

# Plan for today

- Transportation Problem
  - Transportation Simplex Method
    - Basic Feasible Solution
    - Iterations
    - Termination
    - Reasoning for optimality
    - Integral Solutions Property

## Announcements:

- Quiz 2 posted today  
10 questions, 60 mins  
Canvas
- Exam 1 on Thu
- No HW this week
- Next HW due on Mar 29

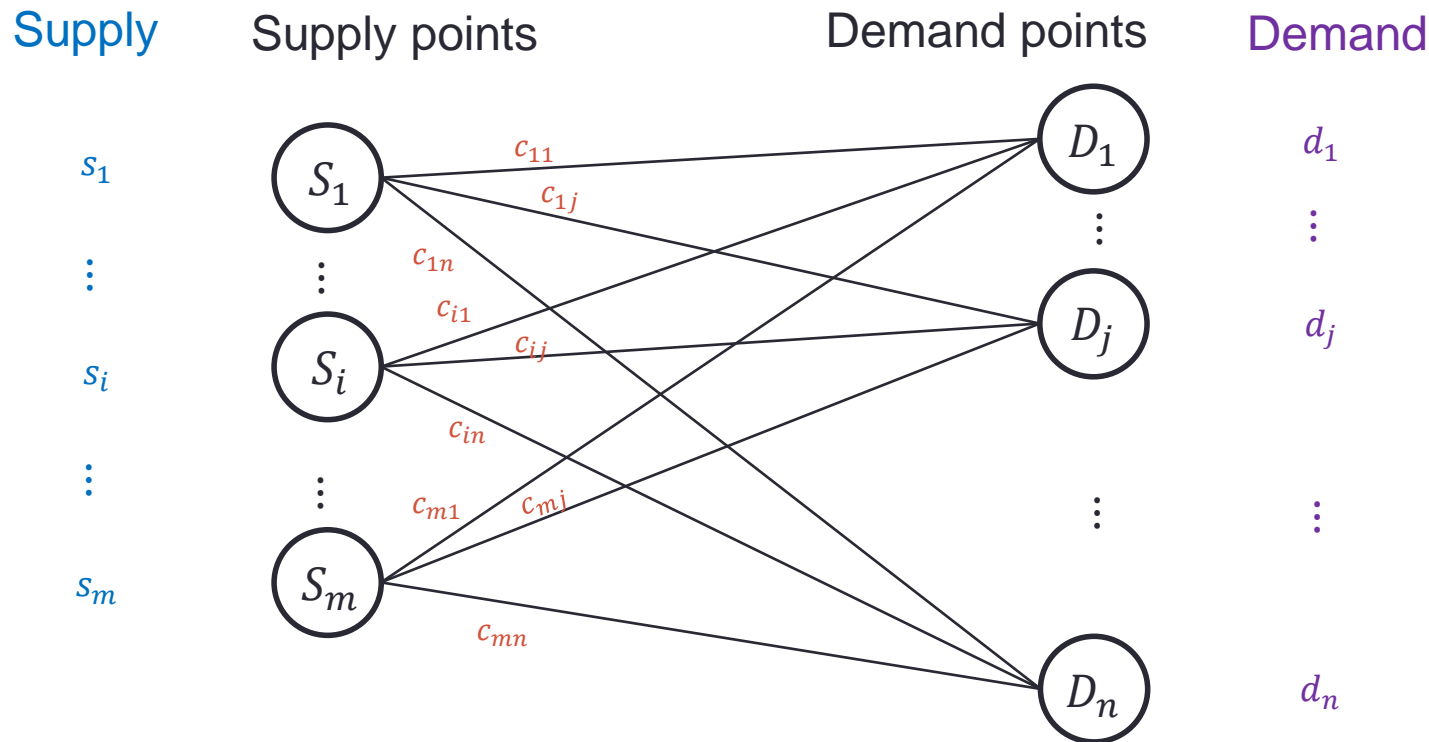


# TRANSPORTATION PROBLEM

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... where we see an algorithm to solve the transportation problem

