Plan for today

- LP Solving in Excel
  - Interpreting reports
- Transportation Problem
  - LP Formulation
  - Transportation Simplex Method
    - Initialization: Northwest corner rule

Announcements:

- Exam 1 Topics: Until the end of Game Theory
- Permitted: Lecture notes/slides/videos, HWs, Quizzes
- **NOT permitted:** collaboration, solution sources, softwares

LP SOLVING IN EXCEL

Outline

- Step 0: Install Solver

- Step 1: Input Instance and Solve
  - Step 1.1: Input Data, Declare Variables, Objective, and Constraints
  - Step 1.2: Set up solver
  - Step 1.3: Solve (and get reports)

- Step 2: Interpret reports

STEP 0: INSTALL SOLVER

… To be done only if your Excel Software does not have it already installed

Step 0: Install Solver

- Windows: https://www.youtube.com/watch?v=g7C3XXyMV4A

- MacOS: https://www.youtube.com/watch?v=g7C3XXyMV4A

STEP 1: INPUT INSTANCE AND SOLVE

Example 2: Formulation

- Tesla makes two models of cars
  - Model I: makes a profit of $3 million per batch
  - Model II: sells for $5 million per batch
- Tesla has three plants with limited working hours
  - Plant 1: Frame I
    - at most 4 working hours per week
    - 1 hour to prepare a batch of Frame I
  - Plant 2: Frame II
    - at most 12 working hours per week
    - 2 hours to prepare a batch of Frame II
  - Plant 3: Assembly
    - at most 18 working hours per week
    - 3 hours to assemble a batch of Model I (using Frame I) and 2 hours to assemble a batch of Model II (using Frame II)

	C_1	C_2	Availability
P_1	1		4
P_2		2	12
P_3	3	2	18
Profit	3	5	

Question: What is the best product mix?

$\max Z = 3x_1 + 5x_2$ (profit)

$x_1 \leq 4$ (hour constraint for plant 1)

$2x_2 \leq 12$ (hour constraint for plant 2)

$3x_1 + 2x_2 \leq 18$ (hour constraint for plant 3)

$x_1 \geq 0$ (non-negative amount of commodity 1)

$x_2 \geq 0$ (non-negative amount of commodity 2)

Step 1.1.1: Input data

	A	B	C	D
1	**Tesla Production Problem**			
2				
3		Model I	Model II	Availability
4	Number of working hours at Plant 1	1	0	4
5	Number of working hours at Plant 2	0	2	12
6	Number of working hours at Plant 3	3	2	18
7	Profit (in millions)	3	5	
8				

Step 1.1.2: Declare variables and objective

	A	B	C	D
1	**Tesla Production Problem**			
2				
3		Model I	Model II	Availability
4	Number of working hours at Plant 1	1	0	4
5	Number of working hours at Plant 2	0	2	12
6	Number of working hours at Plant 3	3	2	18
7	Profit (in millions)	3	5	
8				
9	**Decision Variables**			
10	Number of batches of model I, x_1			
11	Number of batches of model II, x_2			
12				
13	**Objective**			
14	Total profit	0		
15				

- Profit cell B14 is a changing cell
- It is defined as an excel function "=B7*B10+C7*B11"
  - It represents $3x_1 + 5x_2$

