Plan for today

- Integer Programming
  - How to Solve?
    - Branch and Bound
  - Fun with Formulations
    - Project Assistant Allocation
    - Map Coloring Problem
    - …

Announcements:

- Follow University Academic Integrity Policy: https://studentcode.illinois.edu/article1/part4/1-401/
- Exam 2:
  Release: 6pm, Thu, 5 May
  Due: 11pm, Fri, 6 May
  Duration: 3 hours
  Topics: Everything starting from (and including) Transportation Problem
- Review problems in Canvas
- Exam 2: Collaboration with one partner is allowed; both of you should indicate each other's names in your submission;
  (choose your collaborator wisely?)

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

INTEGER PROGRAMMING

… where we see the power of integer variables and algorithms to solve optimization problems involving integer variables

Integer Programming

An **integer programming (IP)** model is similar to a linear programming model, except all variables have to take integer values

- if all variables have to be either 0 or 1, the model is sometimes referred to **Binary IP (BIP)**
- if some variables have to be integers while others can take real values, the model is referred to **Mixed IP (MIP)**
- when both objective and constraints in an IP are linear, the IP model is also referred to as a **Integer Linear Program (ILP) or Mixed ILP (MILP)**
- If objectives or constraints are nonlinear then it is known as **non-linear IP**
- Non-linear IP is usually very difficult to solve (ILP is hard enough)

Linear Programming (LP)

$$\max\ Z = 3x_1 + 5x_2$$
$$3x_1 + 4x_2 \le 25$$
$$x_1 + x_2 \le 20$$
$$x_1 \ge 0, x_2 \ge 0$$

Integer Programming (IP)

$$\max\ Z = 3x_1 + 5x_2$$
$$3x_1 + 4x_2 \le 25$$
$$x_1 + x_2 \le 20$$
$$x_1, x_2 \text{ integers}$$

Binary Integer Programming (BIP)

$$\max\ Z = 3x_1 + 5x_2$$
$$3x_1 + 4x_2 \le 25$$
$$x_1 + x_2 \le 20$$
$$x_1, x_2 \in \{0,1\}$$

Mixed Integer Programming (MIP)

$$\max\ Z = 3x_1 + 5x_2$$
$$3x_1 + 4x_2 \le 25$$
$$x_1 + x_2 \le 20$$
$$x_1 \ge 0, x_2 \text{ integer}$$

Recall:
- A function is **linear** if it satisfies additivity and proportionality
- Constraint $f(x) \le b_i$ is linear if f is linear

$$\max\ Z = 3x_1^2 + 5x_1x_2$$
$$3x_1 + 4x_2 \le 25$$
$$x_1 + x_2 \le 20$$
$$x_1 \ge 0, x_2 \text{ integer}$$

Non-linear IP

POWER OF INTEGER VARIABLES

IP Formulations

… where we see the power of integer variables in mathematical modeling

IP: HOW TO SOLVE?

Caution: IPs are difficult to solve

SOLVING IPS

1. Pre-processing: Variable Fixing

Some simple pre-processing can go a long way in reducing the search space

SOLVING IPS

2. Branch and Bound Algorithm

(For maximization IPs)

Branch and Bound: branching

Suppose our problem is

$$\max Z = 3x_1 + x_2 + 4x_3 + 2x_4$$
$$4x_1 + 2x_2 + 3x_3 + 4x_4 \leq 12.5$$
$$2x_1 - 3x_2 + 4x_3 + 7x_4 \leq 8.2$$
$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

Branch:

- fix $x_1 = 0$ and solve the rest as a LP relaxation (i.e., drop integer constraints)

$$\max Z_0 = x_2 + 4x_3 + 2x_4$$
$$2x_2 + 3x_3 + 4x_4 \leq 12.5$$
$$-3x_2 + 4x_3 + 7x_4 \leq 8.2$$
$$x_2, x_3, x_4 \leq 1$$
$$x_2, x_3, x_4 \geq 0$$

- fix $x_1 = 1$ and solve the rest as a LP relaxation (i.e., drop integer constraints)

