IE 310: Deterministic Models in Optimization

- Linear Programming Formulations
  - Definition
  - Standard Form
  - History
  - More Formulations

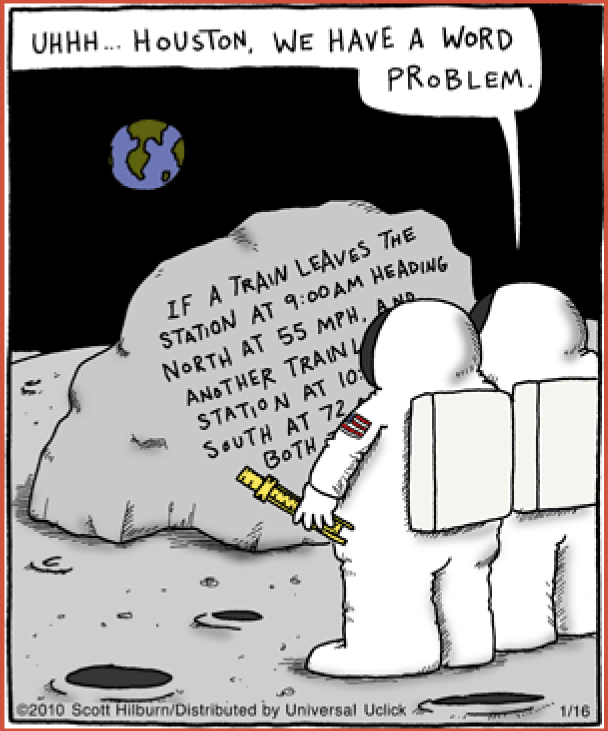
Announcements:

- You have the control on mute/unmute button in Zoom
- HW 1 due on Tue

- Linearization Techniques
  1. Absolute values
  2. Ratio of variables
  3. Maximizing the minimum

- How to solve LPs?
  - Graphical method
    - Plotting constraints

Recap



FORMULATIONS

… where we model problems mathematically

- So that we can use standard solution techniques

SUDOKU

Sudoku

Given: a partially filled table

Each cell to be filled with a number from 1 to 9
Each number can appear exactly once
- in each row
- in each col
- in each subtable

Goal: Fill numbers in cells to satisfy the above rules

Subtable

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Sudoku

Subtable k

$(1,1);k$	$(1,2);k$	$(1,3);k$
$(2,1);k$	$(2,2);k$	$(2,3);k$
$(3,1);k$	$(3,2);k$	$(3,3);k$

(1,1); 1	(1,2); 1	(1,3); 1	(1,1); 2	(1,2); 2	(1,3); 2	(1,1); 3	(1,2); 3	(1,3); 3
(2,1); 1	(2,2); 1	(2,3); 1	(2,1); 2	(2,2); 2	(2,3); 2	(2,1); 3	(2,2); 3	(2,3); 3
(3,1); 1	(3,2); 1	(3,3); 1	(3,1); 2	(3,2); 2	(3,3); 2	(3,1); 3	(3,2); 3	(3,3); 3
(1,1); 4	(1,2); 4	(1,3); 4	(1,1); 5	(1,2); 5	(1,3); 5	(1,1); 6	(1,2); 6	(1,3); 6
(2,1); 4	(2,2); 4	(2,3); 4	(2,1); 5	(2,2); 5	(2,3); 5	(2,1); 6	(2,2); 6	(2,3); 6
(3,1); 4	(3,2); 4	(3,3); 4	(3,1); 5	(3,2); 5	(3,3); 5	(3,1); 6	(3,2); 6	(3,3); 6
(1,1); 7	(1,2); 7	(1,3); 7	(1,1); 8	(1,2); 8	(1,3); 8	(1,1); 9	(1,2); 9	(1,3); 9
(2,1); 7	(2,2); 7	(2,3); 7	(2,1); 8	(2,2); 8	(2,3); 8	(2,1); 9	(2,2); 9	(2,3); 9
(3,1); 7	(3,2); 7	(3,3); 7	(3,1); 8	(3,2); 8	(3,3); 8	(3,1); 9	(3,2); 9	(3,3); 9

Step 1: identify decision variables

$x^m_{(i,j);k}$ ≔ Indicator variable for whether cell $(i,j);k$ contains number m

for $i = 1,2,3$, $j = 1,2,3$, $k = 1,\ldots,9$, and $m = 1,\ldots,9$

Step 2: determine the objective function

$$\max 0x^1_{(1,1);1}$$

Step 3: identify constraints

$\sum_{k=1}^{3}\sum_{j=1}^{3} x^m_{(i,j);k} = 1$ for every $i = 1,2,3$, $m = 1,\ldots,9$

$\sum_{k=4}^{6}\sum_{j=1}^{3} x^m_{(i,j);k} = 1$ for every $i = 1,2,3$, $m = 1,\ldots,9$

$\sum_{k=7}^{9}\sum_{j=1}^{3} x^m_{(i,j);k} = 1$ for every $i = 1,2,3$, $m = 1,\ldots,9$

Every number m appears in each row exactly once

$\sum_{k\in\{1,4,7\}}\sum_{i=1}^{3} x^m_{(i,j);k} = 1$ for every $j = 1,2,3$, $m = 1,\ldots,9$

$\sum_{k\in\{2,5,8\}}\sum_{i=1}^{3} x^m_{(i,j);k} = 1$ for every $j = 1,2,3$, $m = 1,\ldots,9$

$\sum_{k\in\{2,5,8\}}\sum_{i=1}^{3} x^m_{(i,j);k} = 1$ for every $j = 1,2,3$, $m = 1,\ldots,9$

Every number m appears in each col exactly once

… More constraints

Sudoku

Subtable k

$(1,1);k$	$(1,2);k$	$(1,3);k$
$(2,1);k$	$(2,2);k$	$(2,3);k$
$(3,1);k$	$(3,2);k$	$(3,3);k$

$(1,1);1$	$(1,2);1$	$(1,3);1$	$(1,1);2$	$(1,2);2$	$(1,3);2$	$(1,1);3$	$(1,2);3$	$(1,3);3$
$(2,1);1$	$(2,2);1$	$(2,3);1$	$(2,1);2$	$(2,2);2$	$(2,3);2$	$(2,1);3$	$(2,2);3$	$(2,3);3$
$(3,1);1$	$(3,2);1$	$(3,3);1$	$(3,1);2$	$(3,2);2$	$(3,3);2$	$(3,1);3$	$(3,2);3$	$(3,3);3$
$(1,1);4$	$(1,2);4$	$(1,3);4$	$(1,1);5$	$(1,2);5$	$(1,3);5$	$(1,1);6$	$(1,2);6$	$(1,3);6$
$(2,1);4$	$(2,2);4$	$(2,3);4$	$(2,1);5$	$(2,2);5$	$(2,3);5$	$(2,1);6$	$(2,2);6$	$(2,3);6$
$(3,1);4$	$(3,2);4$	$(3,3);4$	$(3,1);5$	$(3,2);5$	$(3,3);5$	$(3,1);6$	$(3,2);6$	$(3,3);6$
$(1,1);7$	$(1,2);7$	$(1,3);7$	$(1,1);8$	$(1,2);8$	$(1,3);8$	$(1,1);9$	$(1,2);9$	$(1,3);9$
$(2,1);7$	$(2,2);7$	$(2,3);7$	$(2,1);8$	$(2,2);8$	$(2,3);8$	$(2,1);9$	$(2,2);9$	$(2,3);9$
$(3,1);7$	$(3,2);7$	$(3,3);7$	$(3,1);8$	$(3,2);8$	$(3,3);8$	$(3,1);9$	$(3,2);9$	$(3,3);9$