Step 1.1.3: Declare constraints

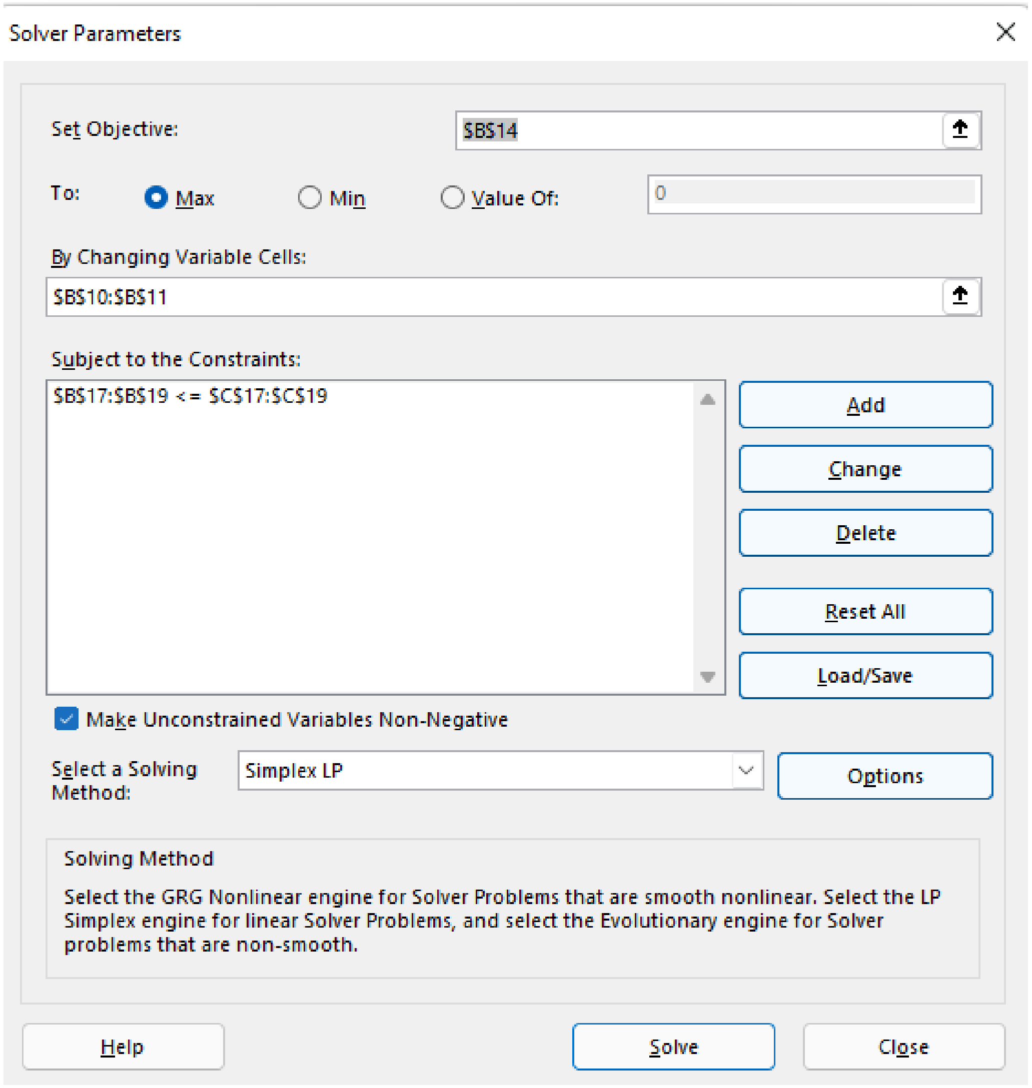
	A	B	C	D
1	**Tesla Production Problem**			
2				
3		Model I	Model II	Availability
4	Number of working hours at Plant 1	1	0	4
5	Number of working hours at Plant 2	0	2	12
6	Number of working hours at Plant 3	3	2	18
7	Profit (in millions)	3	5	
8				
9	**Decision Variables**			
10	Number of batches of model I, x_1			
11	Number of batches of model II, x_2			
12				
13	**Objective**			
14	Total profit	0		
15				
16	**Constraints**	LHS	RHS	
17	Number of working hours at Plant 1	0	4	
18	Number of working hours at Plant 2	0	12	
19	Number of working hours at Plant 3	0	18	

- LHS are changing cells: they are the LHS of constraints
- Examples:
  - B17 cell is defined as an excel function "=B4*B10+C4*B11"
    - It represents 1*x_1 + 0*x_2
  - B18 cell is defined as an excel function "=B5*B10+C5*B11"
    - It represents 0*x_1 + 2x_2

Step 1.2: Setup Solver

- Step 2.1: Open Solver Dialog
  - Data -> Solver
- Step 2.2: Setup Solver
  - Select the cell that represents the “Objective”
    - Choose object cell B14
  - Check “Max” to indicate maximization
  - Select the cells that represent the “Variables”
    - Choose B10:B11 (press “Shift” key to select many cells)
  - Select the cells that represent the “Constraints”
    - Constraints can only involve two adjacent columns
    - Choose B17-B19 and C17-C19
  - Check “Make Unconstrained Variables Non-Negative” (if needed)
  - Select “Simplex LP” to tell the Solver that this is an LP
  - Click ”Solve”

Step 1.2: Setup Solver



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  - Check “Make Unconstrained Variables Non-Negative” (if needed)
  - Select “Simplex LP” to tell the Solver that this is an LP
  - Click ”Solve”

Tips for declaring constraints

The constraint

$B$17:$B$19 <= $C$17:$C$19

is the same as the system of constraints:

$B$17 <= $C$17

$B$18 <= $C$18

$B$19 <= $C$19

But much more convenient to use!

Step 1.3: Solve (and get reports)

