Plan for today

- Transportation Problem
  - Transportation Simplex Method
    - Basic Feasible Solution
    - Iterations
    - Termination
    - Reasoning for optimality
    - Integral Solutions Property

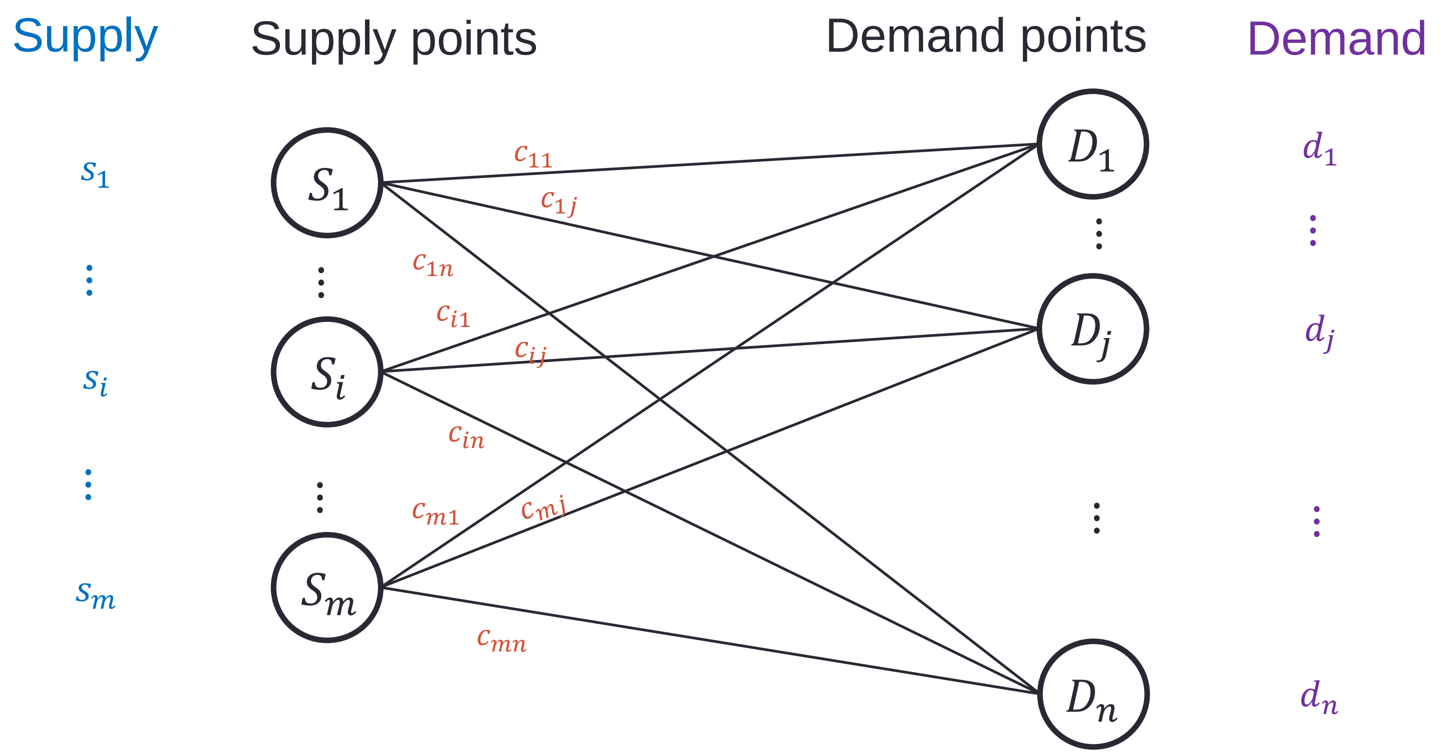
Announcements:

- Quiz 2 posted today
  10 questions, 60 mins
  Canvas
- Exam 1 on Thu
- No HW this week
- Next HW due on Mar 29

TRANSPORTATION PROBLEM

… where we see an algorithm to solve the transportation problem

Transportation Problem: General Case



- Single product to be transported from supply to demand
- Transportation Cost per unit from S_i to D_j is $c_{ij}(\geq 0)$ for all $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$
- Supply constraints: Entire supply at every S_i must be distributed to the demand points
- Demand constraints: Entire demand at every D_j must be received from the supply points

$$\Rightarrow \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

Question: How much to transport from each supply point to each demand point to minimize cost?