Step 1: identify decision variables

$x^m_{(i,j);k}$:= Indicator variable for whether cell $(i,j);k$ contains number m

for $i = 1,2,3$, $j = 1,2,3$, $k = 1,\dots,9$, and $m = 1,\dots,9$

Step 2: determine the objective function

$$\max 0x^1_{(1,1);1}$$

Step 3: identify constraints

… More constraints

$\sum_{i=1}^{3}\sum_{j=1}^{3} x^m_{(i,j);k} = 1$ for every $k, m = 1,\dots,9$ — Every number m appears in each subtable k exactly once

$\sum_{m=1}^{9} x^m_{(i,j);k} = 1$ for every $i,j = 1,2,3$, $k = 1,\dots,9$ — Every cell $(i,j);k$ has a number in it

$x^3_{(1,2);1} = 1$

$\vdots$

— Partially filled table entries

Sudoku

Not a linear programming formulation

owing to these constraints

$$\max 0x^{1}_{(1,1);1}$$

$x^{m}_{(i,j);k} \in \{0,1\}$ for $i,j = 1,2,3, m,k = 1,\dots,9$

$\sum_{k=1}^{3}\sum_{j=1}^{3} x^{m}_{(i,j);k} = 1$ for every $i = 1,2,3,\ m = 1,\dots,9$

$\sum_{k=4}^{6}\sum_{j=1}^{3} x^{m}_{(i,j);k} = 1$ for every $i = 1,2,3,\ m = 1,\dots,9$

$\sum_{k=7}^{9}\sum_{j=1}^{3} x^{m}_{(i,j);k} = 1$ for every $i = 1,2,3,\ m = 1,\dots,9$

$\sum_{k\in\{1,4,7\}}\sum_{i=1}^{3} x^{m}_{(i,j);k} = 1$ for every $j = 1,2,3,\ m = 1,\dots,9$

$\sum_{k\in\{2,5,8\}}\sum_{i=1}^{3} x^{m}_{(i,j);k} = 1$ for every $j = 1,2,3,\ m = 1,\dots,9$

$\sum_{k\in\{2,5,8\}}\sum_{i=1}^{3} x^{m}_{(i,j);k} = 1$ for every $j = 1,2,3,\ m = 1,\dots,9$

$\sum_{i=1}^{3}\sum_{j=1}^{3} x^{m}_{(i,j);k} = 1$ for every $k,m = 1,\dots,9$

$\sum_{m=1}^{9} x^{m}_{(i,j);k} = 1$ for every $i,j = 1,2,3, k = 1,\dots,9$

$x^{3}_{(1,2);1} = 1$

$\vdots$

Partially filled table entries

LINEAR PROGRAMMING FORMULATION

… where we formally define "Linear Programming Formulations" and see its standard form

Linear Programming Formulation

- A Linear Programming Formulation is an optimization problem formulation in which
  - the objective function and
  - all constraints

  are linear functions of decision variables, which are real variables

- More specifically, in a linear programming, the objective function and all constraints satisfy the following four conditions
  - Proportionality
  - Additivity
  - Divisibility
  - Certainty

LINEAR PROGRAMMING FORMULATION EXAMPLES

… where we see more examples of problems that can be formulated as LPs

RADIATION THERAPY PROBLEM

Example 3: Radiation Therapy

- Cancer treatment machine sends ionized radiation beam through a patient's body
- Use two beams, determine dosage (kilorad) to satisfy the following requirements

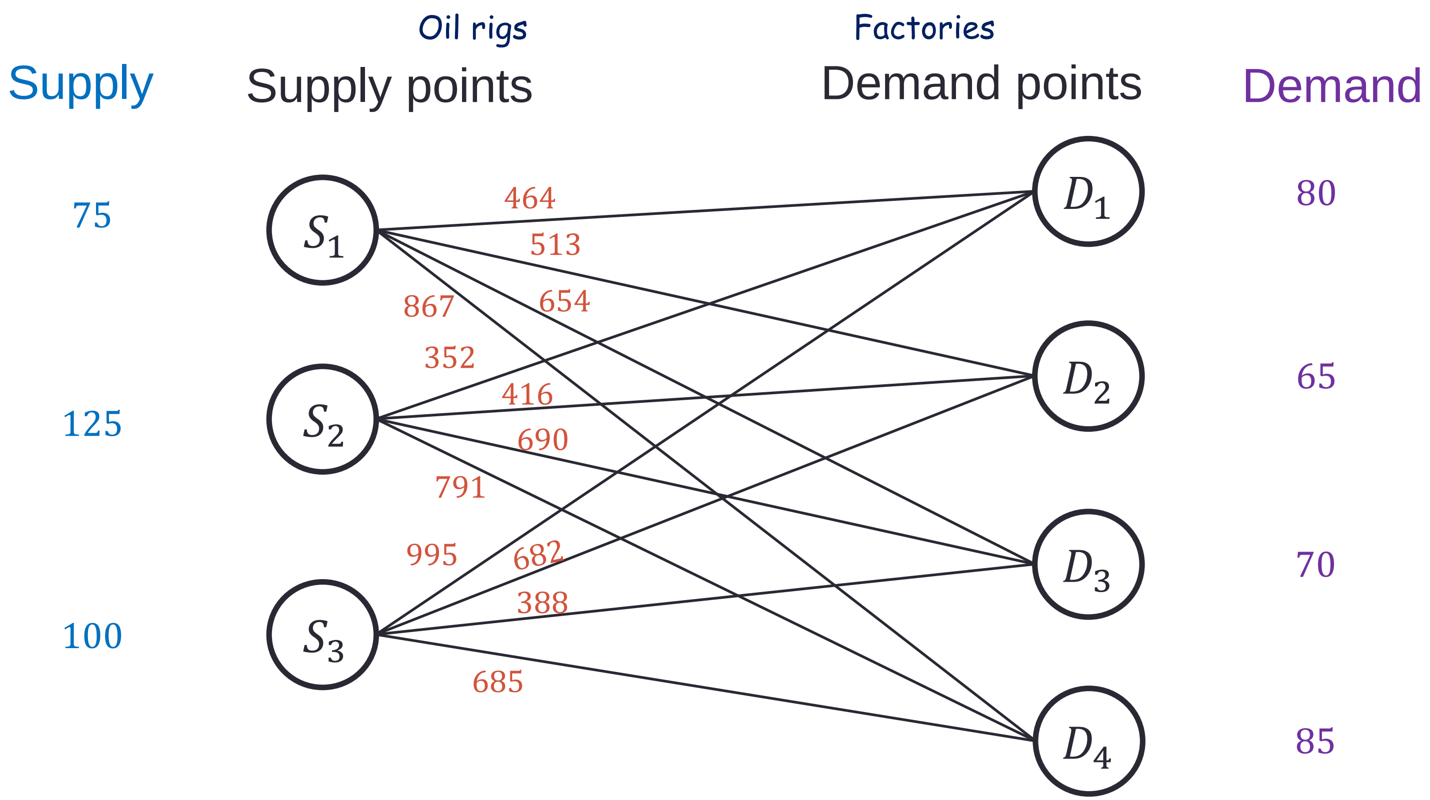
	Fraction of Entry Dose Absorbed by area		
area	Beam 1	Beam 2	Restriction on Total Average Dosage
Healthy Tissues	0.4	0.5	Minimize
Critical Tissues	0.3	0.1	≤2.7
Tumor Tissues	0.5	0.5	=6
Tumor Center Tissues	0.6	0.4	≥6

- Step 1: Decision variables x_1: dosage of beam 1, x_2: dosage of beam 2
- Step 2: Objective function $\min Z = 0.4\, x_1 + 0.5\, x_2$
- Step 3: Constraints

$$
\begin{aligned}
&0.3\, x_1 + 0.1\, x_2 \leq 2.7 \\
&0.5\, x_1 + 0.5\, x_2 = 6 \\
&0.6\, x_1 + 0.4\, x_2 \geq 6 \\
&x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

