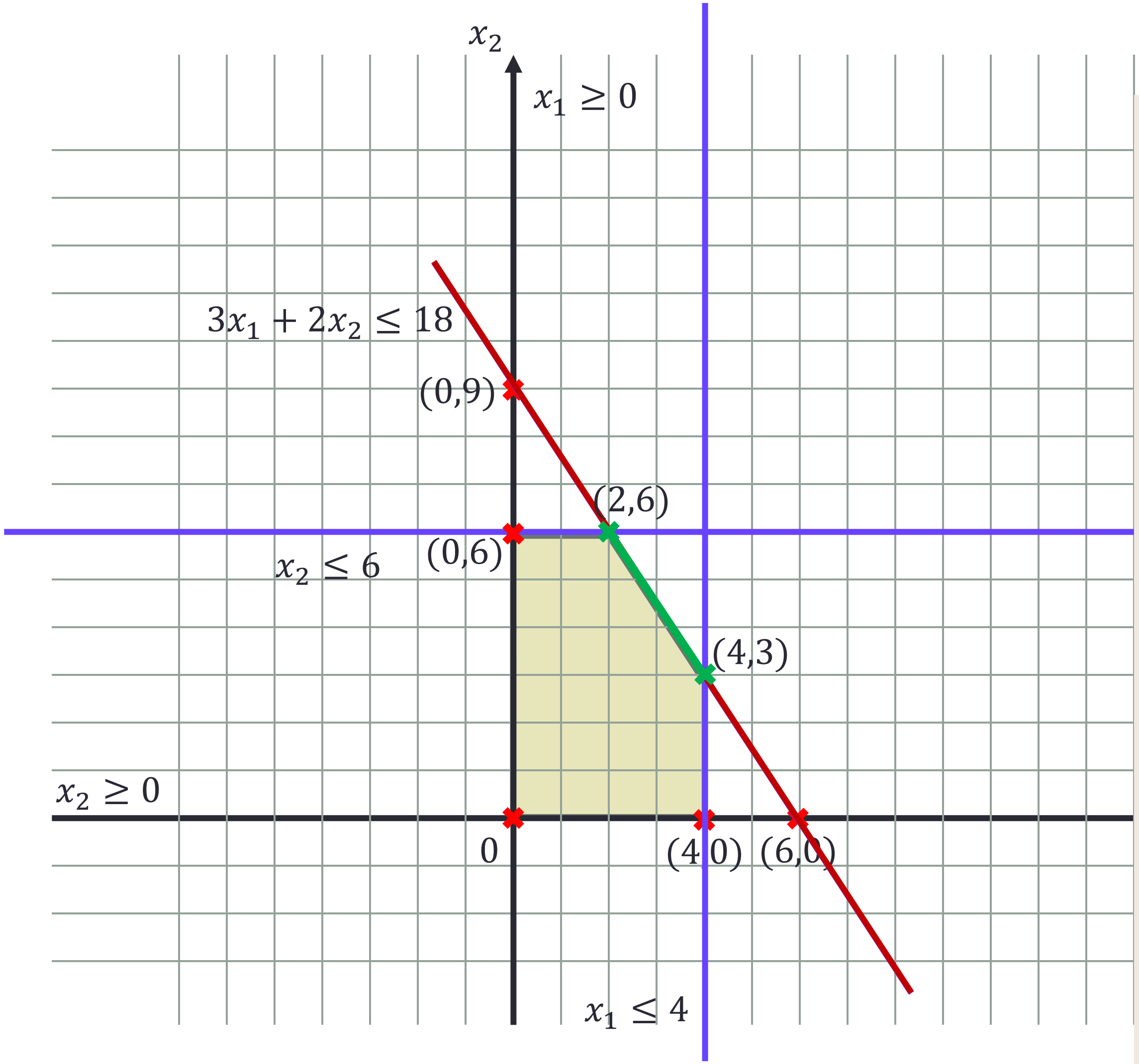
Slope of iso-line is identical to that of $3x_1 + 2x_2 = 18$

Graphic Method
New solution

$$\max\ Z = 7.5x_1 + 5x_2$$

subject to:

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$



Feasible region remains the same

Iso-lines change

Slope of iso-line is identical to that of $3x_1 + 2x_2 = 18$

Optimal solution:
There are many optimal solutions
Such an LP problem is said to be **degenerate**

In particular Corner feasible points $(x_1 = 4, x_2 = 3)$ and $(x_1 = 2, x_2 = 6)$ are optimum

Observation: If we only need one optimal solution, it suffices to restrict our attention to **corner feasible points**

EXAMPLE 4

Graphic Method
(Add a new constraint)

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

New constraint: $3x_1 + 5x_2 \geq 45$

$$x_1 \geq 0$$
$$x_2 \geq 0$$

Feasible region changes

Graphic Method
(Add a new constraint)