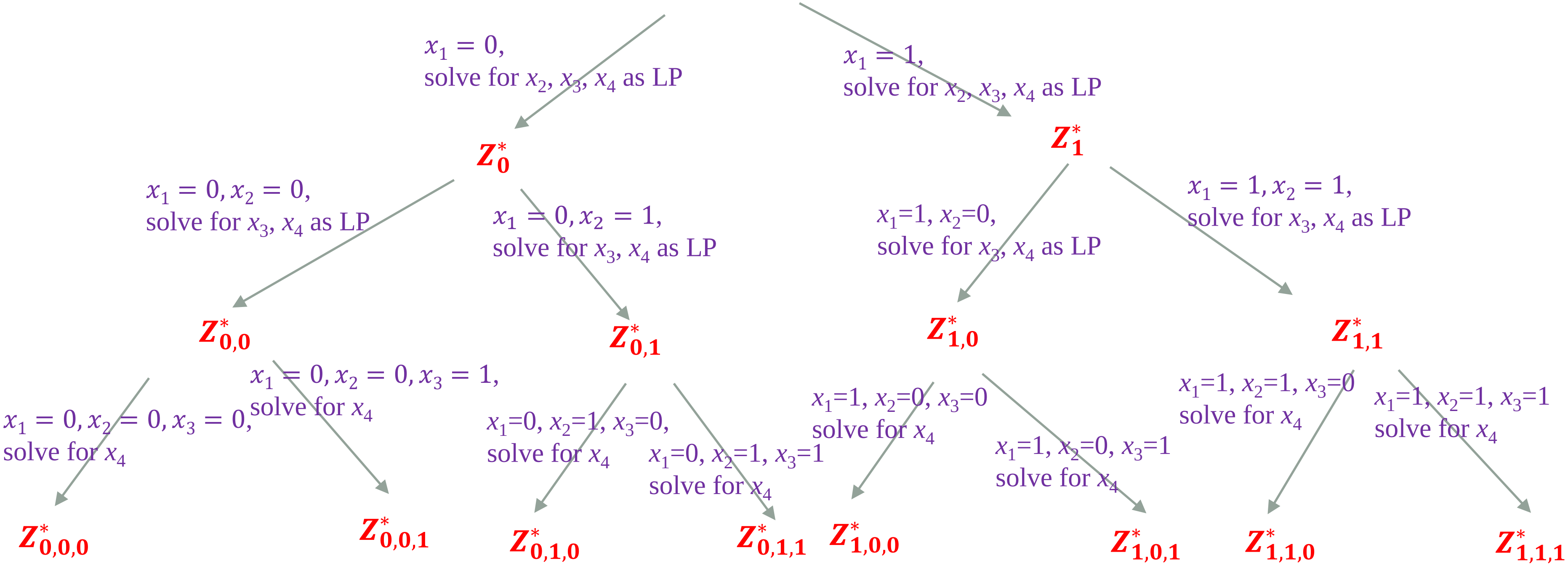
$$\max Z_1 = 3 + x_2 + 4x_3 + 2x_4$$
$$4 + 2x_2 + 3x_3 + 4x_4 \leq 12.5$$
$$2 - 3x_2 + 4x_3 + 7x_4 \leq 8.2$$
$$x_2, x_3, x_4 \leq 1$$
$$x_2, x_3, x_4 \geq 0$$

Effectively dividing the entire integer feasible solution space into two sets:

- Set 0 contains all integer feasible solutions that has $x_1 = 0$, and
  Set 1 contains all integer feasible solutions that has $x_1 = 1$
- Z_0^* is an upper bound on the objective value of all solutions in set 0, and
  Z_1^* is an upper bound on the objective value of all solutions in set 1

Branch and Bound: branching

$$\max Z = 3x_1 + x_2 + 4x_3 + 2x_4$$
$$4x_1 + 2x_2 + 3x_3 + 4x_4 \leq 12.5$$
$$2x_1 - 3x_2 + 4x_3 + 7x_4 \leq 8.2$$
$$x_1, x_2, x_3, x_4 \in \{0,1\}$$



Branch and Bound: fathoming

When can we stop exploring some branches?