TRANSPORTATION PROBLEM

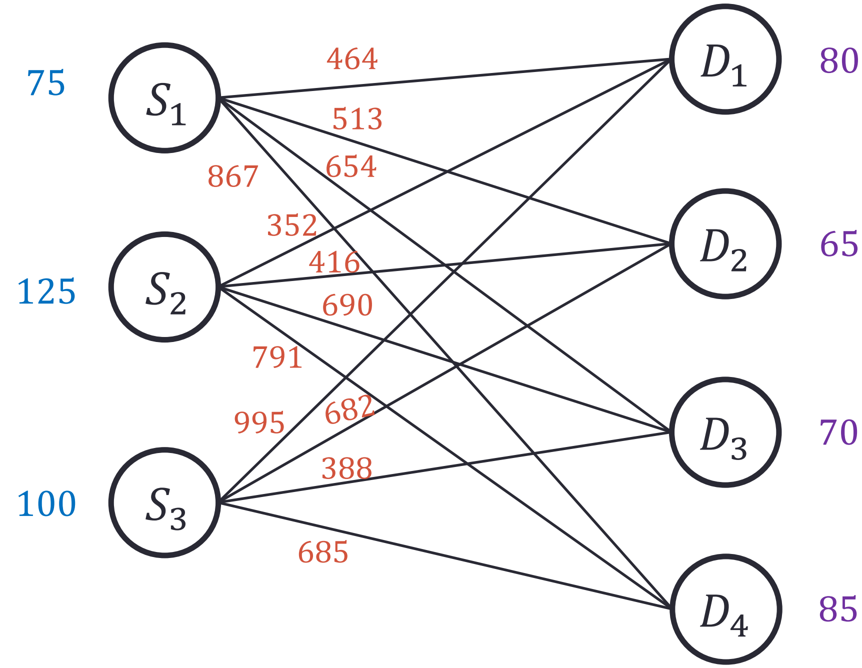
Transportation Problem



- Single product, say oil, needs to be transported from supply to demand
- Supply constraints: Entire supply at every S_i must be distributed among demand points
- Demand constraints: Entire demand at every D_j must be received from supply points
- Transportation Cost per unit from S_i to D_j is $c_{ij} (\geq 0)$ for all $i \in \{1,2,3\}, j \in \{1,2,3,4\}$

Question: How much to transport from each supply point to each demand point to minimize cost?

Formulation



Step 1: identify decision variables

x_{ij}: quantity to be transported
from S_i to D_j, $i \in \{1,2,3\}, j \in \{1,2,3,4\}$

Step 2: determine the objective function

$$\min Z = 464\, x_{11} + 513\, x_{12} + 654\, x_{13} + 867\, x_{14} + 352\, x_{21} + 416\, x_{22} + 690\, x_{23} + 791\, x_{24} + 995\, x_{31} + 682\, x_{32} + 388\, x_{33} + 685\, x_{34}$$

Step 3: identify constraints

$$x_{11} + x_{12} + x_{13} + x_{14} = 75$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 125$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 100$$

Supply constraints: Total outgoing from a supply point to all demand points should be equal to the supply at that point

$$x_{11} + x_{21} + x_{31} = 80$$
$$x_{12} + x_{22} + x_{32} = 65$$
$$x_{13} + x_{23} + x_{33} = 70$$
$$x_{14} + x_{24} + x_{34} = 85$$

Demand constraints: Total incoming into a demand point from all supply points should be equal to the demand requirement at that point

$$x_{ij} \geq 0, i = 1,2,3, j = 1,2,3,4$$

Formulation

$$\min Z = 464\,x_{11} + 513\,x_{12} + 654\,x_{13} + 867\,x_{14}$$
$$+352\,x_{21} + 416\,x_{22} + 690\,x_{23} + 791\,x_{24}$$
$$+995\,x_{31} + 682\,x_{32} + 388\,x_{33} + 685\,x_{34}$$

$$x_{11} + x_{12} + x_{13} + x_{14} = 75$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 125$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 100$$
$$x_{11} + x_{21} + x_{31} = 80$$
$$x_{12} + x_{22} + x_{32} = 65$$
$$x_{13} + x_{23} + x_{33} = 70$$
$$x_{14} + x_{24} + x_{34} = 85$$
$$x_{ij} \geq 0, i = 1,2,3, j = 1,2,3,4$$

BANK LOAN MANAGEMENT

Bank Loan Management Problem

- Bank makes four kinds of loans with the following rates of interest:
  - Level-1 mortgage: 14%
  - Level-2 mortgage: 20%
  - Home Improvement: 20%
  - Personal overdraft: 10%

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- The bank has a lending capability of $250 million and is constrained by the following policies (policies are in place to mitigate risks while investing):
  - Level-1 mortgages must be at least $55 million
  - Level-2 mortgages cannot exceed $125 million

Question: What is the best mix to lend to maximize interest?

Formulation

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- Total available: $250 million
- Level-1 mortgages must be at least 55 million dollars
- Level-2 mortgages cannot exceed 125 million dollars

Step 1: identify decision variables

x_i: amount in millions of dollars loaned in type i for $i = 1,2,3,4$

Step 2: determine the objective function

$$\max Z(x_1, x_2, x_3, x_4) = 0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$$

Step 3: identify constraints

$$x_1 + x_2 + x_3 + x_4 \leq 250$$
$$x_1 \geq 55$$
$$x_2 \leq 125$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Formulation

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- Total available: $250 million
- Level-1 mortgages must be at least 55 million dollars
- Level-2 mortgages cannot exceed 125 million dollars

$$\max Z(x_1, x_2, x_3, x_4) = 0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$$

$$x_1 + x_2 + x_3 + x_4 \leq 250$$
$$x_1 \geq 55$$
$$x_2 \leq 125$$
$$x_1, x_2, x_3, x_4 \geq 0$$

LINEAR PROGRAMMING FORMULATION EXAMPLES

Keep an eye out:

- more formulations throughout the course

STANDARD FORM OF LINEAR PROGRAMMING FORMULATION

"Standard form" so that "standard solution techniques" can be developed and applied

Example:
$x^2 = 5x + 3$ is a quadratic equation but it is not convenient to solve.
To solve, you first bring it into standard form: $x^2 - 5x - 3 = 0$.

Compare the formulations of Examples 3 and 2

$$\begin{aligned} &\min Z = 0.4x_1 + 0.5x_2 \\ \text{subject to:}\quad & 0.3x_1 + 0.1x_2 \leq 2.7 \\ & 0.5x_1 + 0.5x_2 = 6 \\ & 0.6x_1 + 0.4x_2 \geq 6 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Example 3

$$\begin{aligned} &\max\ Z = 3x_1 + 5x_2 \\ \text{subject to:}\quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Example 2

- Differences?
  - Example 3's objective is minimization while example 2's objective is maximization
  - Example 3 has = and ≥ constraints while example 2 has ≤ constraints

- Transform the therapy formulation into the same format as that of Example 2

$$\begin{aligned} &\max Z = -0.4x_1 - 0.5x_2 \\ \text{subject to:}\quad & 0.3x_1 + 0.1x_2 \leq 2.7 \\ & 0.5x_1 + 0.5x_2 \leq 6 \\ & -0.5x_1 - 0.5x_2 \leq -6 \\ & -0.6x_1 - 0.4x_2 \leq -6 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