# Transportation Problem: General Case



- Single product to be transported from supply to demand
- Transportation Cost per unit from  $S_i$  to  $D_j$  is  $c_{ij} (\geq 0)$  for all  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$
- Supply constraints: Entire supply at every  $S_i$  must be distributed to the demand points
- Demand constraints: Entire demand at every  $D_j$  must be received from the supply points

$$\Rightarrow \sum_{i=1}^m s_i = \sum_{j=1}^n d_j$$

**Question: How much to transport from each supply point to each demand point to minimize cost?**

## Formulation of the general case transportation problem

$$\min Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

$$\sum_{j=1}^n x_{ij} = s_i \text{ for every } i \in \{1, \dots, m\}$$

$$\sum_{i=1}^m x_{ij} = d_j \text{ for every } j \in \{1, \dots, n\}$$

$$x_{ij} \geq 0 \text{ for every } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$$

# ALGORITHM FOR THE TRANSPORTATION PROBLEM

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- Can use Simplex method, but it is typically slow
- So we have a special-purpose algorithm
  - ... known as the “Transportation Simplex Method”

## A Convenient Representation of the Transportation Problem

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16	16	13	22	17	50
$S_2$	14	14	13	19	15	60
$S_3$	19	19	20	23	99	50
$S_4$	99	0	99	0	0	50
Demand	30	20	70	30	60	



Per unit shipment cost  $c_{34}$

# A Convenient Representation of the Transportation Problem

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16 20	16 20	13 10	22	17	50
$S_2$	14 10	14	13 50	19	15	60
$S_3$	19	19	20 10	23 30	99 10	50
$S_4$	99	0	99	0	0 50	50
Demand	30	20	70	30	60	$Z = ??$

We will avoid writing the variables and

- simply write the allocations in the respective cells
- simply write the relevant allocations

Test your understanding:  
What is the objective value of this solution?

# Transportation Simplex Method: Two Stage Algorithm

$$\min Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

$$\sum_{j=1}^n x_{ij} = s_i \text{ for every } i \in \{1, \dots, m\}$$

$$\sum_{i=1}^m x_{ij} = d_j \text{ for every } j \in \{1, \dots, n\}$$

$$x_{ij} \geq 0 \text{ for every } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$$

**Initialization:** Identify an initial **allocation**  
(i.e., an initial basic feasible solution)

Northwest corner rule

Allocation of  
values for  $x_{ij}$

**Iterations:** Iterative method to arrive at an optimal solution



# TRANSPORTATION SIMPLEX METHOD

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Initialization: identifying an initial basic feasible solution

# Northwest Corner Rule

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16 30	16 20	13	22	17	<del>50</del> 20
$S_2$	14	14 0	13 60	19	15	60
$S_3$	19	19	20 10	23 30	99 10	<del>50</del> <del>40</del> 10
$S_4$	99	0	99	0	0 50	50
Demand	30	<del>20</del> 0	<del>70</del> 10	30	<del>60</del> 50	$Z = 3460$

1. Start from northwest corner
2. Allocate as much as possible at  $x_{ij}$  while meeting demand and supply constraints
3. If allocation exhausts all supply from  $S_i$ , move one row down
4. Else if  $S_i$  has any supply remaining, move one column to the right

# BASIC FEASIBLE SOLUTION

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... to a transportation problem.

- Needed for iterations of the Transportation Simplex Method

## Basic Feasible Solution to a Transportation Problem

A **basic feasible solution** is an allocation satisfying the following conditions:

1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly  $m + n - 1$  allocations
3. The allocations do not form a **loop**

the corresponding variables are **basic variables**



# Northwest Corner Rule

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16 30	16 20	13	22	17	50
$S_2$	14	14 0	13 60	19	15	60
$S_3$	19	19	20 10	23 30	99 10	50
$S_4$	99	0	99	0	0 50	50
Demand	30	20	70	30	60	$Z = 3460$