1. If the subproblem is infeasible [Fathoming due to infeasibility]

2. If the subproblem has an integer optimal solution [Fathoming due to integrality]

3. If we find an integer feasible solution that dominates (has at least as much objective value as) the solution of a LP relaxation at some point, then the entire branch below that LP relaxation can be eliminated and there is no need to even solve the LPs in that branch

[Fathoming due to bound]

For instance,

If the objective value $Z_{x_1=1,x_2=1,x_3=1,x_4=0} > Z_{x_1=0,x_2=0}$, then there is no need to check any solution with $x_1 = 0, x_2 = 0$

B&B: Example of Fathoming

$Z_I = -\infty$
No incumbent

Solve LP-relaxation
(non-integer soln, $Z^* = 25$)

Solve Subproblem 0
(non-integer soln, $Z^* = 24$)

Solve Subproblem 1
(non-integer soln, $Z^* = 22$)

B&B: Example of Fathoming

$Z_I = -\infty$
No incumbent

$Z_I = 22$
Incumbent from subproblem 0,0

Solve LP-relaxation
(non-integer soln, $Z^* = 25$)

Solve Subproblem 0
(non-integer soln, $Z^* = 24$)

Solve Subproblem 1
(non-integer soln, $Z^* = 22$)

Fathom

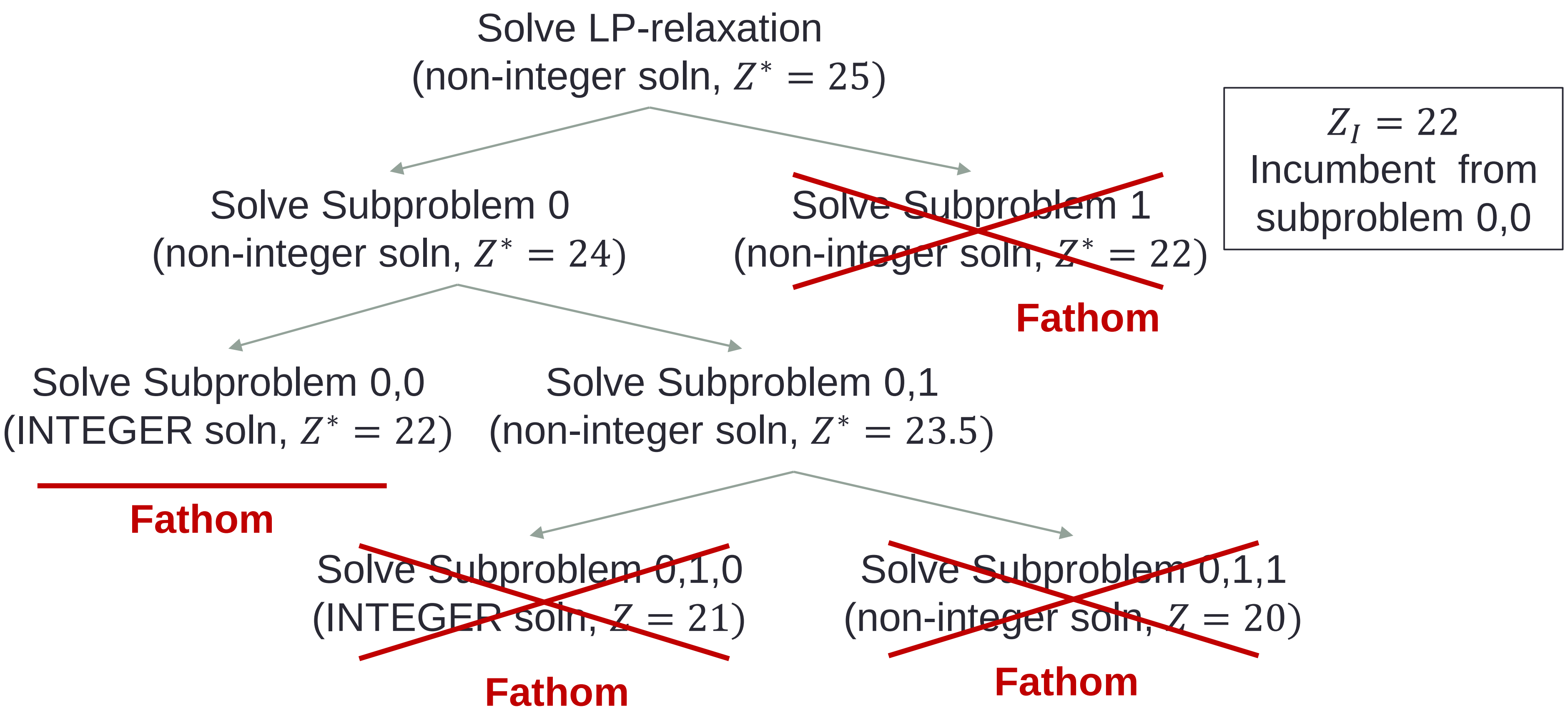
Solve Subproblem 0,0
(INTEGER soln, $Z^* = 22$)