Formulation of the general case transportation problem

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = s_i \text{ for every } i \in \{1, \dots, m\}$$

$$\sum_{i=1}^{m} x_{ij} = d_j \text{ for every } j \in \{1, \dots, n\}$$

$$x_{ij} \geq 0 \text{ for every } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$$

ALGORITHM FOR THE TRANSPORTATION PROBLEM

- Can use Simplex method, but it is typically slow
- So we have a special-purpose algorithm

… known as the "Transportation Simplex Method"

A Convenient Representation of the Transportation Problem

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	16	16	13	22	17	50
S_2	14	14	13	19	15	60
S_3	19	19	20	23	99	50
S_4	99	0	99	0	0	50
Demand	30	20	70	30	60	

Per unit shipment cost c_{34}

A Convenient Representation of the Transportation Problem

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16] 20	[16] 20	[13] 10	[22]	[17]	50
S_2	[14] 10	[14]	[13] 50	[19]	[15]	60
S_3	[19]	[19]	[20] 10	[23] 30	[99] 10	50
S_4	[99]	[0]	[99]	[0]	[0] 50	50
Demand	30	20	70	30	60	$Z = ??$

We will avoid writing the variables and
- simply write the allocations in the respective cells
- simply write the relevant allocations

Test your understanding:
What is the objective value of this solution?

Transportation Simplex Method: Two Stage Algorithm

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = s_i \text{ for every } i \in \{1, \dots, m\}$$

$$\sum_{i=1}^{m} x_{ij} = d_j \text{ for every } j \in \{1, \dots, n\}$$

$$x_{ij} \geq 0 \text{ for every } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$$

Initialization: Identify an initial **allocation** (i.e., an initial basic feasible solution)

Allocation of values for x_{ij}

Northwest corner rule

Iterations: Iterative method to arrive at an optimal solution

TRANSPORTATION SIMPLEX METHOD

Initialization: identifying an initial basic feasible solution

Northwest Corner Rule

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16] 30	[16] 20	[13]	[22]	[17]	~~50~~ 20
S_2	[14]	[14] 0	[13] 60	[19]	[15]	60
S_3	[19]	[19]	[20] 10	[23] 30	[99] 10	~~50~~ ~~40~~ 10
S_4	[99]	[0]	[99]	[0]	[0] 50	50
Demand	30	~~20~~ 0	~~70~~ 10	30	~~60~~ 50	$Z = 3460$

1. Start from northwest corner
2. Allocate as much as possible at x_{ij} while meeting demand and supply constraints
3. If allocation exhausts all supply from S_i, move one row down
4. Else if S_i has any supply remaining, move one column to the right

BASIC FEASIBLE SOLUTION

… to a transportation problem.

- Needed for iterations of the Transportation Simplex Method

Basic Feasible Solution to a Transportation Problem

A **basic feasible solution** is an allocation satisfying the following conditions:

1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly $m + n - 1$ allocations
3. The allocations do not form a loop

the corresponding variables are **basic variables**

Northwest Corner Rule

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16] 30	[16] 20	[13]	[22]	[17]	50
S_2	[14]	[14] 0	[13] 60	[19]	[15]	60
S_3	[19]	[19]	[20] 10	[23] 30	[99] 10	50
S_4	[99]	[0]	[99]	[0]	[0] 50	50
Demand	30	20	70	30	60	$Z = 3460$

1. Supply, demand and non-negativity constraints are satisfied
2. Number of allocations $=$ 8, which is $m + n - 1 = 5 + 4 - 1$