Observe that $0.5x_1 + 0.5x_2 = 6$ is equivalent to

$$0.5x_1 + 0.5x_2 \leq 6$$
$$0.5x_1 + 0.5x_2 \geq 6$$

which are equivalent to

$$0.5x_1 + 0.5x_2 \leq 6$$
$$-0.5x_1 - 0.5x_2 \leq -6$$

Linear Programming Formulation: Standard Form

Maximize $Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

subject to:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \\
x_1 \geq 0, x_2 \geq 0, \ldots, x_n &\geq 0
\end{aligned}
$$

- Objective should be maximization
- Constraints should be less than or equal to
  - With the linear function of variables appearing on the LHS and
  - Constant appearing on the RHS
- All variables have non-negativity constraints

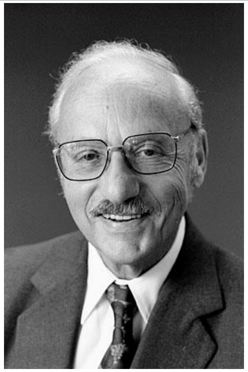
Linear Programming Formulation: Standard Form

Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

- Minimization objective can be converted into maximization objective
- Greater-than-or-equal-to constraints can be converted into less-than-or-equal-to constraints
- Equality constraints can be converted into two less-than-or-equal-to constraints



Linear Programs (LP): History

- Conceived by George B. Dantzig around 1947
- Historically speaking, Kantorovich (1939) was the first but his work was published only in 1959
- Title of Dantzig's first paper:

  "Programming in Linear Structure"
- Koopmans coined the term "Linear Programming" (1949)
- **The first algorithm (Simplex method)** to solve Linear Programming formulations was published by Dantzig (1949)

LINEARIZATION TECHNIQUES

… where we see how to massage certain non-linear formulations into linear formulations

Motivation

- Sometimes it is easy to write non-linear constraints (or objective) for the problem
- The resulting formulation would be a non-linear program
- However, non-linear programs are **extremely** difficult to solve
- So, we want to linearize non-linear constraints/objective whenever possible to obtain a linear program
- We will see three situations where the problem in words immediately suggests a non-linear formulation, but that formulation can subsequently be linearized to obtain a linear program

LINEARIZATION TECHNIQUES

1. Absolute values

Bank Loan Management Problem

- Bank makes four kinds of loans with the following rates of interest:
  - Level-1 mortgage: 14%
  - Level-2 mortgage: 20%
  - Home Improvement: 20%
  - Personal overdraft: 10%
- The bank has a lending capability of $250 million and is constrained by the following policies (policies are in place to mitigate risks while investing):
  - Level-1 mortgages must be at least $55 million
  - Level-2 mortgages cannot exceed $125 million

Question: What is the best mix to lend to maximize interest?

Additional constraint:
Level-2 mortgage and Home Improvement amount cannot differ from each other by more than 5 million dollars

Formulation

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- Total available: \$250 million
- Level-1 mortgages must be at least 55 million dollars
- Level-2 mortgages cannot exceed 125 million dollars

Level-2 mortgage amount and Home Improvement amount cannot differ from each other by more than 5 million dollars

Step 1: identify decision variables

x_i: amount in millions of dollars loaned in type i for $i = 1,2,3,4$

Step 2: determine the objective function

$$\max Z(x_1, x_2, x_3, x_4) = 0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$$

Step 3: identify constraints

$$x_1 + x_2 + x_3 + x_4 \leq 250$$
$$x_1 \geq 55$$
$$x_2 \leq 125$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$$\begin{cases} x_2 - x_3 \leq 5 \\ -x_2 + x_3 \leq 5 \end{cases} \longleftrightarrow |x_2 - x_3| \leq 5$$

Violates proportionality

i.e., any feasible solution to the original constraint is also feasible to the pair of constraints and vice-versa

Linearize the constraint: $|x_2 - x_3| \leq 5$ is equivalent to two constraints $\begin{cases} x_2 - x_3 \leq 5 \\ -x_2 + x_3 \leq 5 \end{cases}$ So, replace it with two constraints

Formulation

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- Total available: $250 million
- Level-1 mortgages must be at least 55 million dollars
- Level-2 mortgages cannot exceed 125 million dollars

Level-2 mortgage amount and Home Improvement amount cannot differ from each other by more than 5 million dollars

$$\max Z(x_1, x_2, x_3, x_4) = 0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$$

$$x_1 + x_2 + x_3 + x_4 \leq 250$$
$$x_1 \geq 55$$
$$x_2 \leq 125$$
$$x_1, x_2, x_3, x_4 \geq 0$$
$$x_2 - x_3 \leq 5$$
$$-x_2 + x_3 \leq 5$$

Caution

- Absolute values cannot be linearized always

Formulation

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- Total available: \$250 million
- Level-1 mortgages must be at least 55 million dollars
- Level-2 mortgages cannot exceed 125 million dollars

Level-2 mortgage amount and Home Improvement amount **have to** differ from each other by **more than** 5 million dollars

Step 1: identify decision variables

x_i: amount in millions of dollars loaned in type i for $i = 1,2,3,4$

Step 2: determine the objective function

$$\max Z(x_1, x_2, x_3, x_4) = 0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$$

Step 3: identify constraints

$$x_1 + x_2 + x_3 + x_4 \leq 250$$
$$x_1 \geq 55$$
$$x_2 \leq 125$$
$$x_1, x_2, x_3, x_4 \geq 0$$
$$|x_2 - x_3| \geq 5$$

Cannot linearize this

$|x_2 - x_3| \geq 5$ is equivalent to saying that either $x_2 - x_3 \geq 5$ OR $-x_2 + x_3 \geq 5$

So, this cannot be formulated as a linear program

LINEARIZATION TECHNIQUES

2. Ratio of variables

Formulation

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- Total available: \$250 million
- Level-1 mortgages must be at least 55 million dollars
- Level-2 mortgages cannot exceed 125 million dollars
- Average interest rate over all loan types cannot exceed 15%

Step 1: identify decision variables

x_i: amount in millions of dollars loaned in type i for $i = 1,2,3,4$

Step 2: determine the objective function

$$\max Z(x_1, x_2, x_3, x_4) = 0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$$

Step 3: identify constraints

$$x_1 + x_2 + x_3 + x_4 \le 250$$
$$x_1 \ge 55$$
$$x_2 \le 125$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Average interest rate $\frac{0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4}{x_1 + x_2 + x_3 + x_4} \le 0.15$

Violates proportionality

Linearize the constraint: $0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4 \le 0.15(x_1 + x_2 + x_3 + x_4)$

Simplify: $-0.01x_1 + 0.05x_2 + 0.05x_3 - 0.05x_4 \le 0$

Formulation

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- Total available: $250 million
- Level-1 mortgages must be at least 55 million dollars
- Level-2 mortgages cannot exceed 125 million dollars
- Average interest rate over all loan types cannot exceed 15%

$$\max Z(x_1, x_2, x_3, x_4) = 0.14x_1 + 0.20x_2 + 0.20x_3 + 0.10x_4$$

$$x_1 + x_2 + x_3 + x_4 \leq 250$$

$$x_1 \geq 55$$

$$x_2 \leq 125$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$-0.01x_1 + 0.05x_2 + 0.05x_3 - 0.05x_4 \leq 0$$

LINEARIZATION TECHNIQUES

3. Maximizing the minimum

Bank Loan Management Problem

- Bank makes four kinds of loans with the following rates of interest:
  - Level-1 mortgage: 14%
  - Level-2 mortgage: 20%
  - Home Improvement: 20%
  - Personal overdraft: 10%
- The bank has a lending capability of $250 million and is constrained by the following policies (policies are in place to mitigate risks while investing):
  - Level-1 mortgages must be at least $55 million
  - Level-2 mortgages cannot exceed $125 million

Question: What is the best mix to lend to ~~maximize interest~~?