Fathom

Solve Subproblem 0,1
(non-integer soln, $Z^* = 23.5$)

New incumbent from subproblem 0,0.
So check all remaining subproblems:
Subproblem 0,0: Fathomed
Subproblem 0,1: Not fathomed
Subproblem 1: Fathomed

B&B: Example of Fathoming



No more unfathomed subproblems
Optimal solution is the incumbent and has objective value $Z_I^* = 22$

B&B: A complete example

$$\max Z = 16x_1 + 22x_2 + 12x_3 + 8x_4$$
$$5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14$$
$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

best integer solution so far:
$(x_1, x_2, x_3, x_4) = (0,1,1,1)$
$Z_I = 42$

$$\max Z = 16x_1 + 22x_2 + 12x_3 + 8x_4$$
$$5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14$$
$$0 \le x_1, x_2, x_3, x_4 \le 1$$
$$(x_1^*, x_2^*, x_3^*, x_4^*) = \left(1,1,\tfrac{1}{2},0\right),\ Z_1^* = 44$$

$x_1 = 0$ | $x_1 = 1$

Branch $x_1 = 0$:

$$\max Z_0 = 22x_2 + 12x_3 + 8x_4$$
$$7x_2 + 4x_3 + 3x_4 \le 14$$
$$0 \le x_2, x_3, x_4 \le 1$$
$$(x_1, x_2^*, x_3^*, x_4^*) = (0,1,1,1),\ Z_0^* = 42$$

- a feasible solution for IP
- best solution for those with x_1=0
- no need to proceed down this branch

Keep 42 as the best integer solution

fathom

Branch $x_1 = 1$:

$$\max Z_1 = 16 + 22x_2 + 12x_3 + 8x_4$$
$$7x_2 + 4x_3 + 3x_4 \le 9$$
$$0 \le x_2, x_3, x_4 \le 1$$
$$(x_1, x_2^*, x_3^*, x_4^*) = \left(1,1,\tfrac{1}{2},0\right),\ Z_1^* = 44$$

- not feasible for IP since x_3 is fractional
- Obj value is better than the current best integer solution

continue to explore this branch

B&B (exploring the branch of $x_1 = 1$)

best integer solution so far:

$(x_1, x_2, x_3, x_4) = (0,1,1,1)$

$Z_I = 42$

$$\max Z_1 = 16 + 22x_2 + 12x_3 + 8x_4$$
$$7x_2 + 4x_3 + 3x_4 \leq 9$$
$$0 \leq x_2, x_3, x_4 \leq 1$$

Branch $x_2 = 0$:

$$\max Z_{1,0} = 16 + 12x_3 + 8x_4$$
$$4x_3 + 3x_4 \leq 9$$
$$0 \leq x_3, x_4 \leq 1$$