Northwest Corner Rule

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	16 $x_{11} = 30$	16 $x_{12} = 20$	13 x_{13}	22 x_{14}	17 x_{15}	50
S_2	14 x_{21}	14 $x_{22} = 0$	13 $x_{23} = 60$	19 x_{24}	15 x_{25}	60
S_3	19 x_{31}	19 x_{32}	20 $x_{33} = 10$	23 $x_{34} = 30$	99 $x_{35} = 10$	50
S_4	99 x_{41}	0 x_{42}	99 x_{43}	0 x_{44}	0 $x_{45} = 50$	50
Demand	30	20	70	30	60	$Z = 3460$

A cell with a value written is an allocation and corresponds to a basic variable
An empty (with no written value) corresponds to a non-basic variable

Basic Feasible Solution to a Transportation Problem

A **basic feasible solution** is an allocation satisfying the following conditions:

1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly $m + n - 1$ allocations
3. The allocations do not form a loop

the corresponding variables are **basic variables**

Loop

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16]	[16]	[13] 30	[22]	[17] 20	50
S_2	[14] 30	[14]	[13] 30	[19]	[15]	60
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50
S_4	[99]	[0]	[99] 10	[0]	[0] 40	50
Demand	30	20	70	30	60	

Loop: a cycle that moves from one allocation to another using **alternate** horizontal and vertical lines

- Note: horizontal and vertical should alternate

Loop: another example

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	16 5	16	13 30	22	17 15	50
S_2	14 25	14	13 35	19	15	60
S_3	19 0	19 20	20	23 30	99	50
S_4	99	0	99 5	0	0 45	50
Demand	30	20	70	30	60	

Loop: a cycle that moves from one allocation to another using **alternate** horizontal and vertical lines

- Note: horizontal and vertical should alternate

Loop: How to break

Two options: 1st option

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16]	[16]	[13] 30 (-1)	[22]	[17] 20 $(+1)$	50
S_2	[14] 30	[14]	[13] 30	[19]	[15]	60
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50
S_4	[99]	[0]	[99] 10 $(+1)$	[0]	[0] 40 (-1)	50
Demand	30	20	70	30	60	

Change in obj $= -13 + 99 - 0 + 17 = +103$

Loop: How to break

Two options: 2nd option

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16]	[16]	[13] 30 $(+1)$	[22]	[17] 20 (-1)	50
S_2	[14] 30	[14]	[13] 30	[19]	[15]	60
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50
S_4	[99]	[0]	[99] 10 (-1)	[0]	[0] 40 $(+1)$	50
Demand	30	20	70	30	60	

Change in obj $= 13 - 99 + 0 - 17 = -103 \Rightarrow$ 2nd option improves objective

So take the second option

Loop: How to break

Take the second option

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16]	[16]	[13] 31 $(+1)$	[22]	[17] 19 (-1)	50
S_2	[14] 30	[14]	[13] 30	[19]	[15]	60
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50
S_4	[99]	[0]	[99] 9 (-1)	[0]	[0] 41 $(+1)$	50
Demand	30	20	70	30	60	

Change in obj $= 13 - 99 + 0 - 17 = -103 \Rightarrow$ Continuing 2nd option improves objective

So take the second option and improve the objective as much as possible

Can add another 9

Loop: How to break

Take the second option and improve the objective as much as possible

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16]	[16]	[13] 40	[22]	[17] 10	50
S_2	[14] 30	[14]	[13] 30	[19]	[15]	60
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50
S_4	[99]	[0]	[99]	[0]	[0] 50	50
Demand	30	20	70	30	60	

Loop: How to break

- A loop can be broken using two options
  - One of them will result in lesser (non-increasing) cost
  - Break the loop and obtain a solution with lesser cost

- A feasible solution with more than $m + n - 1$ allocations will have a loop
  - it can be broken to get a basic feasible solution

Northwest corner rule gives a Basic Feasible Solution

A **basic feasible solution** is an allocation satisfying the following conditions:

1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly $m + n - 1$ allocations
3. The allocations do not form a loop

Northwest corner rule gives a basic feasible solution

- Clearly it satisfies supply, demand and non-negativity constraints
- It will give exactly $m + n - 1$ allocations. Why?