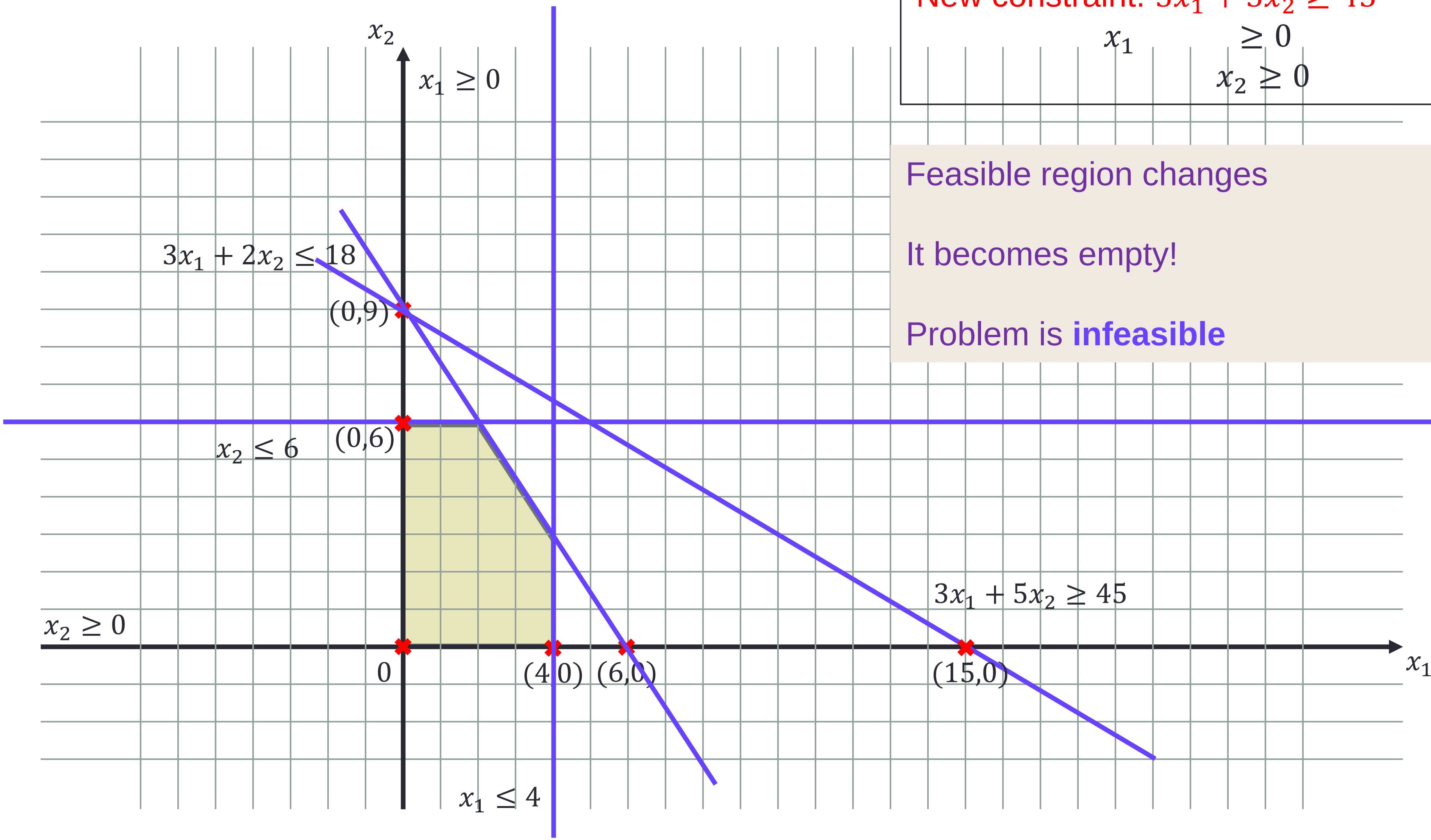
$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

New constraint: $3x_1 + 5x_2 \geq 45$

$$x_1 \geq 0$$
$$x_2 \geq 0$$



Feasible region changes

It becomes empty!

Problem is **infeasible**

EXAMPLE 5

Graphic Method
(Drop constraints)

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$

$$\cancel{2x_2 \leq 12}$$

$$\cancel{3x_1 + 2x_2 \leq 18}$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