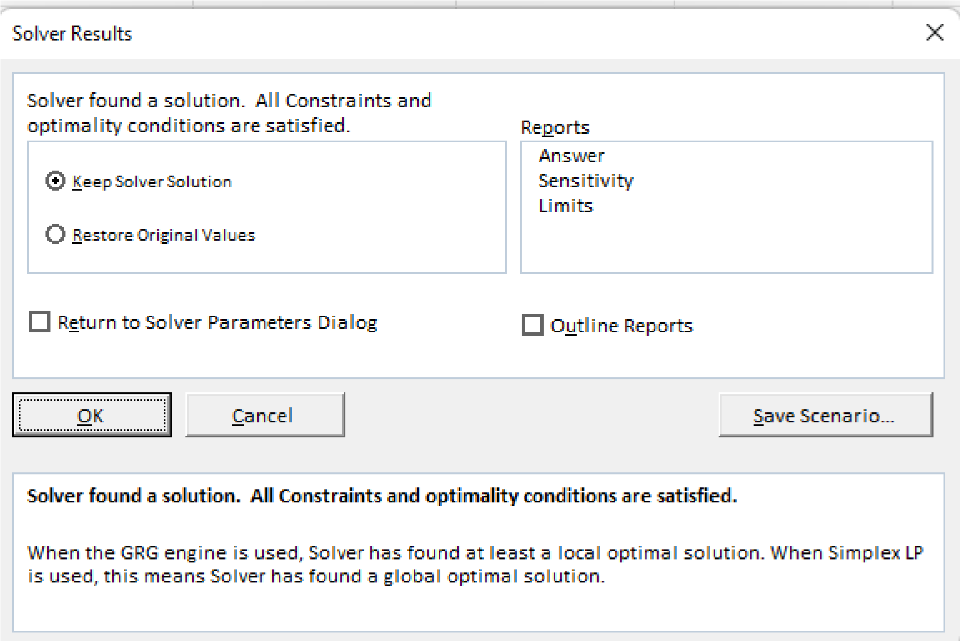
	A	B	C	D
1	**Tesla Production Problem**			
2				
3		Model I	Model II	Availability
4	Number of working hours at Plant 1	1	0	4
5	Number of working hours at Plant 2	0	2	12
6	Number of working hours at Plant 3	3	2	18
7	Profit (in millions)	3	5	
8				
9	**Decision Variables**			
10	Number of batches of model I, x_1	2		
11	Number of batches of model II, x_2	6		
12				
13	**Objective**			
14	Total profit	36		
15				
16	**Constraints**	LHS	RHS	
17	Number of working hours at Plant 1	2	4	
18	Number of working hours at Plant 2	12	12	
19	Number of working hours at Plant 3	18	18	

After clicking solve, see two changes:

- **Change 1:** values in “Decision Variables”, “Objective”, and LHS cells will change
- Optimum decision variable values: (2,6)
- Optimum objective value: 36

Step 1.3: Solve (and get reports)

- **Change 2:** solver window becomes



- There are 3 possible reports. Each one will be a separate tab in the excel file.

STEP 2: INTERPRET REPORTS

1. Answer Report
2. Sensitivity Report
3. Limits Report: No useful information

1. ANSWER REPORT

Answer Report

Microsoft Excel 16.0 Answer Report
Worksheet: [TESLA.xlsx]Sheet1
Report Created: 2/28/2022 4:15:32 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0 Seconds.
Iterations: 2 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$B$14	Total profit Model I	0	36

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$10	Number of batches of model I, x_1 Model I	0	2	Contin
$B$11	Number of batches of model II, x_2 Model I	0	6	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$B$17	Number of working hours at Plant 1 LHS	2	$B$17<=$C$17	Not Binding	2
$B$18	Number of working hours at Plant 2 LHS	12	$B$18<=$C$18	Binding	0
$B$19	Number of working hours at Plant 3 LHS	18	$B$19<=$C$19	Binding	0

Answer Report 1 | Sensitivity Report 1 | Limits Report 1 | Sheet1

- Gives optimum (and initial) values of objective function
- Gives optimum (and initial) values of variables
- For each constraint:
  - Gives the amount of ‘slack’ between LHS and RHS at optimum
  - Whether the constraint is satisfied as an equation (i.e., binding) at optimum

2. SENSITIVITY REPORT

Sensitivity Report

	A	B	C	D	E	F	G	H
1	**Microsoft Excel 16.0 Sensitivity Report**							
2	**Worksheet: [TESLA.xlsx]Sheet1**							
3	**Report Created: 2/28/2022 4:15:32 PM**							
4								
5								
6	Variable Cells							
7				**Final**	**Reduced**	**Objective**	**Allowable**	**Allowable**
8		**Cell**	**Name**	**Value**	**Cost**	**Coefficient**	**Increase**	**Decrease**
9		$B$10	Number of batches of model I, x_1 Model I	2	0	3	4.5	3
10		$B$11	Number of batches of model II, x_2 Model I	6	0	5	1E+30	3
11								
12	Constraints							
13				**Final**	**Shadow**	**Constraint**	**Allowable**	**Allowable**
14		**Cell**	**Name**	**Value**	**Price**	**R.H. Side**	**Increase**	**Decrease**
15		$B$17	Number of working hours at Plant 1 LHS	2	0	4	1E+30	2
16		$B$18	Number of working hours at Plant 2 LHS	12	1.5	12	6	6
17		$B$19	Number of working hours at Plant 3 LHS	18	1	18	6	6
18								