  … whenever we made an allocation, we satisfied either the supply constraint or the demand constraint and eliminated either the supply row or the demand col (but not both)

- It will not form a loop. Why?

  … whenever we made an allocation, we allocated the maximum possible to saturate either the supply or the demand requirement

Allocations satisfying 1 and 2 may still have a loop

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16]	[16]	[13] 30	[22]	[17] 20	50
S_2	[14] 30	[14]	[13] 30	[19]	[15]	60
S_3	[19]	[19] 20	[20]	[23] 30	[99]	50
S_4	[99]	[0]	[99] 10	[0]	[0] 40	50
Demand	30	20	70	30	60	

Degenerate Basic Feasible Solutions

Basic Feasible Solutions where some of the basic variables have an allocation of zero are known as **degenerate basic feasible solutions**

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16]	[16]	[13] 40	[22]	[17] 10	50
S_2	[14] 30	[14]	[13] 30	[19]	[15]	60
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50
S_4	[99]	[0]	[99]	[0]	[0] 50	50
Demand	30	20	70	30	60	

Caution before Iteration Stage

- Initial solutions obtained from Northwest Corner Rule that we saw earlier can be used in Iteration Stage

- Solutions obtained by other rules can be used in Iteration Stage only if they are basic feasible

    - So an arbitrary solution can be used in Iteration Stage by first breaking loops and ensuring that it is a Basic Feasible Solution

TRANSPORTATION SIMPLEX METHOD

Iterations: To arrive at an optimal solution

1. Identify Entering Basic Variable

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16] 2	[16] 2	[13] 40 0	[22] 4	[17] 10 0	50	-5
S_2	[14] 30 0	[14] 0	[13] 30 0	[19] 1	[15] ○ -2	60	-5
S_3	[19] 0 0	[19] 20 0	[20] 2	[23] 30 0	[99] 77	50	0
S_4	[99] 102	[0] 3	[99] 103	[0] -1	[0] 50 0	50	-22
Demand	30	20	70	30	60	$Z = 2570$	
v_j	19	19	18	23	22		

Step 1: Find the row i with largest number of basic variables and set $u_i = 0$

Step 2: Obtain other v_j, u_i from the equations $c_{ij} - u_i - v_j = 0$ <u>for all basic variables</u> x_{ij}

Step 3: Compute $c_{ij} - u_i - v_j$ for all non-basic variables x_{ij}

Step 4: Select non-basic variable x_{ij} for which $c_{ij} - u_i - v_j$ is the most negative

1. Entering variable: x_{25}

1. Identify Entering Basic Variable

Step 1: Find the row i with largest number of basic variables and set $u_i = 0$
(break ties by picking the largest i)
Step 2: Obtain other v_j, u_i from the equations $c_{ij} - u_i - v_j = 0$ for all basic variables x_{ij}
Step 3: Compute $c_{ij} - u_i - v_j$ for all non-basic variables x_{ij}
Step 4: Select non-basic variable x_{ij} for which $c_{ij} - u_i - v_j$ is the most negative
(break ties by picking the least i and least j)

2. Identify Leaving Basic Variable

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 40	[22]	[17] 10	50	-5
S_2	[14] 30	[14]	[13] 30	[19]	[15] ◯	60	-5
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50	0
S_4	[99]	[0]	[99]	[0]	[0] 50	50	-22
Demand	30	20	70	30	60	$Z = 2570$	
v_j	19	19	18	23	22		

Step 1: Find a loop from the entering basic variable x_{ij}

1. Entering variable: x_{25}

2. Identify Leaving Basic Variable

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 40 (+1)	[22]	[17] 10 (−1)	50	−5
S_2	[14] 30	[14]	[13] 30 (−1)	[19]	[15] (+1)	60	−5
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50	0
S_4	[99]	[0]	[99]	[0]	[0] 50	50	−22
Demand	30	20	70	30	60	$Z = 2570$	
v_j	19	19	18	23	22		