1. Supply, demand and non-negativity constraints are satisfied
2. Number of allocations = 8, which is  $m + n - 1 = 5 + 4 - 1$

# Northwest Corner Rule

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16 $x_{11} = 30$	16 $x_{12} = 20$	13 $x_{13}$	22 $x_{14}$	17 $x_{15}$	50
$S_2$	14 $x_{21}$	14 $x_{22} = 0$	13 $x_{23} = 60$	19 $x_{24}$	15 $x_{25}$	60
$S_3$	19 $x_{31}$	19 $x_{32}$	20 $x_{33} = 10$	23 $x_{34} = 30$	99 $x_{35} = 10$	50
$S_4$	99 $x_{41}$	0 $x_{42}$	99 $x_{43}$	0 $x_{44}$	0 $x_{45} = 50$	50
Demand	30	20	70	30	60	$Z = 3460$

A cell with a value written is an allocation and corresponds to a **basic variable**  
 An empty (with no written value) corresponds to a non-basic variable

## Basic Feasible Solution to a Transportation Problem

A **basic feasible solution** is an allocation satisfying the following conditions:

1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly  $m + n - 1$  allocations
3. The allocations do not form a **loop**

the corresponding variables are **basic variables**



# Loop

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16	16	13	22	17	50
$S_2$	14	14	13	19	15	60
$S_3$	19	19	20	23	99	50
$S_4$	99	0	99	0	0	50
Demand	30	20	70	30	60	

**Loop:** a cycle that moves from one allocation to another using **alternate** horizontal and vertical lines

- Note: horizontal and vertical should alternate



# Loop: another example

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16	16	13	22	17	50
$S_2$	14	14	13	19	15	60
$S_3$	19	20	20	23	99	50
$S_4$	99	0	99	0	0	50
Demand	30	20	70	30	60	

**Loop:** a cycle that moves from one allocation to another using **alternate** horizontal and vertical lines

- Note: horizontal and vertical should alternate

# Loop: How to break

Two options: 1<sup>st</sup> option

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16	16	13	22	17	50
$S_2$	14	14	13	19	15	60
$S_3$	19	19	20	23	99	50
$S_4$	99	0	99	0	0	50
<b>Demand</b>	30	20	70	30	60	

$$\text{Change in obj} = -13 + 99 - 0 + 17 = +103$$

# Loop: How to break

Two options: 2<sup>nd</sup> option

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16	16	13	22	17	50
$S_2$	14	14	13	19	15	60
$S_3$	19	19	20	23	99	50
$S_4$	99	0	99	0	0	50
<b>Demand</b>	30	20	70	30	60	

Diagram annotations: A red arrow points from 20 in  $S_1, D_5$  to 30 in  $S_1, D_3$  with a red  $(+1)$  label. A red arrow points from 30 in  $S_1, D_3$  to 10 in  $S_4, D_3$  with a red  $(-1)$  label. A red arrow points from 10 in  $S_4, D_3$  to 40 in  $S_4, D_5$  with a red  $(+1)$  label. A red arrow points from 40 in  $S_4, D_5$  to 20 in  $S_1, D_5$  with a red  $(-1)$  label.

Change in obj =  $13 - 99 + 0 - 17 = -103 \Rightarrow$  2<sup>nd</sup> option improves objective

So take the second option

# Loop: How to break

Take the second option

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16	16	13	22	17	50
$S_2$	14	14	13	19	15	60
$S_3$	19	19	20	23	99	50
$S_4$	99	0	99	0	0	50
Demand	30	20	70	30	60	

The table shows a transportation problem with sources  $S_1, S_2, S_3, S_4$  and demands  $D_1, D_2, D_3, D_4, D_5$ . The current solution is shown with values in boxes. A loop is highlighted with red arrows:  $S_1 \rightarrow D_4 \rightarrow S_4 \rightarrow D_5 \rightarrow S_1$ . The values in the loop are 22, 17, 0, and 17 respectively. The change in the objective function is calculated as  $13 - 99 + 0 - 17 = -103$ . The value 9 is shown in the  $S_4 \rightarrow D_3$  cell, indicating the amount that can be added to the loop.

