# Plan for today

- Sensitivity Analysis
  - Addition of a new constraint
  - Addition of a new variable

- Game Theory
  - Mathematical Framework
  - What would we like to understand?

# SENSITIVITY ANALYSIS

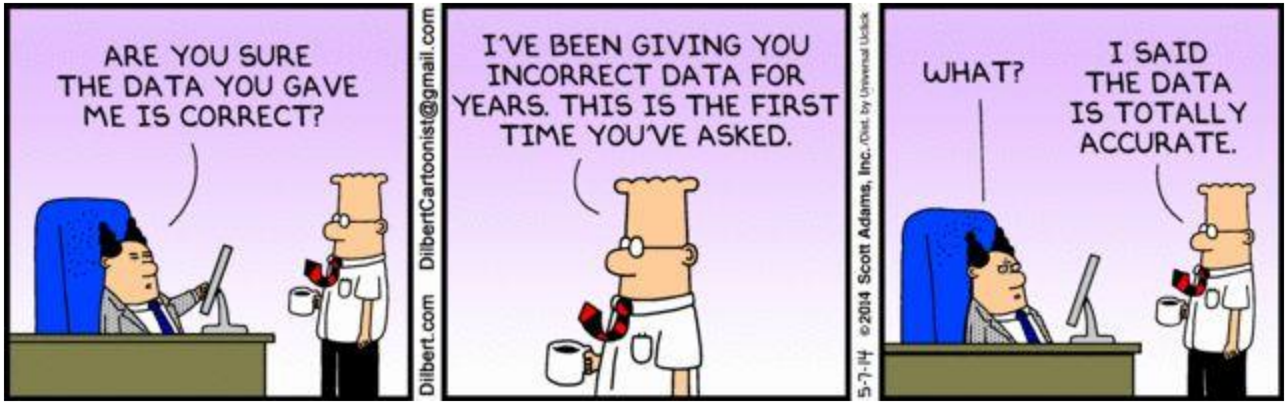
… where we see how to handle change in data after solving the LP

Here is where we will use two tools that we have learnt before:

1. the matrix form of the simplex method and
2. the dual simplex method

# Sensitivity Analysis: Motivation

- Suppose that you have solved the LP via Simplex
- Now, there is some change in data
  - Perhaps, the profit for some item has changed
    or some resource is available more, etc.
  - Perhaps, your boss gave you some incorrect data to begin with!




How do we solve the changed LP?

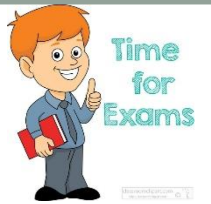
- Obvious approach: Re-solve the LP from scratch
- Advantage of Simplex: Can reuse optimal tableau of the starting problem

# SENSITIVITY ANALYSIS

1. Change in objective

# Recap

## Linear Programming – sensitivity analysis




| Basic Var. | $Z$ | Original Variables | Slack Variables | RHS |
|---|---|---|---|---|
| $Z$ | 1 | $y^T A - c^T$ | $y^T$ | $y^T b$ |
| $x_B$ | $\mathbf{0}$ | $B^{-1}A$ | $B^{-1}$ | $B^{-1}b$ |

When there is a change in **coefficients of the objective function**, the current simplex tableau

- Will be feasible
- May not be optimal
  - If optimality conditions are satisfied, then terminate
  - If not optimal, then continue with simplex iterations

Before applying simplex, remember to ensure that the tableau satisfies:

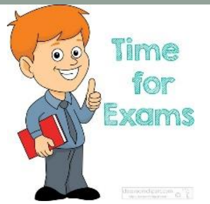
- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one

# SENSITIVITY ANALYSIS

2. Change in RHS

# Recap

## Linear Programming – sensitivity analysis




| Basic Var. | $Z$ | Original Variables | Slack Variables | RHS |
|---|---|---|---|---|
| $Z$ | 1 | $y^T A - c^T$ | $y^T$ | $y^T b$ |
| $x_B$ | $\mathbf{0}$ | $B^{-1} A$ | $B^{-1}$ | $B^{-1} b$ |

When there is a change in **RHS**, the current simplex tableau

- Will satisfy optimality conditions
- May not be feasible
  - If feasible, then terminate
  - If infeasible, then continue with <u>dual simplex iterations</u>