$(x_1, x_2, x_3^*, x_4^*) = (1,0,1,1)$, $Z_{1,0}^* = 36$

- feasible for IP
- Obj is below the current best solution
- no need to proceed down this branch

fathom

Branch $x_2 = 1$:

$$\max Z_{1,1} = 38 + 12x_3 + 8x_4$$
$$4x_3 + 3x_4 \leq 2$$
$$0 \leq x_3, x_4 \leq 1$$

$(x_1, x_2, x_3^*, x_4^*) = \left(1,1,\frac{1}{2},0\right)$, $Z_{1,1}^* = 44$

- not feasible for IP since x_3 is fractional
- Obj is better than the current best integer solution

continue to explore this branch

B&B (exploring the branch of $x_1 = 1, x_2 = 1$)

best integer solution so far:
$(x_1, x_2, x_3, x_4) = (0,1,1,1)$
$Z_I = 42$

$\max Z_{1,1} = 38 + 12x_3 + 8x_4$
$4x_3 + 3x_4 \leq 2$
$0 \leq x_3, x_4 \leq 1$

$x_3 = 0$

$\max Z_{1,1,0} = 38 + 8x_4$
$3x_4 \leq 2$
$0 \leq x_4 \leq 1$

$(x_1, x_2, x_3, x_4^*) = (1,1,0,\frac{2}{3}),\ Z_{1,1,0}^* = 44.33$

- not feasible for IP
- obj above the current best solution

continue to explore this branch

x_4=0

$(x_1, x_2, x_3, x_4) = (1,1,0,0)$
$Z_{1,1,0,0}^* = 38$

x_4=1

Infeasible

$x_3 = 1$

$\max Z_{1,1,1} = 50 + 8x_4$
$3x_4 \leq -2$
$0 \leq x_4 \leq 1$

LP relaxation is infeasible

- no need to further explore

fathom

Optimum solution is
$(x_1, x_2, x_3, x_4) = (0,1,1,1)$
$Z_I^* = 42$

Branch and Bound Algorithm

- For a IP that is a maximization problem

- The B & B algorithm stores the best integer solution found so far
  - This solution is called the ***incumbent***, and its objective is Z_I
  - When a new incumbent is found, check all remaining subproblems to see if they can be fathomed (i.e., their objective is not better than Z_I)

- Initialization: There is no incumbent, so $Z_I = -\infty$
  - Solve the LP-relaxation, suppose Z^* is its objective
    - If it has an integer solution, then algorithm terminates
    - If it does not have an integer solution, label the LP relaxation as an unbranched subproblem with objective value Z^*

Branch and Bound Algorithm

- Iteration of B&B:

1. Branch on the most recently created subproblem (if there is a tie, choose the one with the larger bound) and solve the two new branched subproblems
2. Four cases for subproblem j (for $j = 0,1$); suppose its optimal objective value is Z_j^*:
    1. If it has no feasible solutions, fathom it
    2. If it has $Z_j^* \leq Z_I$, fathom it
    3. If its solution is integral, and has $Z_j^* > Z_I$
        - This solution is the new incumbent, so let $Z_I = Z_j^*$
        - For every remaining subproblem, fathom it if it has objective value $\leq Z_I$
        - Fathom the incumbent
    4. If its solution is non-integer, but it has $Z_j^* > Z_I$, then do not fathom it
3. If any unfathomed subproblems remain, return to step 1
4. Otherwise terminate and return the incumbent as the optimum

Other ways to branch

- Instead of branching on $x_1, x_2, x_3, \ldots$ in a sequential fashion we could also branch on the variable that takes the most fractional value in the LP optimum solution
- Various other branching rules are also practiced
- Note: Branch and Bound could potentially explore all 2^n possibilities, but it terminates with much fewer explorations for IPs encountered in practice

FUN WITH FORMULATIONS

Flaunt (or test) your formulation skills!

PROJECT ASSISTANT ALLOCATION PROBLEM

Project Assistant Allocation Problem

- Three teams work in parallel to complete a project
- Each team may fail
- The entire project fails if “all” 3 teams fail
- Two new scientists are available
- Adding new scientist(s) to a team reduces its failure probability (as shown)

Question: How many scientists should join each team to minimize the project’s failure probability?