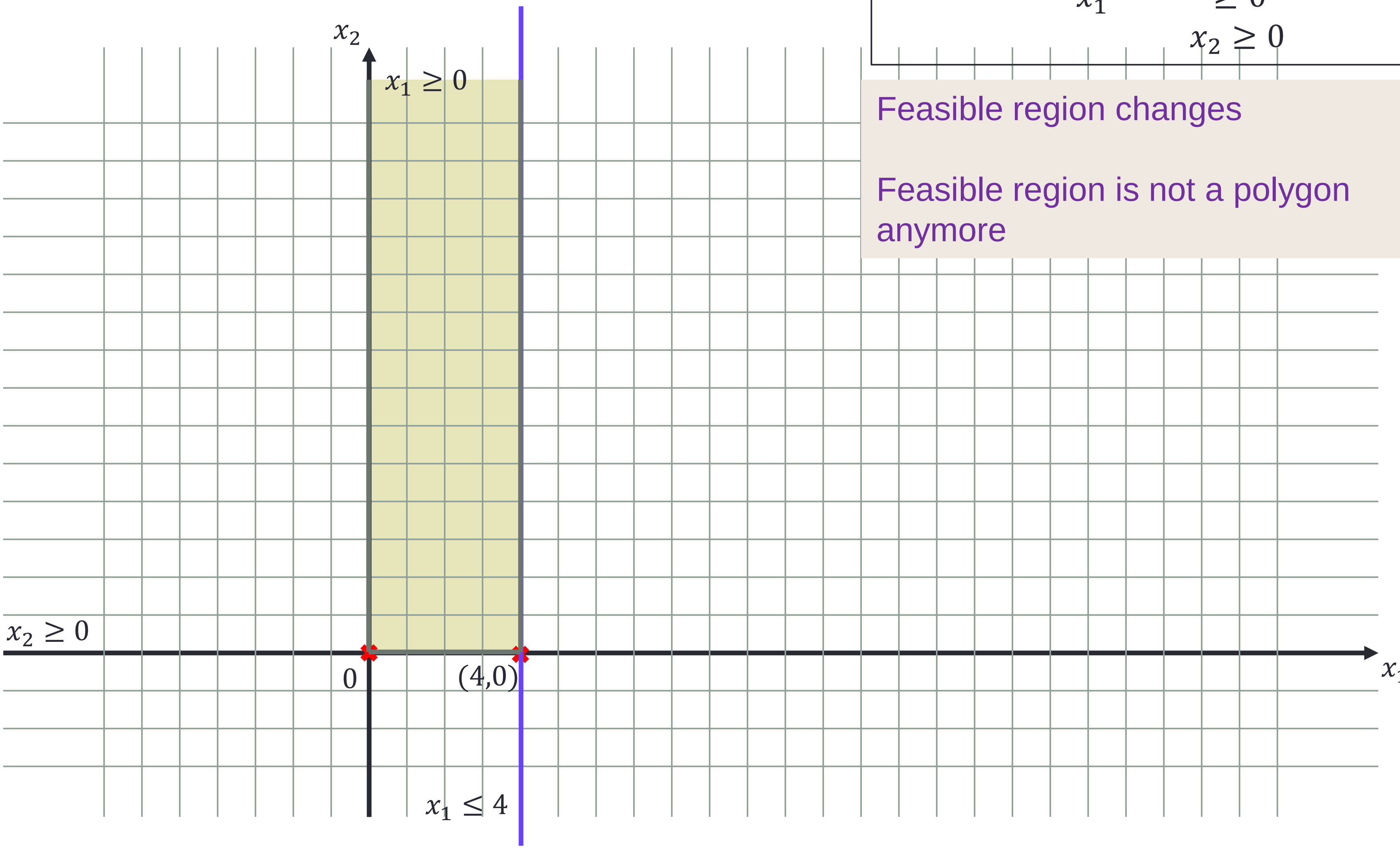
Feasible region changes

Graphic Method (Drop constraints)

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \le 4$$
$$\cancel{2x_2 \le 12}$$
$$\cancel{3x_1 + 2x_2 \le 18}$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$



Feasible region changes

Feasible region is not a polygon anymore

Graphic Method (Drop constraints)