Step 1: Find a loop from the entering basic variable x_{ij}

Step 2: Increasing the entering basic variable, find the basic variable that first decreases to zero

1. Entering variable: x_{25}
2. Leaving variable: x_{15}

2. Identify Leaving Basic Variable

Step 1: Find the loop from the entering basic variable x_{ij}

Step 2: Increasing the entering basic variable, find the basic variable that first decreases to zero
(break ties by picking the least i and least j)

- **Donor:** Basic variables in the loop that are at odd distance from the entering variable
- **Receiver:** Basic variables in the loop that are at even distance from the entering variable
- Pick the donor with the smallest x_{ij} value

3. Modify to get the new Basic Feasible Solution

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 40 $(+1)$	[22]	[17] 10 (-1)	50	-5
S_2	[14] 30	[14]	[13] 30 (-1)	[19]	[15] $(+1)$	60	-5
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50	0
S_4	[99]	[0]	[99]	[0]	[0] 50	50	-22
Demand	30	20	70	30	60	$Z = 2570$	
v_j	19	19	18	23	22		

Step 1: $\Delta =$ value of the leaving basic variable

Step 2: Add Δ to the values of the receivers and subtract Δ from the values of the donors

Step 3: Update Z

1. Entering variable: x_{25}
2. Leaving variable: x_{15}
3. $\Delta = 10$

3. Modify to get the new Basic Feasible Solution

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 50	[22]	[17]	50	-5
S_2	[14] 30	[14]	[13] 20	[19]	[15] 10	60	-5
S_3	[19] 0	[19] 20	[20]	[23] 30	[99]	50	0
S_4	[99]	[0]	[99]	[0]	[0] 50	50	-22
Demand	30	20	70	30	60	$Z = 2550$	
v_j	19	19	18	23	22		

… after modifying

3. Modify to get the new Basic Feasible Solution

Step 1. $\Delta =$ value of the leaving basic variable

Step 2. Add Δ to the values of the receivers and subtract Δ from the values of the donors

Step 3. Remember to update the cost Z

Note 1: The donor leaving variable becomes zero and is not a basic variable anymore, so we do not write the allocation zero explicitly

Note 2: If $\Delta = 0$, then the receiver entering variable which was an implicit zero, now becomes an explicit basic variable, so we write the allocation zero explicitly

All other zero allocations (receiver as well as donor) are also explicitly written except the donor leaving variable

Example: Iteration 2

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16] 2	[16] 2	[13] 50 0	[22] 4	[17] 2	50	-5
S_2	[14] 30 0	[14] 0	[13] 20 0	[19] 1	[15] 10 0	60	-5
S_3	[19] 0 0	[19] 20 0	[20] 2	[23] 30 0	[99] 79	50	0
S_4	[99] 100	[0] 1	[99] 101	[0] -3	[0] 50 0	50	-20
Demand	30	20	70	30	60	$Z = 2550$	
v_j	19	19	18	23	20		

Step 1: Find the row i with largest number of basic variables and set $u_i = 0$

Step 2: Obtain other v_j, u_i from the equations $c_{ij} - u_i - v_j = 0$ <u>for all basic variables</u> x_{ij}

Step 3: Compute $c_{ij} - u_i - v_j$ for all non-basic variables x_{ij}

Step 4: Select non-basic variable x_{ij} for which $c_{ij} - u_i - v_j$ is the most negative

1. Entering variable: x_{44}

Example: Iteration 2

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 50	[22]	[17]	50	-5
S_2	[14] 30 (-1)	[14]	[13] 20	[19]	[15] 10 $(+1)$	60	-5
S_3	[19] 0 $(+1)$	[19] 20	[20]	[23] 30 (-1)	[99]	50	0
S_4	[99]	[0]	[99]	[0] $(+1)$	[0] 50 (-1)	50	-20
Demand	30	20	70	30	60	$Z = 2550$	
v_j	19	19	18	23	20		

Step 1: Find a loop from the entering basic variable x_{ij}

Step 2: Increasing the entering basic variable, find the basic variable that first decreases to zero