No. of new Scientists	**Probability of Failure**		
	Team		
	1	2	3
0	0.40	0.60	0.80
1	0.20	0.40	0.50
2	0.15	0.20	0.30

Project Assistant Allocation Problem

No. of new Scientists	Probability of Failure		
	Team		
	1	2	3
0	0.40	0.60	0.80
1	0.20	0.40	0.50
2	0.15	0.20	0.30

Step 1: identify decision variables

x_{ij} ≔ Indicator variable to indicate if team i is allocated j scientists for $i = 1, 2, 3$, and $j = 0, 1, 2$

Step 2: determine the objective function

$$\min Z = (0.4^{x_{10}} 0.2^{x_{11}} 0.15^{x_{12}})(0.6^{x_{20}} 0.4^{x_{21}} 0.2^{x_{22}})(0.8^{x_{30}} 0.5^{x_{31}} 0.3^{x_{32}})$$

Obj function is non-linear!

How to linearize?

Obs. Equivalent to minimizing $\log(Z)$

$$\min \log Z = \sum_{i=1}^{3} \sum_{j=0}^{2} (\log p_{ij}) x_{ij}$$

where p_{ij} is the probability of failure of team i with j allocated scientists

Step 3: identify constraints

$x_{ij} \in \{0,1\}$ for every $i = 1, 2, 3$ and $j = 0, 1, 2$

$\sum_{j=0}^{2} x_{ij} = 1$ for every $i = 1, 2, 3$ } Each team i is assigned either 0 or 1 or 2 scientists

$$0x_{10} + 1x_{11} + 2x_{12} + 0x_{20} + 1x_{21} + 2x_{22} + 0x_{30} + 1x_{31} + 2x_{32} \leq 2$$

The total number of allocated scientists is at most 2

Project Assistant Allocation Problem

No. of new Scientists	Probability of Failure		
	Team		
	1	2	3
0	0.40	0.60	0.80
1	0.20	0.40	0.50
2	0.15	0.20	0.30

$$\min \sum_{i=1}^{3} \sum_{j=0}^{2} (\log p_{ij}) x_{ij}$$

$x_{ij} \in \{0,1\}$ for every $i = 1, 2, 3$ and $j = 0, 1, 2$

$\sum_{j=0}^{2} x_{ij} = 1$ for every $i = 1, 2, 3$

$$0x_{10} + 1x_{11} + 2x_{12} + 0x_{20} + 1x_{21} + 2x_{22} + 0x_{30} + 1x_{31} + 2x_{32} \leq 2$$

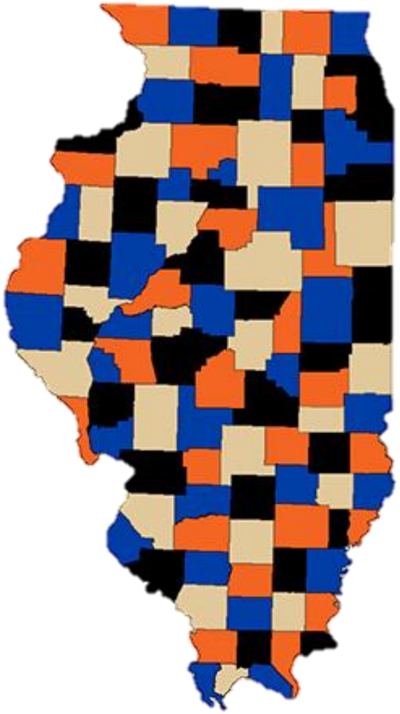
← BILP formulation

MAP COLORING

… for cartographers

Map Coloring

- Here is the county map of the state of Illinois



Goal: minimum number of colors to color the counties so that no two counties that share a border have the same color

Map Coloring

Step 1: identify decision variables

Step 2: determine the objective function

Step 3: identify constraints

	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_1		×	×		×		×
E_2	×				×	×	×
E_3	×					×	
E_4							×
E_5	×	×					
E_6		×	×				×
E_7		×				×	