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

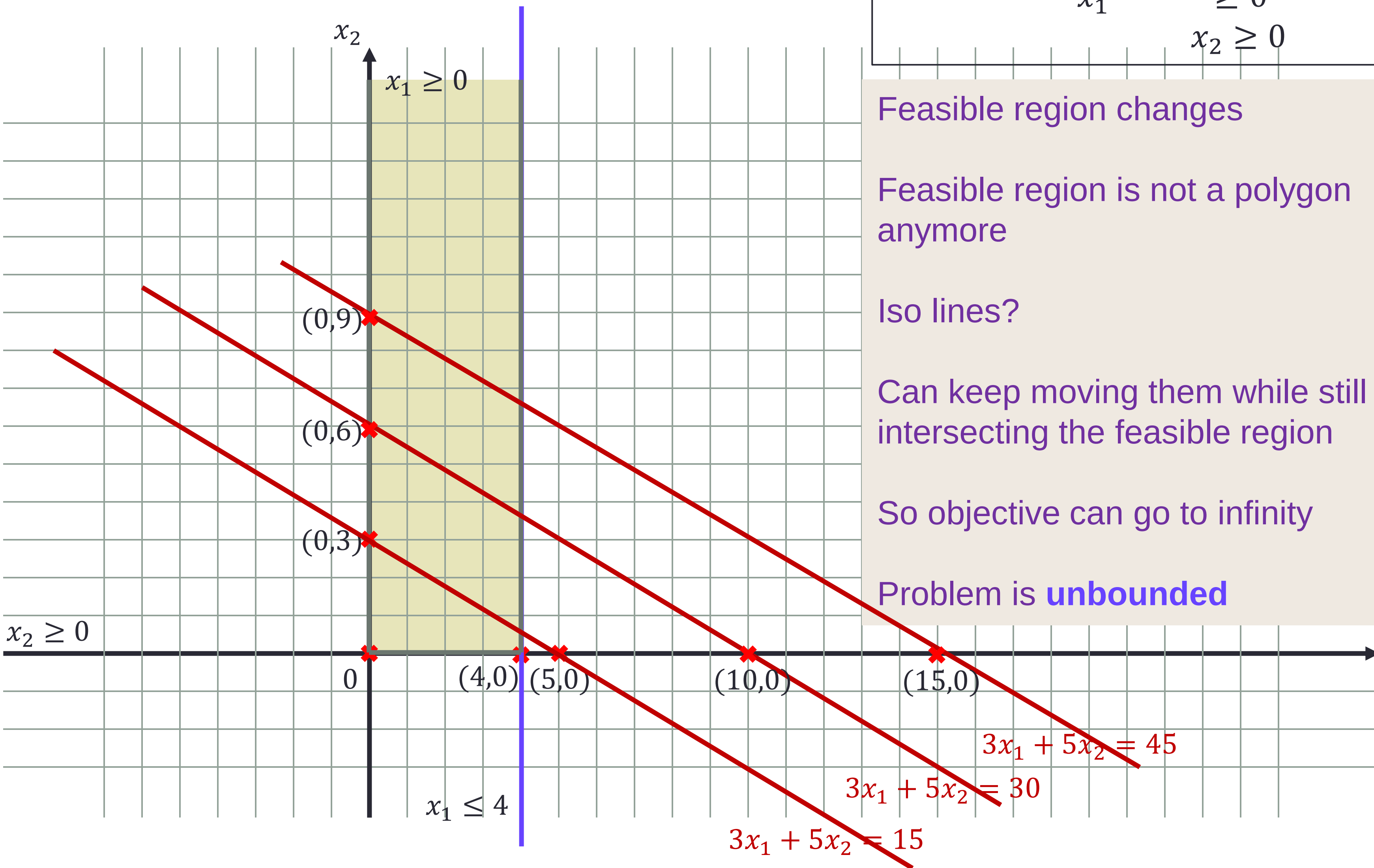
$$x_1 \le 4$$

$$\cancel{2x_2 \le 12}$$

$$\cancel{3x_1 + 2x_2 \le 18}$$

$$x_1 \ge 0$$

$$x_2 \ge 0$$



Feasible region changes

Feasible region is not a polygon anymore

Iso lines?

Can keep moving them while still intersecting the feasible region

So objective can go to infinity

Problem is **unbounded**

Summary (what does the graphic method tell us?)

An LP problem can be either

1. **Infeasible** (empty feasible region, i.e., no point satisfying all constraints) or
2. **Unbounded** (there exist feasible solutions whose objective value is unbounded) or
3. Has an optimal solution

   If there are multiple optimal solutions, then the problem is said to be **degenerate**

If there is one optimal solution, then it must be a **corner feasible point (CFP)**
Note: If there are many optimal solutions, then one of them must be a **corner feasible point**

Important: Although an LP has a corner feasible point as an optimal solution, **NOT** all corner feasible points are optimal solutions (in fact, most of them are not)

Questions:

1. Can we find an optimum corner feasible point without plotting a figure?

Implication: Can find the optimal solution by focusing on CFPs and ignoring the rest of the feasible region