Change in obj =  $13 - 99 + 0 - 17 = -103 \Rightarrow$  Continuing 2<sup>nd</sup> option improves objective

So take the second option and improve the objective as much as possible

Can add another 9

# Loop: How to break

Take the second option and improve the objective as much as possible

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16	16	13	22	17	50
$S_2$	14	14	13	19	15	60
$S_3$	19	19	20	23	99	50
$S_4$	99	0	99	0	0	50
Demand	30	20	70	30	60	

Diagram illustrating a loop in a transportation problem. Red arrows indicate the path for a loop: from  $S_1$  to  $D_5$  (10 units), from  $D_5$  to  $S_4$  (50 units), from  $S_4$  to  $D_3$  (99 units), and from  $D_3$  to  $S_1$  (40 units). The value 30 is shown in the  $S_2$  row,  $D_3$  column, and the value 20 is shown in the  $S_3$  row,  $D_2$  column.

# Loop: How to break

- A loop can be broken using two options
  - One of them will result in lesser (non-increasing) cost
  - Break the loop and obtain a solution with lesser cost
  
- A feasible solution with more than  $m + n - 1$  allocations will have a loop
  - it can be broken to get a basic feasible solution

# Northwest corner rule gives a Basic Feasible Solution

A **basic feasible solution** is an allocation satisfying the following conditions:

1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly  $m + n - 1$  allocations
3. The allocations do not form a loop

## Northwest corner rule gives a basic feasible solution

- Clearly it satisfies supply, demand and non-negativity constraints
- It will give exactly  $m + n - 1$  allocations. Why?
  - ... whenever we made an allocation, we satisfied either the supply constraint or the demand constraint and eliminated either the supply row or the demand col (but not both)
- It will not form a loop. Why?
  - ... whenever we made an allocation, we allocated the maximum possible to saturate either the supply or the demand requirement

## Allocations satisfying 1 and 2 may still have a loop

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16	16	13	22	17	50
$S_2$	14	14	13	19	15	60
$S_3$	19	19	20	23	99	50
$S_4$	99	0	99	0	0	50
<b>Demand</b>	30	20	70	30	60	




# Degenerate Basic Feasible Solutions

Basic Feasible Solutions where some of the basic variables have an allocation of zero are known as **degenerate basic feasible solutions**

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply
$S_1$	16	16	13	22	17	50
		40			10	
$S_2$	14	14	13	19	15	60
	30		30			
$S_3$	19	19	20	23	99	50
	0	20		30		
$S_4$	99	0	99	0	0	50
					50	
Demand	30	20	70	30	60	

# Caution before Iteration Stage

- Initial solutions obtained from Northwest Corner Rule that we saw earlier can be used in Iteration Stage

-  • Solutions obtained by other rules can be used in Iteration Stage only if they are basic feasible

- So an arbitrary solution can be used in Iteration Stage by first breaking loops and ensuring that it is a Basic Feasible Solution

# TRANSPORTATION SIMPLEX METHOD

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Iterations: To arrive at an optimal solution

# 1. Identify Entering Basic Variable

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-5
		2	2	0	4	10	0
$S_2$	14	14	13	19	15	60	-5
	30		30				-2
	0	0	0	1			
$S_3$	19	19	20	23	99	50	0
	0	20		30			77
	0	0	2	0			
$S_4$	99	0	99	0	0	50	-22
		102	3	103	-1	50	0
<b>Demand</b>	30	20	70	30	60	$Z = 2570$	
$v_j$	19	19	18	23	22		

Step 1: Find the row  $i$  with largest number of basic variables and set  $u_i = 0$

Step 2: Obtain other  $v_j, u_i$  from the equations  $c_{ij} - u_i - v_j = 0$  for all basic variables  $x_{ij}$

Step 3: Compute  $c_{ij} - u_i - v_j$  for all non-basic variables  $x_{ij}$

Step 4: Select non-basic variable  $x_{ij}$  for which  $c_{ij} - u_i - v_j$  is the most negative