Different objective:
Maximize the minimum among the four possible interests

Maximizing the minimum

Formulation

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- Total available: \$250 million
- Level-1 mortgages must be at least 55 million dollars
- Level-2 mortgages cannot exceed 125 million dollars

Different objective:
Maximize the minimum among the four possible interests

Step 1: identify decision variables

x_i: amount in millions of dollars loaned in type i for $i = 1,2,3,4$

Violates proportionality

Step 2: determine the objective function

$$\max Z(x_1, x_2, x_3, x_4)$$

where $Z(x_1, x_2, x_3, x_4) = \min\{0.14x_1, 0.20x_2, 0.20x_3, 0.10x_1\}$

This is equivalent to

$$\max z$$
$$z \leq 0.14x_1$$
$$z \leq 0.20x_2$$
$$z \leq 0.20x_3$$
$$z \leq 0.10x_4$$

Step 3: identify constraints

$$x_1 + x_2 + x_3 + x_4 \leq 250$$
$$x_1 \geq 55$$
$$x_2 \leq 125$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Formulation

	Interest rate
T_1	0.14
T_2	0.20
T_3	0.20
T_4	0.10

- Total available: $250 million
- Level-1 mortgages must be at least 55 million dollars
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$$x_1 + x_2 + x_3 + x_4 \leq 250$$
$$x_1 \geq 55$$
$$x_2 \leq 125$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$$\max z$$

$$x_1 + x_2 + x_3 + x_4 \leq 250$$
$$x_1 \geq 55$$
$$x_2 \leq 125$$
$$x_1, x_2, x_3, x_4 \geq 0$$
$$z \leq 0.14x_1$$
$$z \leq 0.20x_2$$
$$z \leq 0.20x_3$$
$$z \leq 0.10x_4$$

Any feasible solution to the LHS problem also gives a feasible solution to the RHS problem with the same objective value and vice-versa

RHS problem can be simplified to be a linear program

Similar technique also works to linearize min of max objective

$\max \mathrm{Z}(x_1, x_2, x_3, x_4)$
where $\mathrm{Z}(x_1, x_2, x_3, x_4) = \min\{0.14x_1, 0.20x_2, 0.20x_3, 0.10x_1\}$

This is equivalent to

$$\max z$$
$$z \leq 0.14x_1$$
$$z \leq 0.20x_2$$
$$z \leq 0.20x_3$$
$$z \leq 0.10x_4$$

$\min \mathrm{Z}(x_1, x_2, x_3, x_4)$
where $\mathrm{Z}(x_1, x_2, x_3, x_4) = \max\{0.14x_1, 0.20x_2, 0.20x_3, 0.10x_1\}$

This is equivalent to

$$\min z$$
$$z \geq 0.14x_1$$
$$z \geq 0.20x_2$$
$$z \geq 0.20x_3$$
$$z \geq 0.10x_4$$

HOW TO SOLVE A LINEAR PROGRAMMING PROBLEM

Method 1: Graphic

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

GRAPHIC METHOD

… where we

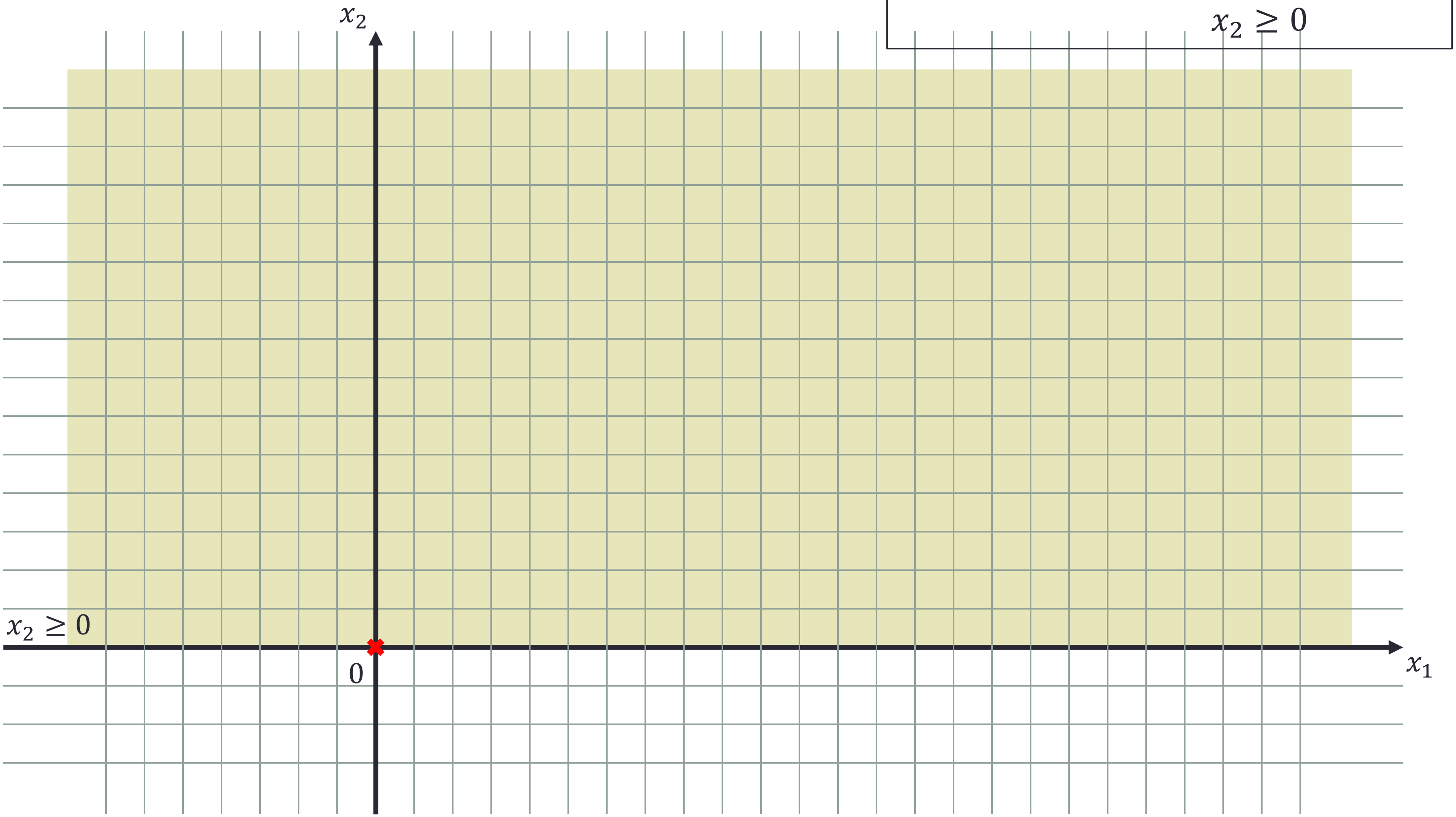
- see how to solve an LP using the Graphic Method and
- understand the scenarios that can happen with an LP