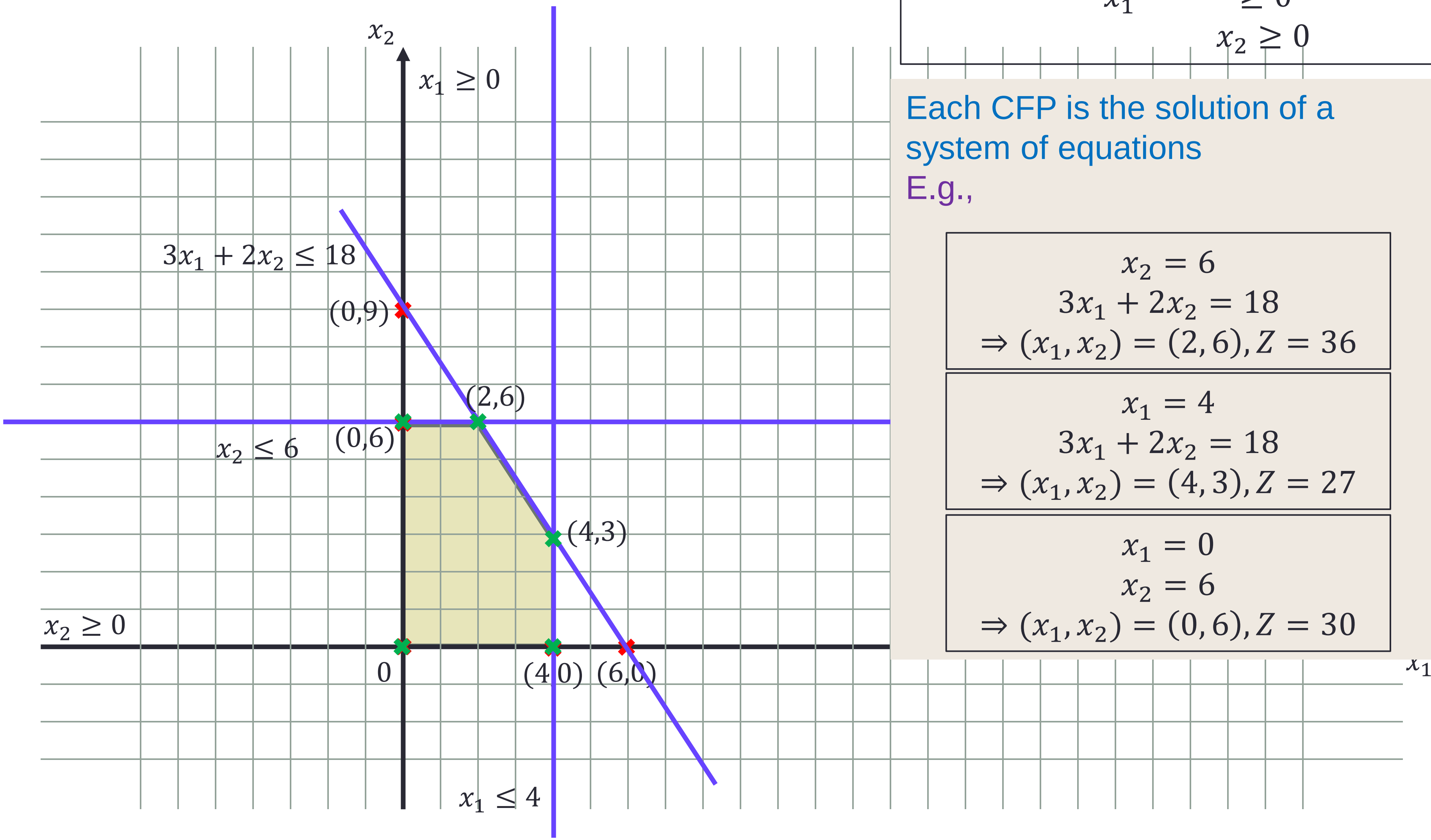
How to find CFPs?

How to find CFPs …

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \le 4$$
$$x_2 \le 6$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$



Each CFP is the solution of a system of equations

E.g.,

$$x_2 = 6$$
$$3x_1 + 2x_2 = 18$$
$$\Rightarrow (x_1, x_2) = (2, 6), Z = 36$$

$$x_1 = 4$$
$$3x_1 + 2x_2 = 18$$
$$\Rightarrow (x_1, x_2) = (4, 3), Z = 27$$

$$x_1 = 0$$
$$x_2 = 6$$
$$\Rightarrow (x_1, x_2) = (0, 6), Z = 30$$

Summary (what the graphic method tells us?)

Exactly one of the following scenarios can happen with a LP problem

1. **Infeasible** (empty feasible region)
2. **Unbounded**
3. Has an optimal solution
   If there are multiple optimal solutions, then the problem is said to be **degenerate**

If there is one optimal solution, then it must be a **corner feasible point (CFP)**
Note: If there are many optimal solutions, then one of them must be a **corner feasible point**

Questions:
1. Can we find an optimum corner feasible point without plotting a figure?

Implication: Can find the optimal solution by focusing on CFPs and ignoring the rest

How to find CFPs?

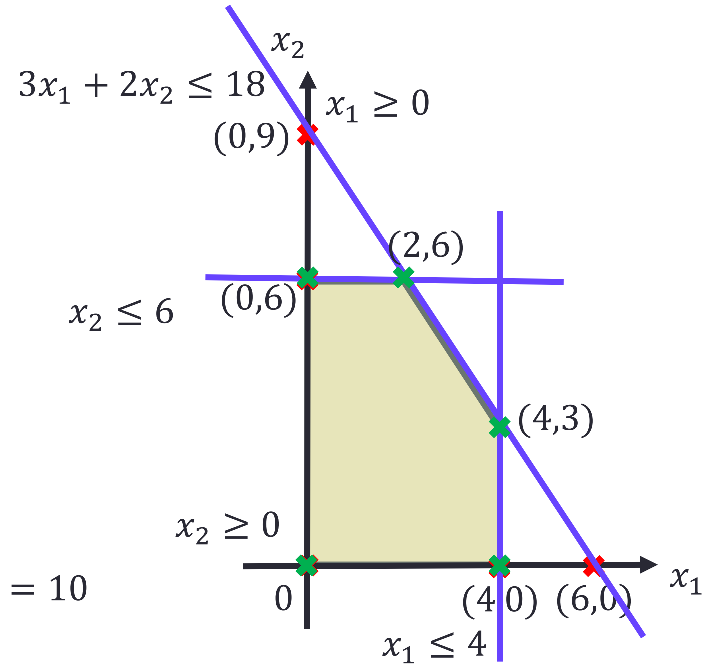
✓

A Solution Procedure for solving LP: Find all CFPs and output the one with the best objective value