1. Entering variable:  $x_{25}$

# 1. Identify Entering Basic Variable

Step 1: Find the row  $i$  with largest number of basic variables and set  $u_i = 0$   
(break ties by picking the largest  $i$ )

Step 2: Obtain other  $v_j, u_i$  from the equations  $c_{ij} - u_i - v_j = 0$  for all basic variables  $x_{ij}$

Step 3: Compute  $c_{ij} - u_i - v_j$  for all non-basic variables  $x_{ij}$

Step 4: Select non-basic variable  $x_{ij}$  for which  $c_{ij} - u_i - v_j$  is the most negative  
(break ties by picking the least  $i$  and least  $j$ )

## 2. Identify Leaving Basic Variable

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-5
$S_2$	14	14	13	19	15	60	-5
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-22
Demand	30	20	70	30	60	$Z = 2570$	
$v_j$	19	19	18	23	22		

Diagram illustrating the identification of the leaving basic variable. A red circle highlights the cell at the intersection of  $S_2$  and  $D_5$ . Blue arrows show a loop: from the red circle to  $D_5$  (value 10), then to  $S_1$  (value 17), then to  $D_3$  (value 13), then to  $S_2$  (value 13), and finally back to the red circle (value 15). The values 40 and 30 are also shown near the  $D_3$  column.

Step 1: Find a loop from the entering basic variable  $x_{ij}$

1. Entering variable:  $x_{25}$

## 2. Identify Leaving Basic Variable

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-5
$S_2$	14	14	13	19	15	60	-5
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-22
Demand	30	20	70	30	60	$Z = 2570$	
$v_j$	19	19	18	23	22		

Diagram annotations: A blue arrow points from the entering variable cell (row  $S_1$ , column  $D_5$ ) to the cell (row  $S_1$ , column  $D_3$ ) with a red  $(+1)$  above it. Another blue arrow points from the same entering variable cell to the cell (row  $S_2$ , column  $D_5$ ) with a red  $(+1)$  below it. A green arrow points from the cell (row  $S_2$ , column  $D_3$ ) to the cell (row  $S_1$ , column  $D_3$ ) with a green  $(-1)$  above it. The value 10 in the cell (row  $S_1$ , column  $D_5$ ) is circled in green. The value 30 in the cell (row  $S_2$ , column  $D_5$ ) is circled in red.

Step 1: Find a loop from the entering basic variable  $x_{ij}$

Step 2: Increasing the entering basic variable, find the basic variable that first decreases to zero

1. Entering variable:  $x_{25}$
2. Leaving variable:  $x_{15}$

## 2. Identify Leaving Basic Variable

Step 1: Find the loop from the entering basic variable  $x_{ij}$

Step 2: Increasing the entering basic variable, find the basic variable that first decreases to zero

(break ties by picking the least  $i$  and least  $j$ )

- **Donor:** Basic variables in the loop that are at odd distance from the entering variable
- **Receiver:** Basic variables in the loop that are at even distance from the entering variable
- Pick the **donor** with the smallest  $x_{ij}$  value



### 3. Modify to get the new Basic Feasible Solution

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-5
$S_2$	14	14	13	19	15	60	-5
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-22
Demand	30	20	70	30	60	$Z = 2570$	
$v_j$	19	19	18	23	22		

Diagram annotations: A blue arrow points from the value 10 in the  $S_1$  row,  $D_5$  column to the value 30 in the  $S_2$  row,  $D_5$  column. A red circle highlights the value 10, and a green circle highlights the value 30. A red arrow points from the 30 to the 10, labeled  $(+1)$ . A green arrow points from the 10 to the 30, labeled  $(-1)$ .