1. Entering variable: x_{44}
2. Leaving variable: x_{21}

Example: Iteration 2

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 50	[22]	[17]	50	-5
S_2	[14] 30 (−1)	[14]	[13] 20	[19]	[15] 10 (+1)	60	-5
S_3	[19] 0 (+1)	[19] 20	[20]	[23] 30 (−1)	[99]	50	0
S_4	[99]	[0]	[99]	[0] (+1)	[0] 50 (−1)	50	-20
Demand	30	20	70	30	60	$Z = 2550$	
v_j	19	19	18	23	20		

Step 1: Δ = value of the leaving basic variable

Step 2: Add Δ to the values of the receivers and subtract Δ from the values of the donors

Step 3: Update Z

1. Entering variable: x_{44}
2. Leaving variable: x_{21}
3. $\Delta = 30$

Example: Iteration 2

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 50	[22]	[17]	50	-5
S_2	[14]	[14]	[13] 20	[19]	[15] 40	60	-5
S_3	[19] 30	[19] 20	[20]	[23] 0	[99]	50	0
S_4	[99]	[0]	[99]	[0] 30	[0] 20	50	-20
Demand	30	20	70	30	60	$Z = 2460$	
v_j	19	19	18	23	20		

… after modifying to get the new basic feasible solution

Example: Iteration 3

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16] 5	[16] 5	[13] 50 0	[22] 7	[17] 2	50	-8
S_2	[14] 3	[14] 3	[13] 20 0	[19] 4	[15] 40 0	60	-8
S_3	[19] 30 0	[19] 20 0	[20] ◯ -1	[23] 0 0	[99] 76	50	0
S_4	[99] 103	[0] 4	[99] 101	[0] 30 0	[0] 20 0	50	-23
Demand	30	20	70	30	60	$Z = 2460$	
v_j	19	19	21	23	23		

Step 1: Find the row i with largest number of basic variables and set $u_i = 0$

Step 2: Obtain other v_j, u_i from the equations $c_{ij} - u_i - v_j = 0$ for all basic variables x_{ij}

Step 3: Select x_{ij} for which $c_{ij} - u_i - v_j$ is the most negative

1. Entering variable: x_{33}

Example: Iteration 3

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 50	[22]	[17]	50	-8
S_2	[14]	[14]	[13] 20 (-1)	[19]	[15] 40 $(+1)$	60	-8
S_3	[19] 30	[19] 20	[20] $(+1)$	[23] 0 (-1)	[99]	50	0
S_4	[99]	[0]	[99]	[0] 30 $(+1)$	[0] 20 (-1)	50	-23
Demand	30	20	70	30	60	$Z = 2460$	
v_j	19	19	21	23	23		

Step 1: Find a loop from the entering basic variable x_{ij}

Step 2: Increasing the entering basic variable, find the basic variable that first decreases to zero

1. Entering variable: x_{33}
2. Leaving variable: x_{34}

Example: Iteration 3

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 50	[22]	[17]	50	-8
S_2	[14]	[14]	[13] 20 (-1)	[19]	[15] 40 $(+1)$	60	-8
S_3	[19] 30	[19] 20	[20] $(+1)$	[23] 0 (-1)	[99]	50	0
S_4	[99]	[0]	[99]	[0] 30 $(+1)$	[0] 20 (-1)	50	-23
Demand	30	20	70	30	60	$Z = 2460$	
v_j	19	19	21	23	23		

Step 1: $\Delta =$ value of the leaving basic variable

Step 2: Add Δ to the values of the receivers and subtract Δ from the values of the donors

Step 3: Update Z

1. Entering variable: x_{33}
2. Leaving variable: x_{34}
3. $\Delta = 0$

Example: Iteration 3

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16]	[16]	[13] 50	[22]	[17]	50	-8
S_2	[14]	[14]	[13] 20	[19]	[15] 40	60	-8
S_3	[19] 30	[19] 20	[20] 0	[23]	[99]	50	0
S_4	[99]	[0]	[99]	[0] 30	[0] 20	50	-23
Demand	30	20	70	30	60	$Z = 2460$	
v_j	19	19	21	23	23		