Part 1: Constraints

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$17	Number of working hours at Plant 1 LHS	2	0	4	1E+30	2
$B$18	Number of working hours at Plant 2 LHS	12	1.5	12	6	6
$B$19	Number of working hours at Plant 3 LHS	18	1	18	6	6

- **Constraint R.H. Side:** refers to the constant in the RHS of each constraint
- **Final Value:** refers to the total value of LHS at optimum
- **Shadow price**: rate of increase of optimal objective when RHS changes

  The shadow price is valid only for its **allowable increase** and **allowable decrease**

  - The **allowable increase (decrease)** on a constraint RHS is the maximum amount the RHS can increase (decrease) **without** the shadow price changing
  - If RHS changes to a value in [RHS - allowable decrease, RHS + allowable decrease], then the change in optimum objective value can be directly computed based on shadow price

To see the impact of changing more than one constraint or going beyond the allowable range, we need to re-run the problem

Exercise

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$21	number of working days at Plant 1 1k Model 3	2	0	4	1E+30	2
$B$22	number of working days at Plant 2 1k Model 3	12	1.5	12	6	6
$B$23	number of working days at Plant 3 1k Model 3	18	1	18	6	6

Q1: The range of the RHS b_1 for which the shadow prices are optimal is

Q2: Change b_1 from 4 to 2; what are the new shadow prices?

Q3: Change b_1 from 4 to 1000; what are the new shadow prices?

Q4: Change b_1 from 4 to 1.5; what are the new shadow prices?

Part 2: Variables

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$10	Number of batches of model I, x_1 Model I	2	0	3	4.5	3
$B$11	Number of batches of model II, x_2 Model I	6	0	5	1E+30	3

- **Final value:** the value of each variable at optimum
- **Reduced cost:** will discuss later
- **Objective coefficient** (from problem): coefficient of the objective function of each variable

- **Allowable increase**:

  How much can the objective coefficient increase before the optimal solution changes (in this example, c_1 can go up to $3 + 4.5 = 7.5$)
- **Allowable decrease**:

  How much can the objective coefficient decrease before the optimal solution changes (in this example: c_1 can go down to $3 - 3 = 0$)

Remark: Allowable increase/decrease data are correct only for changes made to coefficient of one variable at a time. If you change more than one variable, need to re-solve the LP.

Exercise

Q1. If the profit per batch of Model I changes from $3 to $6
(i.e., c_1 changes from 3 to 6), will the optimal solution change?

Q2. If the profit per batch of Model II changes from $3 to $8
(i.e., c_1 changes from 3 to 8), will the optimal solution change?

Q3. If the profit per batch of Model II changes to $1000
(i.e., c_2 changes from 5 to 1000), will the optimal solution change?

Exercise: Change the profits for (1) and (3) above, and run Excel to verify.

"Reduced cost": slack at dual constraints

6 Variable Cells

Row	Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
9	$B$10	Number of batches of model I, x_1 Model I	2	0	3	4.5	3
10	$B$11	Number of batches of model II, x_2 Model I	6	0	5	1E+30	3

11

$$\max\ Z = \boldsymbol{c}^T \boldsymbol{x}$$
$$\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$$
$$\boldsymbol{x} \ge \boldsymbol{0}$$

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y}^T\boldsymbol{A} - \boldsymbol{c}^T$	$\boldsymbol{y}^T$	$\boldsymbol{y}^T\boldsymbol{b}$
$\boldsymbol{x}_B$	$\boldsymbol{0}$	$\boldsymbol{B}^{-1}\boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

Reduced cost: $\boldsymbol{c}^T - \boldsymbol{y}^T\boldsymbol{A}$ (negation of slack variable values of the dual LP at optimum)