Step 1:  $\Delta$  = value of the leaving basic variable

Step 2: Add  $\Delta$  to the values of the **receivers** and subtract  $\Delta$  from the values of the **donors**

Step 3: Update  $Z$

1. Entering variable:  $x_{25}$
2. Leaving variable:  $x_{15}$
3.  $\Delta = 10$

### 3. Modify to get the new Basic Feasible Solution

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-5
$S_2$	14	14	13	19	15	60	-5
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-22
Demand	30	20	70	30	60	$Z = 2550$	
$v_j$	19	19	18	23	22		

... after modifying

### 3. Modify to get the new Basic Feasible Solution

Step 1.  $\Delta$  = value of the leaving basic variable

Step 2. Add  $\Delta$  to the values of the **receivers** and subtract  $\Delta$  from the values of the **donors**

Step 3. Remember to update the cost  $Z$

Note 1: The **donor** leaving variable becomes zero and is not a basic variable anymore, so we do not write the allocation zero explicitly

Note 2: If  $\Delta = 0$ , then the **receiver** entering variable which was an implicit zero, now becomes an explicit basic variable, so we write the allocation zero explicitly

All other zero allocations (**receiver** as well as **donor**) are also explicitly written except the **donor** leaving variable

# Example: Iteration 2

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-5
	2	2	0	4	2		
$S_2$	14	14	13	19	15	60	-5
	30	0	0	0	10		
$S_3$	19	19	20	23	99	50	0
	0	20	0	30	0		
$S_4$	99	0	99	0	0	50	-20
	100	1	101	0	50		
Demand	30	20	70	30	60	$Z = 2550$	
$v_j$	19	19	18	23	20		

Step 1: Find the row  $i$  with largest number of basic variables and set  $u_i = 0$

Step 2: Obtain other  $v_j, u_i$  from the equations  $c_{ij} - u_i - v_j = 0$  for all basic variables  $x_{ij}$

Step 3: Compute  $c_{ij} - u_i - v_j$  for all non-basic variables  $x_{ij}$

Step 4: Select non-basic variable  $x_{ij}$  for which  $c_{ij} - u_i - v_j$  is the most negative

1. Entering variable:  $x_{44}$

# Example: Iteration 2

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-5
$S_2$	14	14	13	19	15	60	-5
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-20
<b>Demand</b>	30	20	70	30	60	$Z = 2550$	
$v_j$	19	19	18	23	20		

Diagram illustrating the identification of the leaving variable in Iteration 2. A loop is formed by the entering variable  $x_{44}$  (circled in red) and the leaving variable  $x_{21}$  (circled in green). The loop consists of the following cells:  $(S_4, D_4)$ ,  $(S_4, D_5)$ ,  $(S_2, D_5)$ ,  $(S_2, D_1)$ , and  $(S_4, D_1)$ . The flow of the loop is indicated by blue arrows: from  $(S_4, D_4)$  to  $(S_4, D_5)$  (+1), from  $(S_4, D_5)$  to  $(S_2, D_5)$  (-1), from  $(S_2, D_5)$  to  $(S_2, D_1)$  (+1), and from  $(S_2, D_1)$  to  $(S_4, D_1)$  (-1). The value 30 is shown in the  $(S_2, D_1)$  cell, and 10 is shown in the  $(S_2, D_5)$  cell.

Step 1: Find a loop from the entering basic variable  $x_{ij}$

Step 2: Increasing the entering basic variable, find the basic variable that first decreases to zero

1. Entering variable:  $x_{44}$
2. Leaving variable:  $x_{21}$

# Example: Iteration 2

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-5
$S_2$	14	14	13	19	15	60	-5
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-20
<b>Demand</b>	30	20	70	30	60	$Z = 2550$	
$v_j$	19	19	18	23	20		

Diagram illustrating the modification of the solution in Iteration 2. The table shows the current solution with the following adjustments:

- Entering variable:  $x_{44}$  (circled in red, +1)
- Leaving variable:  $x_{21}$  (circled in green, -1)
- $\Delta = 30$  (circled in green)

Flow of adjustments:

- From  $S_2$  to  $S_3$ : +10 (circled in red, +1)
- From  $S_3$  to  $S_4$ : -30 (circled in green, -1)
- From  $S_4$  to  $S_2$ : +30 (circled in red, +1)