Graphic Method: Plot Constraints

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$



Graphic Method: Plot Constraints

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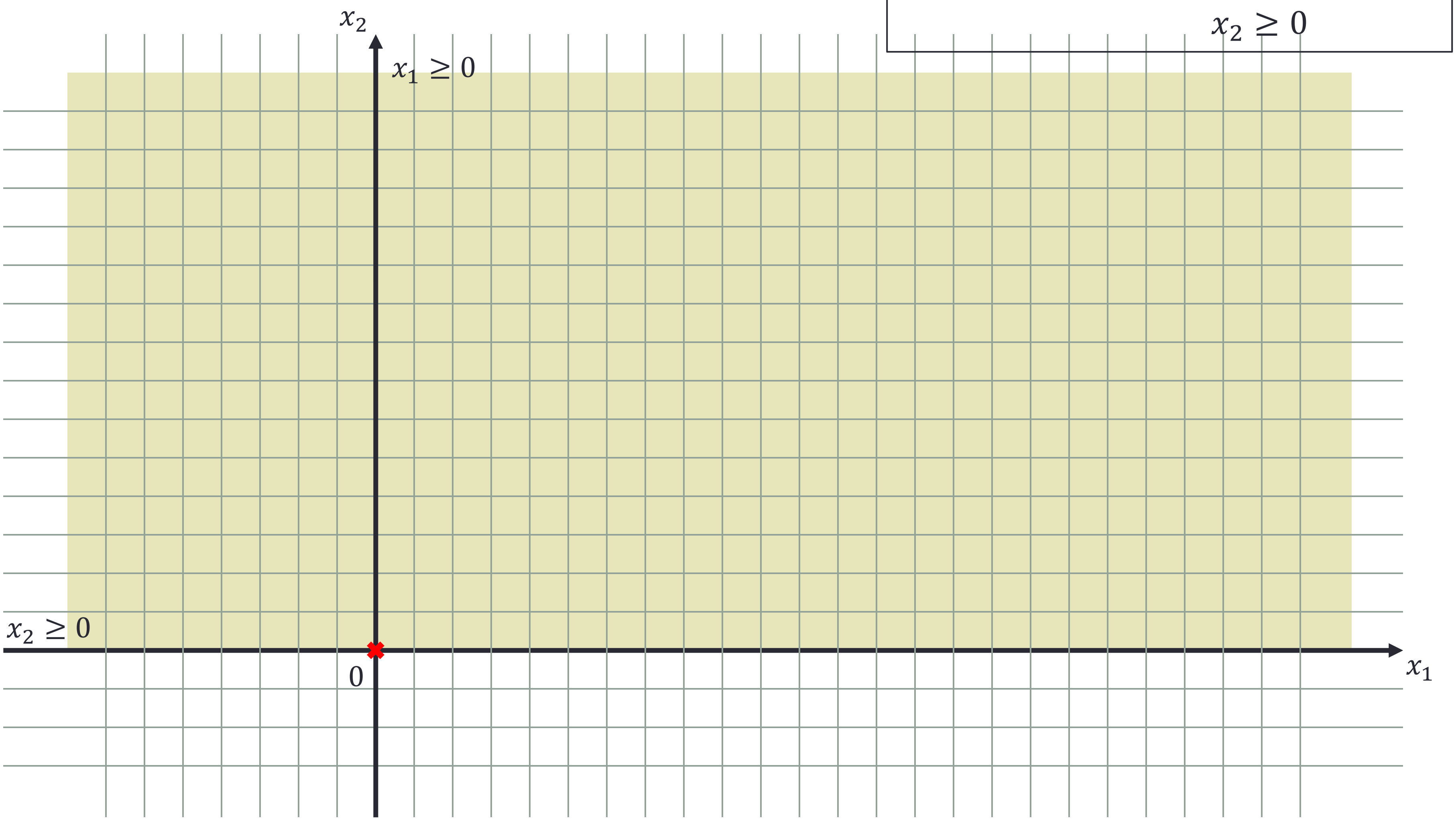

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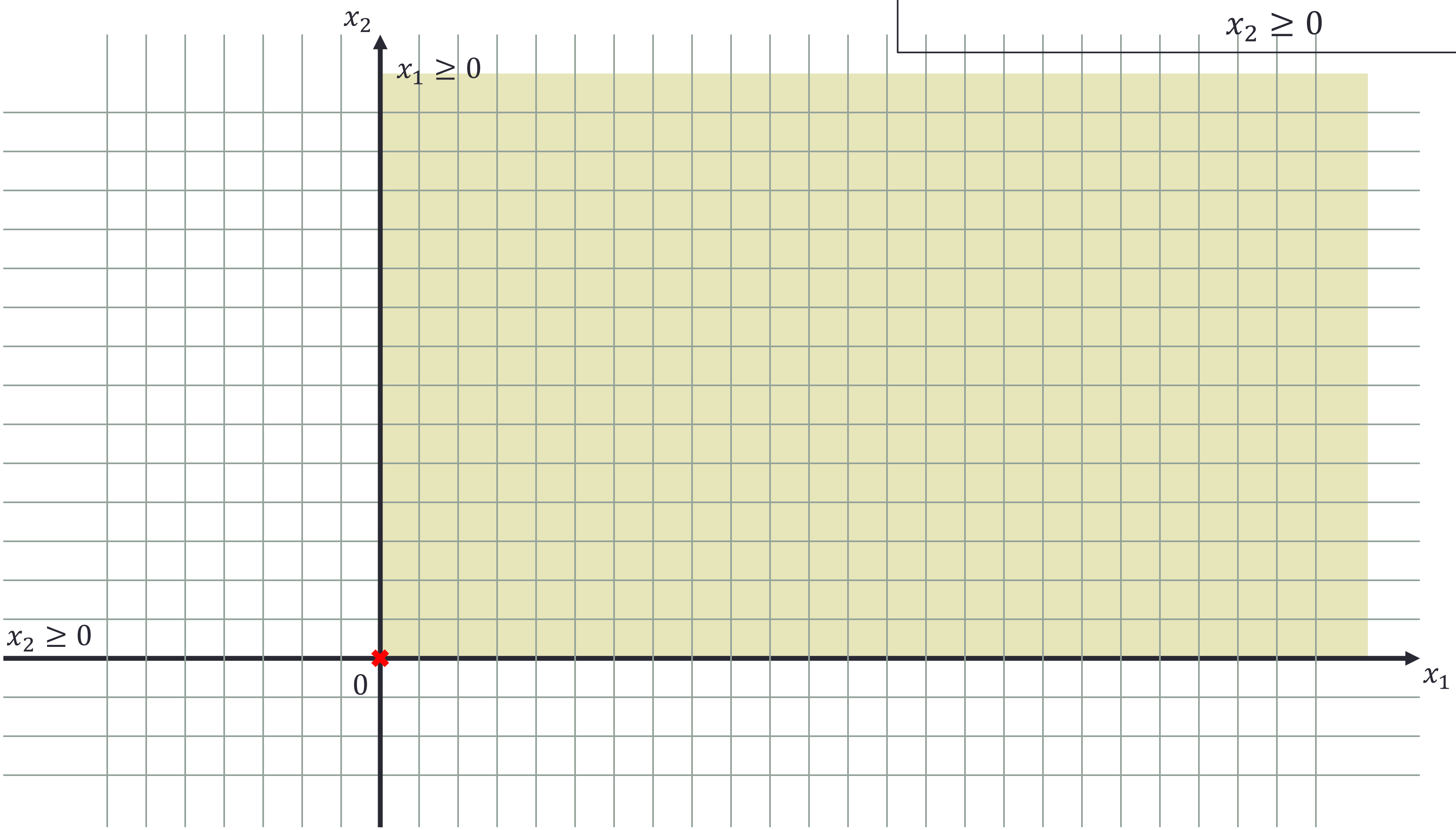