--appear in Z-row of simplex tableau

--coefficients of original variables in the objective equation

The column in red are two 0's because of

A Strong duality

B Weak duality

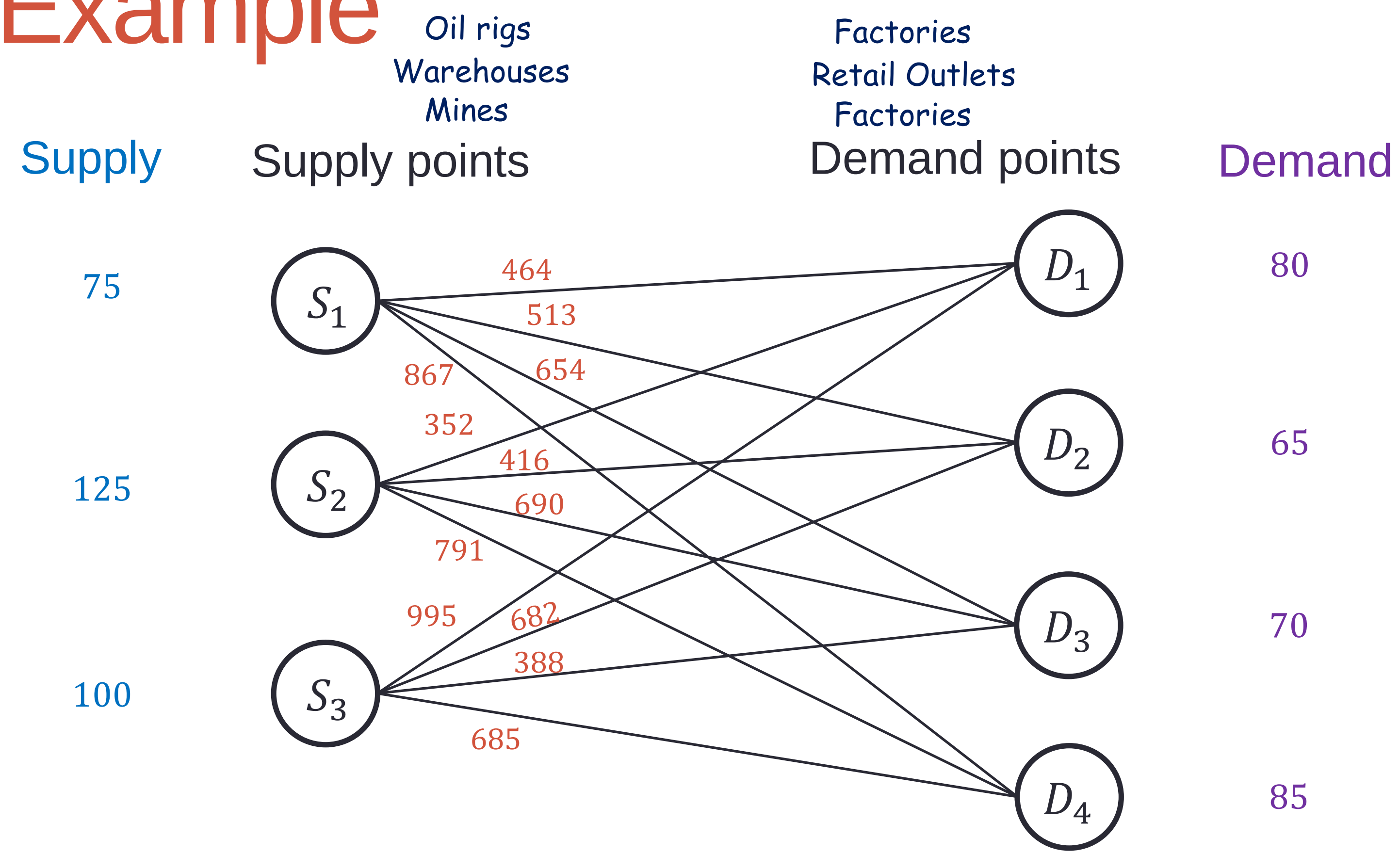
C Complementary slackness Conditions

D Symmetry of LP

TRANSPORTATION PROBLEM

… where we see an algorithm to solve the transportation problem

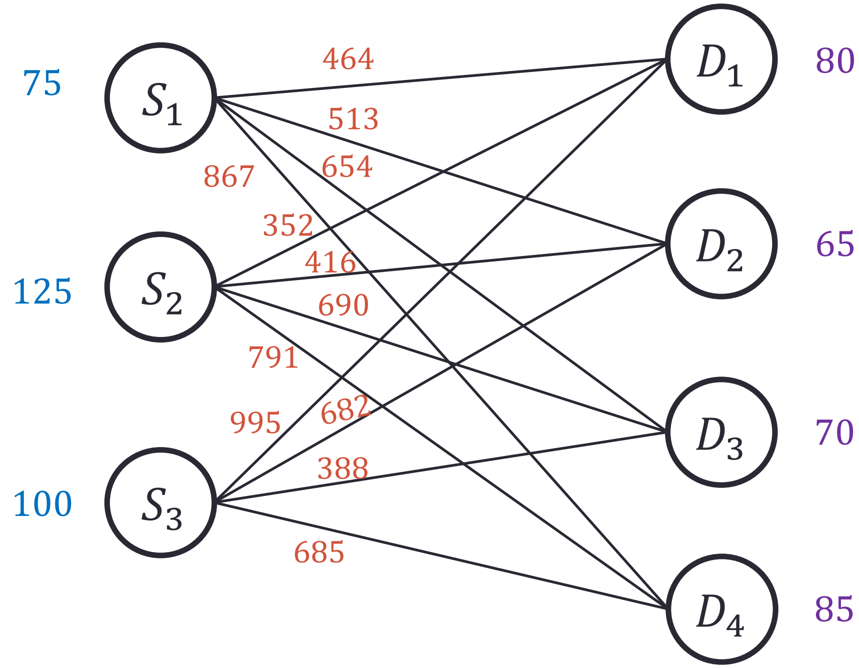
Example



- Single product, say oil, needs to be transported from supply to demand
- Supply constraints: Entire supply at every S_i must be distributed to the demand points
- Demand constraints: Entire demand at every D_j must be received from the supply points
- Transportation Cost per unit from S_i to D_j is $c_{ij} (\geq 0)$ for all $i \in \{1,2,3\}, j \in \{1,2,3,4\}$

Question: How much to transport from each supply point to each demand point to minimize cost?