… after modifying to get the new basic feasible solution

Example: Iteration 4

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16] 4	[16] 4	[13] 50 0	[22] 7	[17] 2	50	-7
S_2	[14] 2	[14] 2	[13] 20 0	[19] 4	[15] 40 0	60	-7
S_3	[19] 30 0	[19] 20 0	[20] 0 0	[23] 1	[99] 77	50	0
S_4	[99] 102	[0] 3	[99] 101	[0] 30 0	[0] 20 0	50	-22
Demand	30	20	70	30	60	$Z = 2460$	
v_j	19	19	20	22	22		

Step 1: Find the row i with largest number of basic variables and set $u_i = 0$

Step 2: Obtain other v_j, u_i from the equations $c_{ij} - u_i - v_j = 0$ for all basic variables x_{ij}

Step 3: Compute $c_{ij} - u_i - v_j$ for all non-basic variables x_{ij}

Step 4: Select non-basic variable x_{ij} for which $c_{ij} - u_i - v_j$ is the most negative

No more entering variable
⇒ attained Optimality

Example: Termination

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	[16] 4	[16] 4	[13] 50 0	[22] 7	[17] 2	50	-7
S_2	[14] 2	[14] 2	[13] 20 0	[19] 4	[15] 40 0	60	-7
S_3	[19] 30 0	[19] 20 0	[20] 0 0	[23] 1	[99] 77	50	0
S_4	[99] 102	[0] 3	[99] 101	[0] 30 0	[0] 20 0	50	-22
Demand	30	20	70	30	60	$Z = 2460$	
v_j	19	19	20	22	22		

Optimality test:
$c_{ij} - u_i - v_j \geq 0$ for all non-basic variables

Termination criterion (Optimality Test)

- If $c_{ij} - u_i - v_j \geq 0$ for all non-basic variables, then terminate

TRANSPORTATION PROBLEM

Reasoning for Optimality?

Reasoning for optimality

Primal LP

$$\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = s_i \text{ for every } i \in \{1,\dots,m\} \quad \rightarrow \quad u_i$$

$$\sum_{i=1}^{m} x_{ij} = d_j \text{ for every } j \in \{1,\dots,n\} \quad \rightarrow \quad v_j$$

$$x_{ij} \geq 0 \text{ for every } i \in \{1,\dots,m\}, j \in \{1,\dots,n\}$$

Dual LP

$$\max \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j$$

$$u_i + v_j \leq c_{ij} \text{ for every } i \in \{1,\dots,m\}, j \in \{1,\dots,n\}$$

u_i is unrestricted for every $i \in \{1,\dots,m\}$

v_j is unrestricted for every $j \in \{1,\dots,n\}$

- The allocation x is a primal feasible solution
- (u, v) is a dual feasible solution
  - For basic variables x_{ij}: we computed (u, v) such that the dual constraint $u_i + v_j = c_{ij}$ is satisfied as an equality
  - For non-basic variables x_{ij}, the termination criteria (i.e., $c_{ij} - u_i - v_j \geq 0$ for all i, j) ensures dual feasibility

By the complementary slackness property, x and (u, v) are optimal primal and optimal dual solutions

- Further, x and (u, v) satisfy complementary slackness conditions
  - If x_{ij} is a basic variable, then we ensured that the dual constraint $u_i + v_j = c_{ij}$ is satisfied as an equality

TRANSPORTATION PROBLEMS

A useful observation: Integral Optimal Solution Property

Transportation Problem Instances have Integral Opt Solutions

Observation: If the supplies and demands are integers, then the transportation problem always has an integral optimum solution.

- In fact, the Transportation Simplex Method always terminates with an optimum solution whose values are integers

Proof:

- Starting allocation (initial basic feasible solution) values obtained by Northwest Corner rule are integers
- Each iteration of the transportation simplex method transforms an allocation with integer values to another allocation with integer values
- So final optimum solution values are integers