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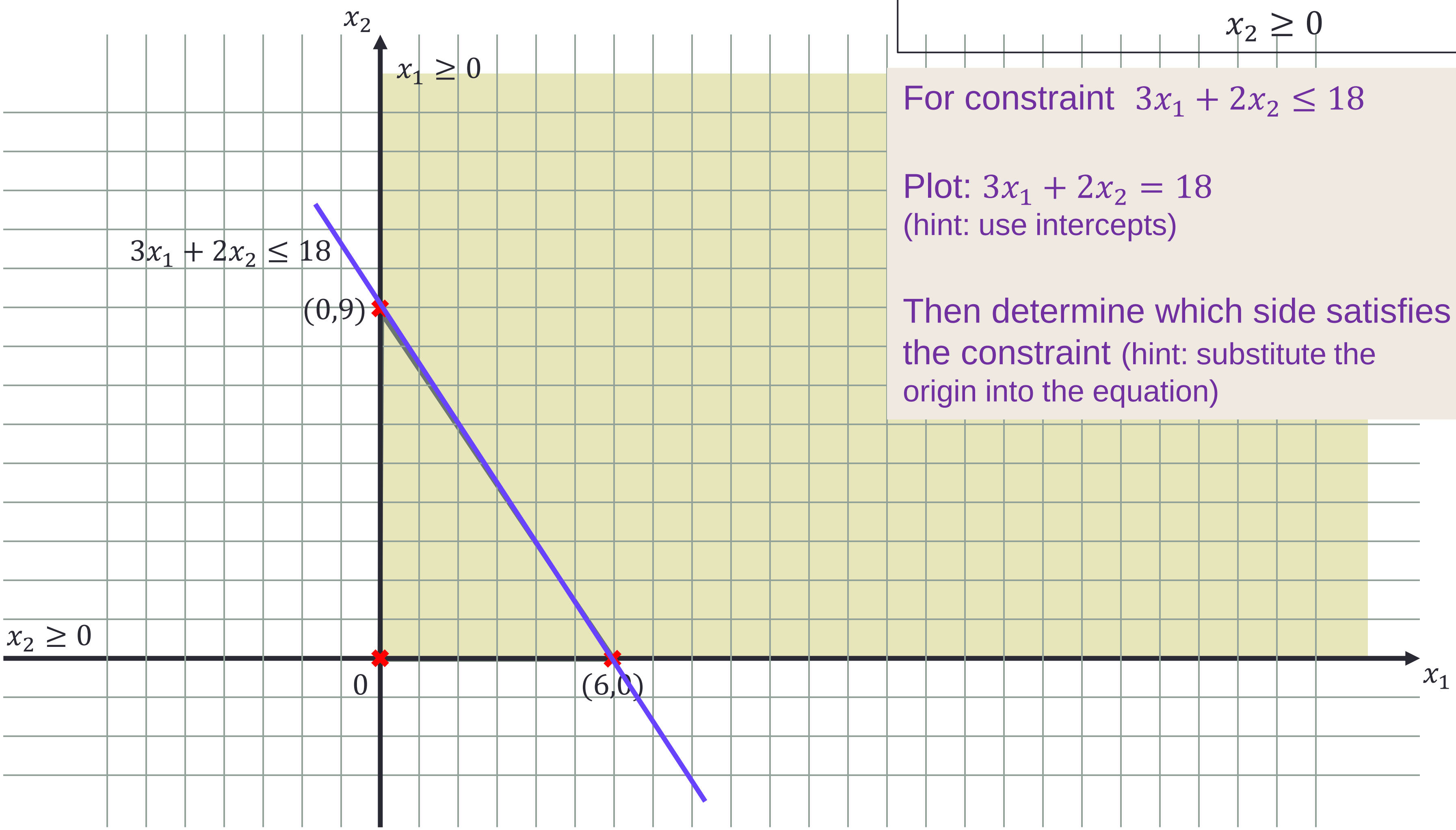


For constraint $3x_1 + 2x_2 \leq 18$

Plot: $3x_1 + 2x_2 = 18$
(hint: use intercepts)

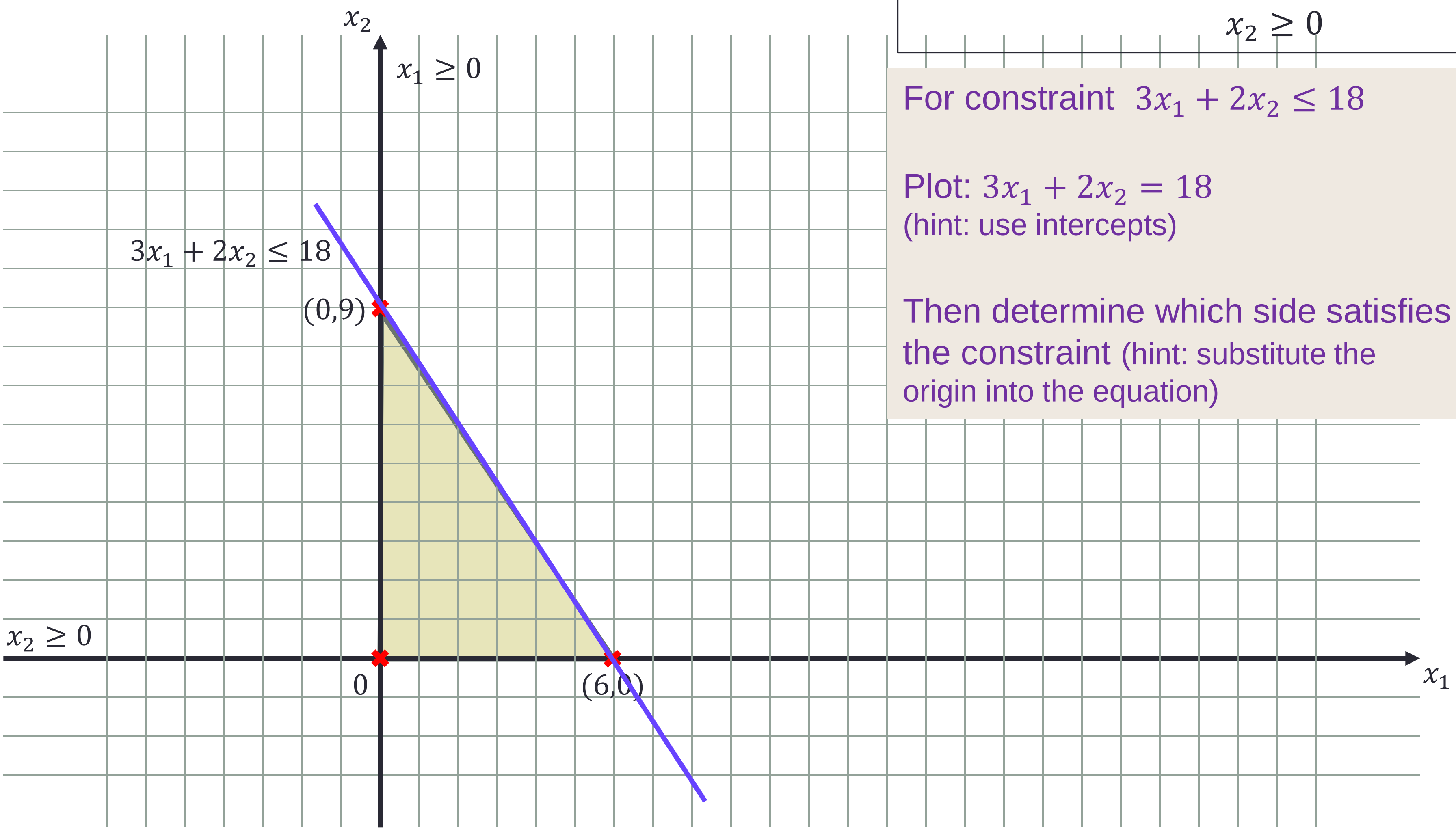
Then determine which side satisfies the constraint (hint: substitute the origin into the equation)

Graphic Method: Plot Constraints

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$



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Plot: $3x_1 + 2x_2 = 18$
(hint: use intercepts)

Then determine which side satisfies the constraint (hint: substitute the origin into the equation)