Formulation of the Example



Step 1: identify decision variables

x_{ij}: quantity to be transported from S_i to D_j, $i \in \{1,2,3\}, j \in \{1,2,3,4\}$

Step 2: determine the objective function

$$\min Z = 464\,x_{11} + 513\,x_{12} + 654\,x_{13} + 867\,x_{14} + 352\,x_{21} + 416\,x_{22} + 690\,x_{23} + 791\,x_{24} + 995\,x_{31} + 682\,x_{32} + 388\,x_{33} + 685\,x_{34}$$

Step 3: identify constraints

$$x_{11} + x_{12} + x_{13} + x_{14} = 75$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 125$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 100$$

Supply constraints: Total outgoing from a supply point to all demand points should be equal to the supply at that point

$$x_{11} + x_{21} + x_{31} = 80$$
$$x_{12} + x_{22} + x_{32} = 65$$
$$x_{13} + x_{23} + x_{33} = 70$$
$$x_{14} + x_{24} + x_{34} = 85$$

Demand constraints: Total incoming into a demand point from all supply points should be equal to the demand requirement at that point

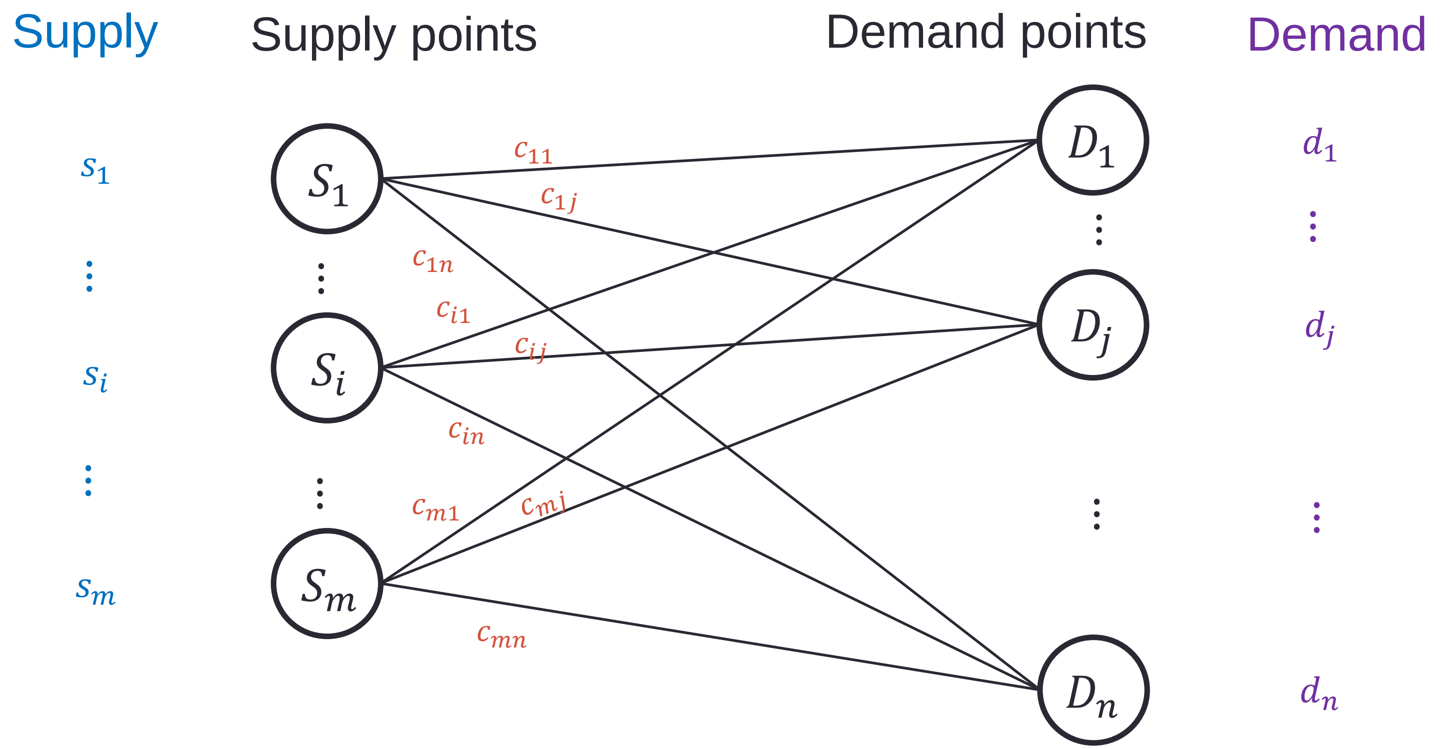
$$x_{ij} \geq 0, i = 1,2,3, j = 1,2,3,4$$