Step 1:  $\Delta$  = value of the leaving basic variable

Step 2: Add  $\Delta$  to the values of the **receivers** and subtract  $\Delta$  from the values of the **donors**

Step 3: Update  $Z$

1. Entering variable:  $x_{44}$
2. Leaving variable:  $x_{21}$
3.  $\Delta = 30$

# Example: Iteration 2

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-5
$S_2$	14	14	13	19	15	60	-5
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-20
Demand	30	20	70	30	60	$Z = 2460$	
$v_j$	19	19	18	23	20		

... after modifying to get the new basic feasible solution

# Example: Iteration 3

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-8
	5	5	0	7	2		
$S_2$	14	14	13	19	15	60	-8
	3	3	0	4	0		
$S_3$	19	19	20	23	99	50	0
	30	20	-1	0	76		
$S_4$	99	0	99	0	0	50	-23
	103	4	101	30	20		
Demand	30	20	70	30	60	$Z = 2460$	
$v_j$	19	19	21	23	23		

Step 1: Find the row  $i$  with largest number of basic variables and set  $u_i = 0$

1. Entering variable:  $x_{33}$

Step 2: Obtain other  $v_j, u_i$  from the equations  $c_{ij} - u_i - v_j = 0$  for all basic variables  $x_{ij}$

Step 3: Select  $x_{ij}$  for which  $c_{ij} - u_i - v_j$  is the most negative



# Example: Iteration 3

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-8
$S_2$	14	14	13	19	15	60	-8
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-23
<b>Demand</b>	30	20	70	30	60	$Z = 2460$	
$v_j$	19	19	21	23	23		

Diagram illustrating the identification of the leaving variable during Iteration 3. The entering variable is  $x_{33}$  (circled in green). A loop is formed by the cells  $(S_3, D_3)$ ,  $(S_3, D_4)$ ,  $(S_4, D_4)$ , and  $(S_4, D_3)$ . The values in these cells are 20, 23, 0, and 99, respectively. The change in the value of the entering variable is +1 (circled in red). The change in the value of the leaving variable is -1 (circled in green). The leaving variable is  $x_{34}$ .

Step 1: Find a loop from the entering basic variable  $x_{ij}$

Step 2: Increasing the entering basic variable, find the basic variable that first decreases to zero

1. Entering variable:  $x_{33}$
2. Leaving variable:  $x_{34}$

# Example: Iteration 3

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-8
$S_2$	14	14	13	19	15	60	-8
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-23
<b>Demand</b>	30	20	70	30	60	$Z = 2460$	
$v_j$	19	19	21	23	23		

Diagram illustrating the modification of the solution in Iteration 3. The table shows the current solution with values in boxes. Blue arrows indicate the flow of the leaving variable  $\Delta$  from the leaving variable cell (20 in  $S_3$ ) to the receiving cells (20 in  $S_2$ , 30 in  $S_4$ , and 40 in  $D_5$ ). Red and green annotations show the changes:  $+1$  (red) and  $-1$  (green) are added to the receiving cells, and  $-1$  (green) is subtracted from the donor cell. The value 0 in  $S_3$  is circled in green, and the value 20 in  $S_3$  is circled in red.

Step 1:  $\Delta$  = value of the leaving basic variable

Step 2: Add  $\Delta$  to the values of the **receivers** and subtract  $\Delta$  from the values of the **donors**

Step 3: Update  $Z$

1. Entering variable:  $x_{33}$
2. Leaving variable:  $x_{34}$
3.  $\Delta = 0$

# Example: Iteration 3

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-8
$S_2$	14	14	13	19	15	60	-8
$S_3$	19	19	20	23	99	50	0
$S_4$	99	0	99	0	0	50	-23
Demand	30	20	70	30	60	$Z = 2460$	
$v_j$	19	19	21	23	23		