A Solution Procedure for solving LP: Find all CFPs and output the one with the best objective value

Drawback: the number of CFPs can be hopelessly large

$$
\begin{array}{lrl}
 & \max\ Z = 3x_1 + 5x_2 & \\
\text{subject to:} & x_1 & \le 4 \\
 & x_2 & \le 6 \\
 & 3x_1 + 2x_2 & \le 18 \\
 & x_1 & \ge 0 \\
 & x_2 & \ge 0
\end{array}
$$

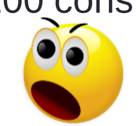


- What is the number of CFPs in the example? 5
- What is the number of possible CFPs in this case? (i.e., # of possible ways to choose two constraints) $\frac{5 \times 4}{2} = 10$

(question: why do we see only 5 CFPs)

- What is the maximum possible number of CFPs if there are 20 constraints? $\frac{20 \times 19}{2} = 190$
- What is the maximum possible number of CFPs if there are 30 variables and 100 constraints?

$$\frac{100 \times 99 \times \cdots \times 71}{30!} = 2.93 \times 10^{25}$$



- What if there are 10,000 variables and several thousand constraints?

Optimality Test

Consider an LP problem that possesses at least one optimal solution
(i.e., the problem is not infeasible, and not unbounded)

Optimality Test: If a CFP has no adjacent CFP with better objective value, then it must be an optimal solution

Graphic illustration: Any CFP that is not optimal must have an adjacent CFP with better objective value

- Suppose that A is a CFP and the objective is higher if the iso-line moves north-east
- Assume that another CFP, B, is the optimum, i.e., better than A (Fig 1)

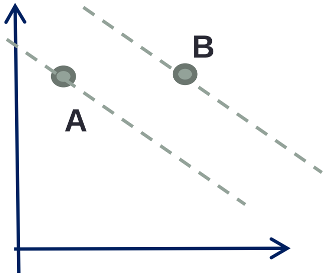


Fig 1

- If B is directly adjacent to A, then A has a better adjacent CFP (Fig 2)

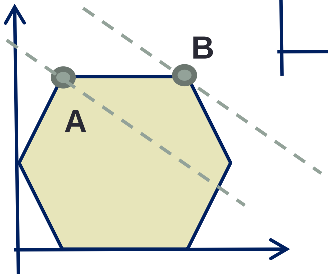


Fig 2

- Otherwise
  - Since A, B are on a polygon, A must have an adjacent CFP, say C, between A, B
  - The iso-line of C is to the north-east of that of A, so C has better objective value than A
  - Thus any point that is not optimal must have an adjacent CFP with better objective value

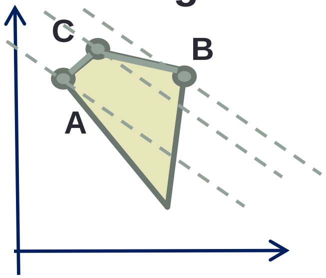


Fig 3

Optimality Test

Consider an LP problem that possesses at least one optimal solution
(i.e., the problem is not infeasible, and not unbounded)

Optimality Test: If a CFP has no adjacent CFP with better objective value, then it must be an optimal solution

Optimality test indicates that there is no need to compute the objective of all CFPs

- just find the one that is locally optimal (obj value at least as good as all its neighbors)

tremendous reduction in search space!

A New Solution Procedure

Consider an LP problem that possesses at least one optimal solution
(i.e., the problem is not infeasible, and not unbounded)

Optimality Test: If a CFP has no adjacent CFP with better objective value, then it must be an optimal solution

Optimality test indicates that there is no need to compute the objective of all CFPs

- just find the one that is locally optimal (obj value at least as good as all its neighbors)

tremendous reduction in search space!

Corresponding Solution Procedure:

Repeat

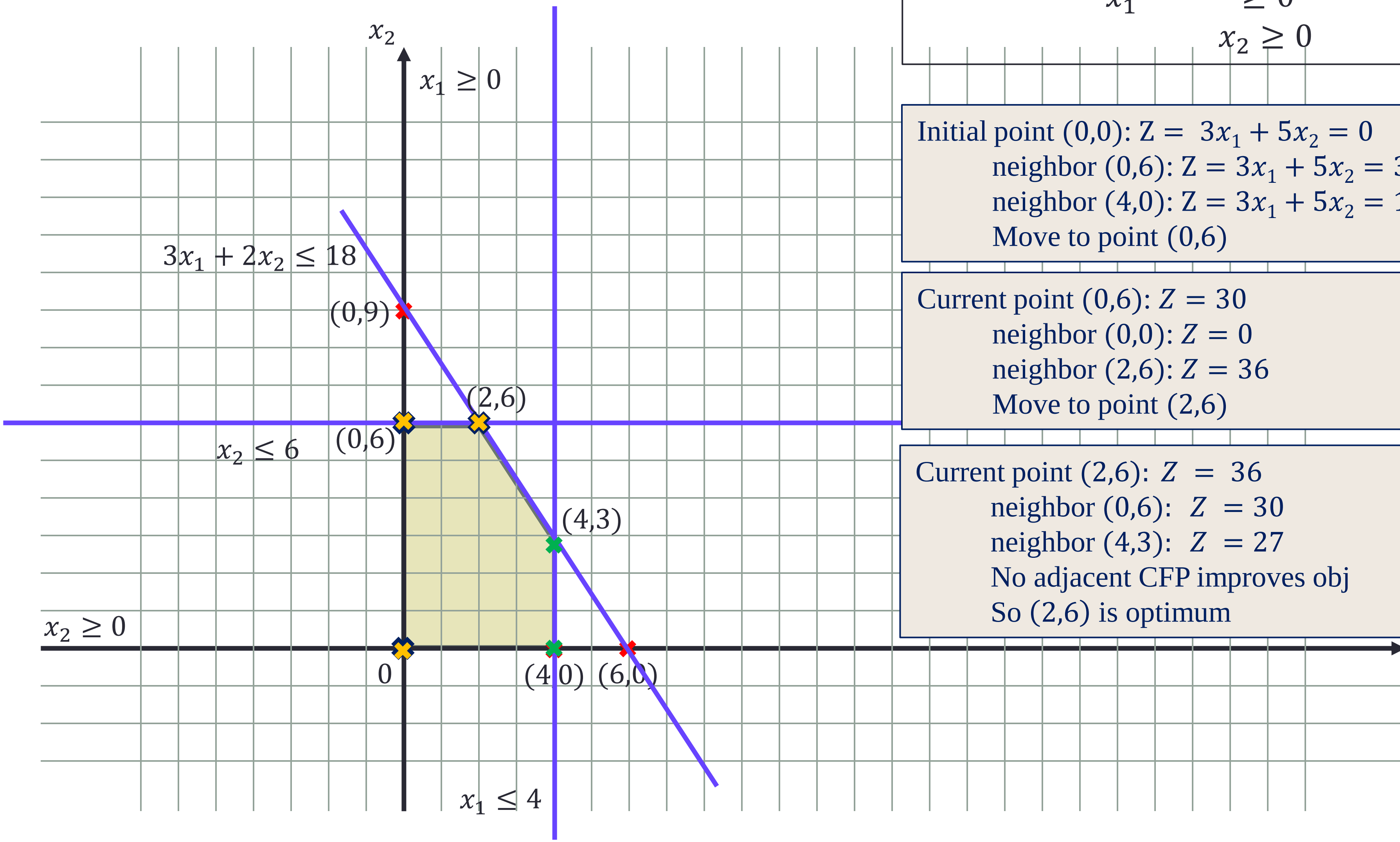
1. start from a CFP
2. compare current CFP's objective value with those of adjacent CFPs
3. if the current CFP has the best objective value in comparison to all its neighbors, then it is an optimal solution
   - STOP and output it
4. otherwise, move to the neighbor that has the highest objective value

New solution procedure

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$



Initial point (0,0): Z = $3x_1 + 5x_2 = 0$
neighbor (0,6): Z = $3x_1 + 5x_2 = 30$
neighbor (4,0): Z = $3x_1 + 5x_2 = 12$
Move to point (0,6)

Current point (0,6): $Z = 30$
neighbor (0,0): $Z = 0$
neighbor (2,6): $Z = 36$
Move to point (2,6)

Current point (2,6): $Z = 36$
neighbor (0,6): $Z = 30$
neighbor (4,3): $Z = 27$
No adjacent CFP improves obj
So (2,6) is optimum

SIMPLEX METHOD

… where we generalize this solution procedure for larger number of variables

Basic idea behind the Simplex Method

- Focus on CFPs
- Iterative: move from one CFP to another, repeat the process
- Initialization: start from a CFP (typically the origin)
- Compare objective only with adjacent CFPs (neighbors)
- Move to an adjacent CFP that yields the largest improvement in the objective value
  - Each move leads to an improvement, i.e., getting closer to the optimum
  - So the process will stop and the optimum (if it exists) will be found

1. What is a CFP?
2. How do we obtain adjacent CFPs?
3. Which adjacent CFP should we move to?
4. How to determine the adjacent CFP and its objective value?