Formulation of the Example

$$\min Z = 464\,x_{11} + 513\,x_{12} + 654\,x_{13} + 867\,x_{14}$$
$$+352\,x_{21} + 416\,x_{22} + 690\,x_{23} + 791\,x_{24}$$
$$+995\,x_{31} + 682\,x_{32} + 388\,x_{33} + 685\,x_{34}$$

$$x_{11} + x_{12} + x_{13} + x_{14} = 75$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 125$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 100$$
$$x_{11} + x_{21} + x_{31} = 80$$
$$x_{12} + x_{22} + x_{32} = 65$$
$$x_{13} + x_{23} + x_{33} = 70$$
$$x_{14} + x_{24} + x_{34} = 85$$
$$x_{ij} \geq 0, i = 1,2,3, j = 1,2,3,4$$

Transportation Problem: General Case

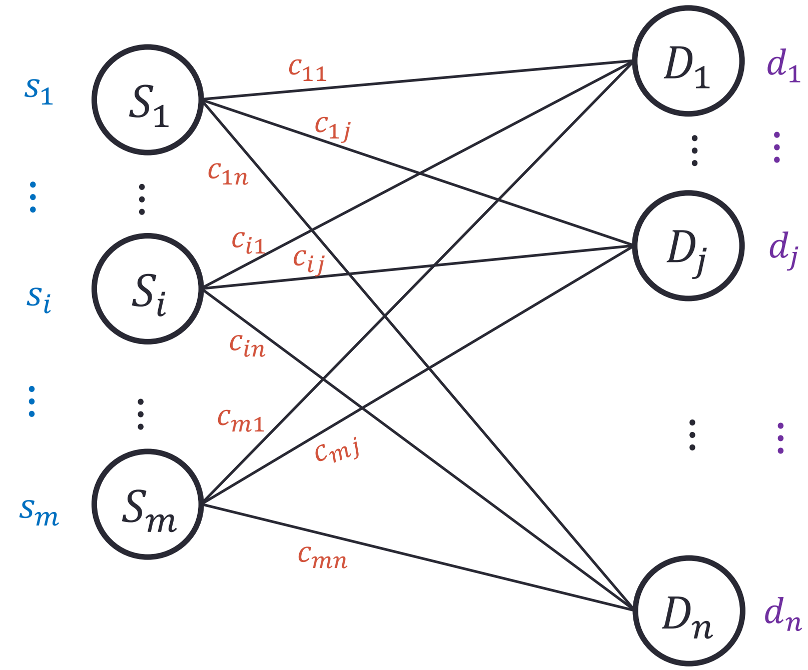


- Single product to be transported from supply to demand
- Transportation Cost per unit from S_i to D_j is $c_{ij} (\geq 0)$ for all $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$
- Supply constraints: Entire supply at every S_i must be distributed to the demand points
- Demand constraints: Entire demand at every D_j must be received from the supply points

$$\Rightarrow \sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$$

Question: How much to transport from each supply point to each demand point to minimize cost?

Formulation of the Example



Step 1: identify decision variables

x_{ij}: quantity to be transported from S_i to D_j, $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$

Step 2: determine the objective function

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Step 3: identify constraints

$$\sum_{j=1}^{n} x_{ij} = s_i \text{ for every } i \in \{1, \dots, m\}$$

$$\sum_{i=1}^{m} x_{ij} = d_j \text{ for every } j \in \{1, \dots, n\}$$

$$x_{ij} \geq 0 \text{ for every } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$$

Supply constraints: Total outgoing from a supply point to all demand points should be equal to the supply at that point

Demand constraints: Total incoming into a demand point from all supply points should be equal to the demand requirement at that point

Formulation of the general case transportation problem

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = s_i \text{ for every } i \in \{1, \dots, m\}$$

$$\sum_{i=1}^{m} x_{ij} = d_j \text{ for every } j \in \{1, \dots, n\}$$

$$x_{ij} \geq 0 \text{ for every } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$$

ALGORITHM FOR THE TRANSPORTATION PROBLEM

- Can use Simplex method, but it is typically slow
- So we have a special-purpose algorithm

… known as the “Transportation Simplex Method”

A Convenient Representation of the Transportation Problem

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	16	16	13	22	17	50
S_2	14	14	13	19	15	60
S_3	19	19	20	23	99	50
S_4	99	0	99	0	0	50
Demand	30	20	70	30	60	