... after modifying to get the new basic feasible solution

# Example: Iteration 4

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-7
		4	4	0	7	2	
$S_2$	14	14	13	19	15	60	-7
		2	2	0	4	0	
$S_3$	19	19	20	23	99	50	0
	30	20	0	0	1	77	
	0	0	0	0	0	0	
$S_4$	99	0	99	0	0	50	-22
		102	3	101	30	20	
				0	0	0	
<b>Demand</b>	30	20	70	30	60	$Z = 2460$	
$v_j$	19	19	20	22	22		

Step 1: Find the row  $i$  with largest number of basic variables and set  $u_i = 0$

Step 2: Obtain other  $v_j, u_i$  from the equations  $c_{ij} - u_i - v_j = 0$  for all basic variables  $x_{ij}$

Step 3: Compute  $c_{ij} - u_i - v_j$  for all non-basic variables  $x_{ij}$

Step 4: Select non-basic variable  $x_{ij}$  for which  $c_{ij} - u_i - v_j$  is the most negative

No more entering variable  
 $\Rightarrow$  attained Optimality

# Example: Termination

	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	Supply	$u_i$
$S_1$	16	16	13	22	17	50	-7
		4	4	0	7	2	
$S_2$	14	14	13	19	15	60	-7
		2	2	0	4	0	
$S_3$	19	19	20	23	99	50	0
	30	20	0	0	1	77	
	0	0	0	0	0	0	
$S_4$	99	0	99	0	0	50	-22
		102	3	101	30	20	
				0	0	0	
Demand	30	20	70	30	60	$Z = 2460$	
$v_j$	19	19	20	22	22		

Optimality test:

$$c_{ij} - u_i - v_j \geq 0 \text{ for all non-basic variables}$$

# Termination criterion (Optimality Test)

- If  $c_{ij} - u_i - v_j \geq 0$  for all non-basic variables, then terminate



# TRANSPORTATION PROBLEM

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Reasoning for Optimality?

# Reasoning for optimality

## Primal LP

$$\min Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

$$\sum_{j=1}^n x_{ij} = s_i \text{ for every } i \in \{1, \dots, m\}$$

$$\sum_{i=1}^m x_{ij} = d_j \text{ for every } j \in \{1, \dots, n\}$$

$$x_{ij} \geq 0 \text{ for every } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$$

→  $u_i$

→  $v_j$

## Dual LP

$$\max \sum_{i=1}^m s_i u_i + \sum_{j=1}^n d_j v_j$$

$$u_i + v_j \leq c_{ij} \text{ for every } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$$

$$u_i \text{ is unrestricted for every } i \in \{1, \dots, m\}$$

$$v_j \text{ is unrestricted for every } j \in \{1, \dots, n\}$$

- The allocation  $x$  is a primal feasible solution
- $(u, v)$  is a dual feasible solution
  - For basic variables  $x_{ij}$ : we computed  $(u, v)$  such that the dual constraint  $u_i + v_j = c_{ij}$  is satisfied as an equality
  - For non-basic variables  $x_{ij}$ , the termination criteria (i.e.,  $c_{ij} - u_i - v_j \geq 0$  for all  $i, j$ ) ensures dual feasibility
- Further,  $x$  and  $(u, v)$  satisfy complementary slackness conditions
  - If  $x_{ij}$  is a basic variable, then we ensured that the dual constraint  $u_i + v_j = c_{ij}$  is satisfied as an equality

By the complementary slackness property,  $x$  and  $(u, v)$  are optimal primal and optimal dual solutions



# TRANSPORTATION PROBLEMS

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A useful observation: Integral Optimal Solution Property

## Transportation Problem Instances have Integral Opt Solutions

**Observation:** If the supplies and demands are integers, then the transportation problem always has an integral optimum solution.

- In fact, the Transportation Simplex Method always terminates with an optimum solution whose values are integers



### Proof:

- Starting allocation (initial basic feasible solution) values obtained by Northwest Corner rule are integers
- Each iteration of the transportation simplex method transforms an allocation with integer values to another allocation with integer values
- So final optimum solution values are integers