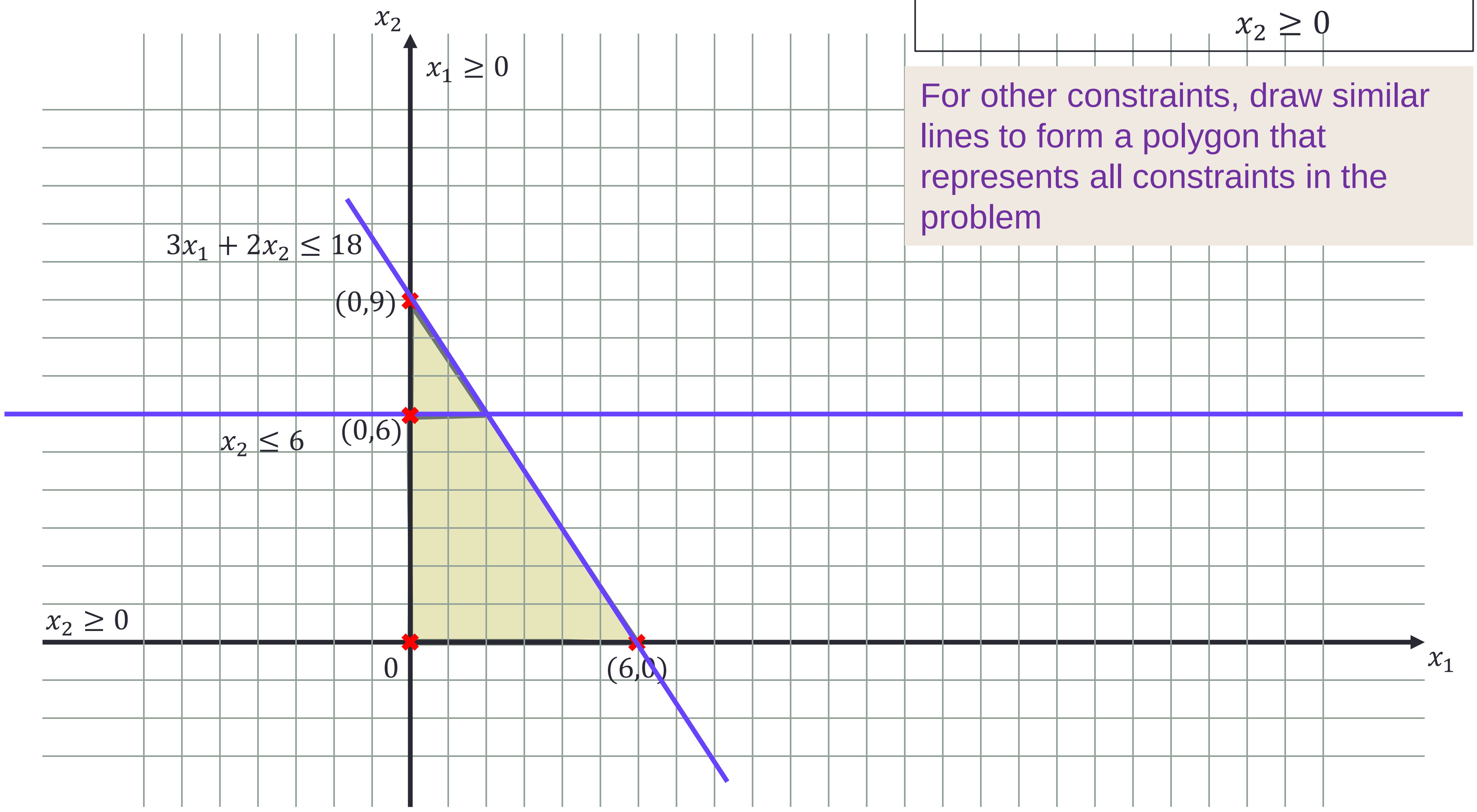
Graphic Method: Plot Constraints

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$

For other constraints, draw similar lines to form a polygon that represents all constraints in the problem



Graphic Method: Plot Constraints

$$\max\ Z = 3x_1 + 5x_2$$

subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
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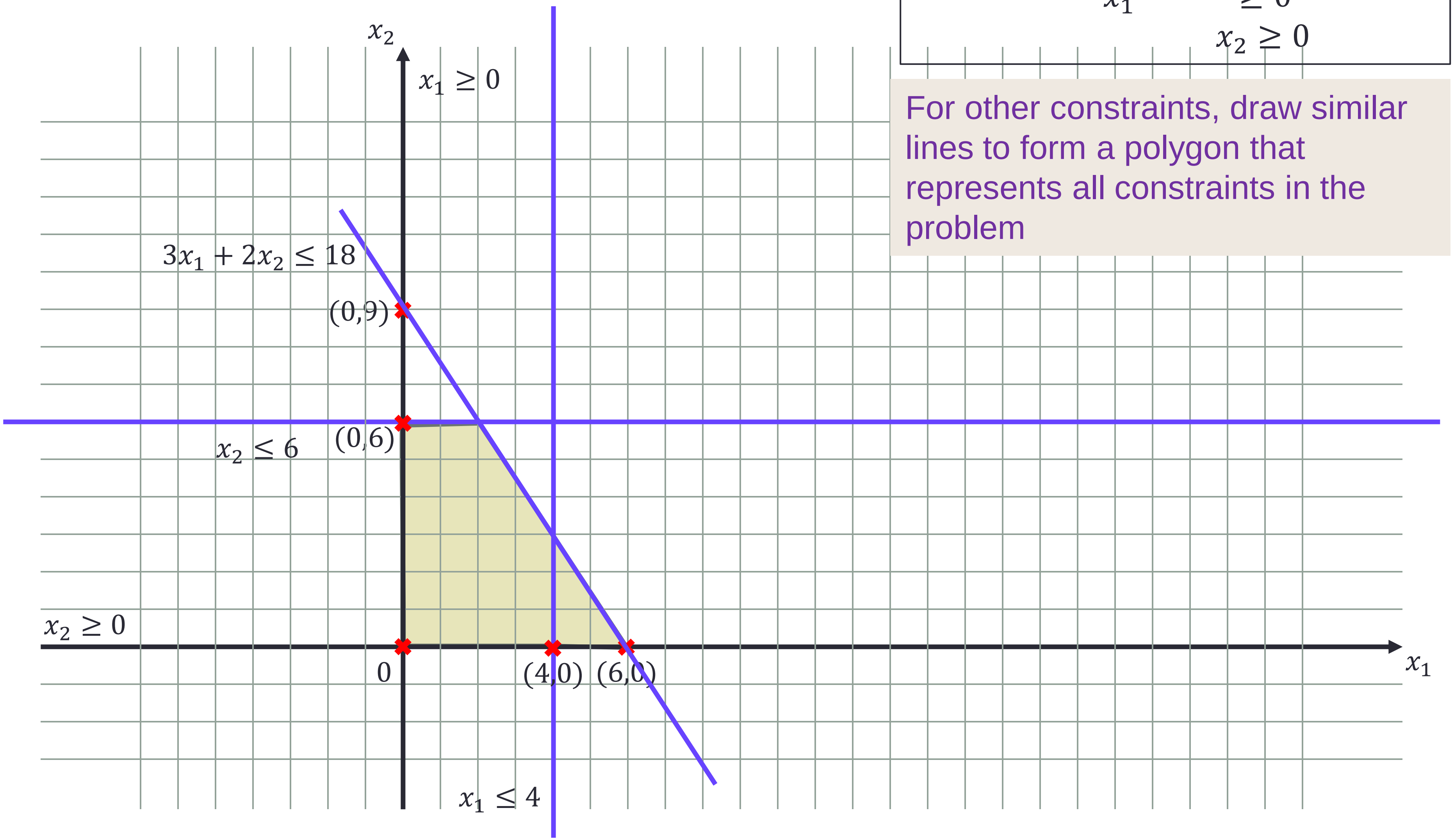

Graphic Method: Plot Constraints

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subject to:

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$
$$x_1 \geq 0$$
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The shaded area is referred to as the **feasible region**
- every point in this area satisfies all constraints of the problem