Per unit shipment cost c_{34}

A Convenient Representation of the Transportation Problem

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	16 $x_{11} = 20$	16 $x_{12} = 20$	13 $x_{13} = 10$	22 $x_{14} = 0$	17 $x_{15} = 0$	50
S_2	14 $x_{21} = 10$	14 $x_{22} = 0$	13 $x_{23} = 50$	19 $x_{24} = 0$	15 $x_{25} = 0$	60
S_3	19 $x_{31} = 0$	19 $x_{32} = 0$	20 $x_{33} = 10$	23 $x_{34} = 30$	99 $x_{35} = 10$	50
S_4	99 $x_{41} = 0$	0 $x_{42} = 0$	99 $x_{43} = 0$	0 $x_{44} = 0$	0 $x_{45} = 50$	50
Demand	30	20	70	30	60	

A Convenient Representation of the Transportation Problem

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16] 20	[16] 20	[13] 10	[22] 0	[17] 0	50
S_2	[14] 10	[14] 0	[13] 50	[19] 0	[15] 0	60
S_3	[19] 0	[19] 0	[20] 10	[23] 30	[99] 10	50
S_4	[99] 0	[0] 0	[99] 0	[0] 0	[0] 50	50
Demand	30	20	70	30	60	

We will avoid writing the variables and

- simply write the allocations in the respective cells

Allocation of values for x_{ij}

A Convenient Representation of the Transportation Problem

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	16 / 20	16 / 20	13 / 10	22	17	50
S_2	14 / 10	14	13 / 50	19	15	60
S_3	19	19	20 / 10	23 / 30	99 / 10	50
S_4	99	0	99	0	0 / 50	50
Demand	30	20	70	30	60	$Z = ??$

We will avoid writing the variables and

- simply write the allocations in the respective cells
- simply write the relevant allocations

Test your understanding:
What is the objective value of this solution?

Transportation Simplex Method: Two Stage Algorithm

$$\min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = s_i \text{ for every } i \in \{1, \dots, m\}$$

$$\sum_{i=1}^{m} x_{ij} = d_j \text{ for every } j \in \{1, \dots, n\}$$

$$x_{ij} \geq 0 \text{ for every } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$$

Initialization: Identify an initial **allocation** (i.e., an initial basic feasible solution)

… Via Northwest corner rule

Allocation of values for x_{ij}

Iterations: Iterative method to arrive at an optimal solution

TRANSPORTATION SIMPLEX METHOD

Initialization: identifying an initial basic feasible solution

Northwest Corner Rule

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	[16] 30	[16] 20	[13]	[22]	[17]	~~50~~ 20
S_2	[14]	[14] 0	[13] 60	[19]	[15]	60
S_3	[19]	[19]	[20] 10	[23] 30	[99] 10	~~50~~ ~~40~~ 10
S_4	[99]	[0]	[99]	[0]	[0] 50	50
Demand	30	~~20~~ 0	~~70~~ 10	30	~~60~~ 50	$Z = 3460$

1. Start from northwest corner
2. Allocate as much as possible at x_{ij} while meeting demand and supply constraints
3. If allocation exhausts all supply from S_i, move one row down
4. Else if S_i has any supply remaining, move one column to the right

Initialization Stage: Which rule to use?

- Northwest corner rule is quick and easy
- But it does not take cost into account
  - So, it does not provide an optimum solution
- Other rules that take cost into account exist
  - But none of them provide an optimum solution
  - They only provide an initial basic feasible solution

BASIC FEASIBLE SOLUTION

… to a transportation problem.

- Needed for iterations of the Transportation Simplex Method

Basic Feasible Solution to a Transportation Problem

A **basic feasible solution** is an allocation satisfying the following conditions:

1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly $m + n - 1$ allocations
3. The allocations do not form a loop

the corresponding variables are **basic variables**

Northwest Corner Rule

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	16 / 30	16 / 20	13	22	17	50
S_2	14	14 / 0	13 / 60	19	15	60
S_3	19	19	20 / 10	23 / 30	99 / 10	50
S_4	99	0	99	0	0 / 50	50
Demand	30	20	70	30	60	$Z = 3460$

1. Supply, demand and non-negativity constraints are satisfied
2. Number of allocations $=$ 8, which is $m + n - 1 = 5 + 4 - 1$

Northwest Corner Rule

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	16 $x_{11} = 30$	16 $x_{12} = 20$	13 x_{13}	22 x_{14}	17 x_{15}	50
S_2	14 x_{21}	14 $x_{22} = 0$	13 $x_{23} = 60$	19 x_{24}	15 x_{25}	60
S_3	19 x_{31}	19 x_{32}	20 $x_{33} = 10$	23 $x_{34} = 30$	99 $x_{35} = 10$	50
S_4	99 x_{41}	0 x_{42}	99 x_{43}	0 x_{44}	0 $x_{45} = 50$	50
Demand	30	20	70	30	60	$Z = 3460$

A cell with a value written is an allocation and corresponds to a basic variable
An empty (with no written value) corresponds to a non-basic variable

Basic Feasible Solution to a Transportation Problem

A **basic feasible solution** is an allocation satisfying the following conditions:

1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly $m + n - 1$ allocations
3. The allocations do not form a loop

the corresponding variables are **basic variables**