These questions are trivial if you have a two-dimensional figure to look at, but not so obvious if all you have are symbols and inequalities

Linear Programming Formulation: Standard Form

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \\
x_1 \ge 0, x_2 \ge 0, \dots, x_n &\ge 0
\end{aligned}$$

- Objective should be maximization
- Constraints should be less than or equal to
  - With the linear function of variables appearing on the LHS and
  - Constant appearing on the RHS
- All variables have non-negativity constraints

1. WHAT IS A CFP?

Setting up for Simplex: Augment LP

- Start with LP in standard form

LP

$$\begin{aligned} &\max\ Z = 3x_1 + 5x_2 \\ \text{subject to: } & x_1 \le 4 \\ & x_2 \le 6 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

- Divide constraints into two categories: non-negativity and others
- For each of the other constraints, add a *non-negative variable* to the LHS and set it to equality

$$x_1 \le 4 \quad \longrightarrow x_1 + x_3 = 4, x_3 \ge 0$$

$$x_2 \le 6 \quad \longrightarrow x_2 + x_4 = 6, x_4 \ge 0$$

$$3x_1 + 2x_2 \le 18 \quad \longrightarrow 3x_1 + 2x_2 + x_5 = 18, x_5 \ge 0$$

  - The newly introduced variables, x_3, x_4, x_5 in this case, are referred to as **slack variables**
- Do not forget: Add non-negativity constraints for the slack variables

augmented LP

$$\begin{aligned} &\max\ Z = 3x_1 + 5x_2 \\ \text{subject to: } & x_1 + x_3 = 4 \\ & x_2 + x_4 = 6 \\ & 3x_1 + 2x_2 + x_5 = 18 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0 \end{aligned}$$

- Constraints are either non-negativity constraints or equalities
  - The resulting LP is referred to as **augmented LP**

Augmented LP solves original LP

Introducing slack variables and augmenting the LP does not change the problem

- If $(x_1, x_2, x_3, x_4, x_5)$ is feasible for the augmented LP, then (x_1, x_2) is feasible for the original problem
- If (x_1, x_2) is feasible for the original LP, then $(x_1, x_2, 4 - x_1, 6 - x_2, 18 - 3x_1 - 2x_2)$ is feasible for the augmented LP

original LP

$$
\begin{array}{ll}
 & \max\ Z = 3x_1 + 5x_2 \\
\text{s. t.:} & x_1 \le 4 \\
 & x_2 \le 6 \\
 & 3x_1 + 2x_2 \le 18 \\
 & x_1 \ge 0, x_2 \ge 0
\end{array}
$$

augmented LP

$$
\begin{array}{ll}
 & \max\ Z = 3x_1 + 5x_2 \\
\text{s. t.:} & x_1 + x_3 = 4 \\
 & x_2 + x_4 = 6 \\
 & 3x_1 + 2x_2 + x_5 = 18 \\
 & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0
\end{array}
$$

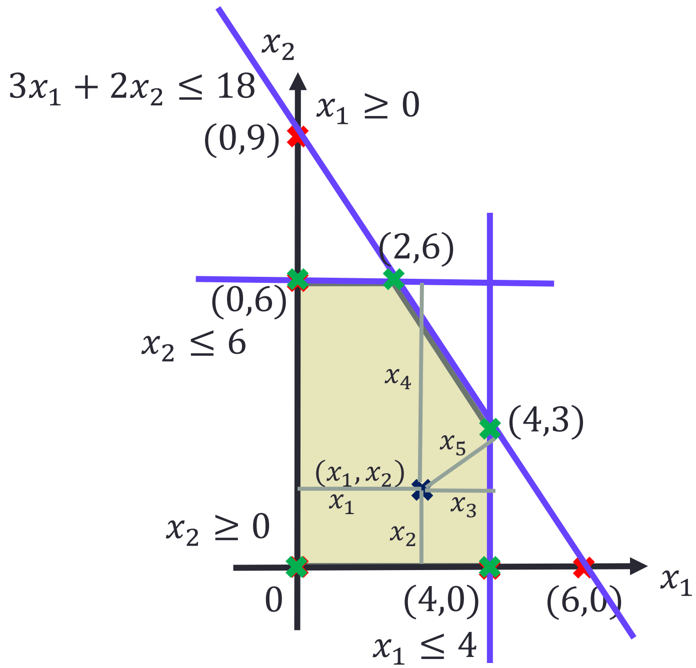
Slack variables (geometric intuition)

original LP

$$\begin{aligned} \max\ Z = 3x_1 + 5x_2 \\ \text{s. t.:} \quad x_1 \le 4 \\ x_2 \le 6 \\ 3x_1 + 2x_2 \le 18 \\ x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

augmented LP

$$\begin{array}{lllll} & & \max\ Z = 3x_1 + 5x_2 & & \\ \text{s. t.:} & x_1 & +x_3 & & = 4 \\ & x_2 & & +x_4 & = 6 \\ & 3x_1 + 2x_2 & & +x_5 & = 18 \\ & \multicolumn{4}{l}{x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0, x_5 \ge 0} \end{array}$$



- A variable x_i represents the distance from the line $x_i = 0$
- The slack variable for a constraint represents the distance from the line associated with the constraint

Note: This is only an intuition. It does not give the exact Euclidean distance.