Before applying simplex, remember to ensure that the tableau satisfies:

- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
- When there is a change in RHS, this property will NOT be violated

# SENSITIVITY ANALYSIS

3. Addition of a new constraint

# sensitivity analysis: adding a new constraint

$$\begin{gathered} \max\ Z = 3x_1 + 5x_2 \\ x_1 \le 4 \\ 2x_2 \le 12 \\ 3x_1 + 2x_2 \le 18 \\ x_1 \ge 0, x_2 \ge 0 \end{gathered} \Longrightarrow \begin{gathered} \max\ Z = 3x_1 + 5x_2 \\ x_1 + x_3 = 4 \\ 2x_2 + x_4 = 12 \\ 3x_1 + 2x_2 + x_5 = 18 \\ x_1 + x_2 + x_6 = 4 \\ x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{gathered}$$

Suppose that in Example 2, SIMPLEX method is applied, leading to $x_1^* = 2, x_2^* = 6$

| Basic Var. | $Z$ | $x_1$ (Original) | $x_2$ (Original) | $x_3$ (Slack) | $x_4$ (Slack) | $x_5$ (Slack) | $RHS$ | ratio |
|---|---|---|---|---|---|---|---|---|
| $Z$ | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 | |
| $x_1$ | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 | |
| $x_2$ | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 | |
| $x_3$ | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 | |

Suppose we have an additional constraint: $x_1 + x_2 \le 4$

# sensitivity analysis: adding a new constraint

| Basic Var. | $Z$ | Original Variables | Slack Variables | RHS |
|---|---|---|---|---|
| $Z$ | 1 | $\boldsymbol{y}^T A - \boldsymbol{c}^T$ | $\boldsymbol{y}^T$ | $\boldsymbol{y}^T \boldsymbol{b}$ |
| $x_B$ | $\mathbf{0}$ | $\boldsymbol{B}^{-1} \boldsymbol{A}$ | $\boldsymbol{B}^{-1}$ | $\boldsymbol{B}^{-1} \boldsymbol{b}$ |

Suppose we have an additional constraint: $x_1 + x_2 \leq 4$

Our current optimal solution $x_1^* = 2, x_2^* = 6, x_3^* = 0$ violates this constraint

Qn: What is the new optimal solution?

Q1: which values in the above table change, which ones remain the same?

Q2: is the same solution still feasible?

Q3: are any optimality conditions violated?

# sensitivity analysis: adding a new constraint

| Basic Var. | $Z$ | Original Variables | Slack Variables | RHS |
|---|---|---|---|---|
| $Z$ | 1 | $\boldsymbol{y}^T A - \boldsymbol{c}^T$ | $\boldsymbol{y}^T$ | $\boldsymbol{y}^T \boldsymbol{b}$ |
| $\boldsymbol{x}_B$ | $\mathbf{0}$ | $\boldsymbol{B}^{-1}\boldsymbol{A}$ | $\boldsymbol{B}^{-1}$ | $\boldsymbol{B}^{-1}\boldsymbol{b}$ |

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

Converted to augmented form, we have $x_1 + x_2 + x_6 = 4$

To continue, we must add this constraint to the tableau
- Add a new row for the constraint
- Add a new col for the new slack variable
- Make the new slack variable to be a basic variable

# sensitivity analysis: adding a new constraint

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | | Original | | Slack | | | | |
| $Z$ | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 0 | 36 |
| $x_1$ | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 2 |
| $x_2$ | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 0 | 6 |
| $x_3$ | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | 2 |
| $x_6$ | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 4 |

$$x_1 = 2 - \left(-\frac{1}{3}x_4 + \frac{1}{3}x_5\right)$$

$$x_2 = 6 - \frac{1}{2}x_4$$

$$x_1 + x_2 + x_6 = 4$$

Again tableau is not in proper format!
… since new constraint has more than one basic variable on the LHS

Update the tableau to get it to satisfy:

- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one

… substitute for $x_1$ and $x_2$ into the last constraint and rewrite

# sensitivity analysis: adding a new constraint

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | | Original | | Slack | | | | |
| $Z$ | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 0 | 36 |
| $x_1$ | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 2 |
| $x_2$ | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 0 | 6 |
| $x_3$ | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | 2 |
| $x_6$ | 0 | 0 | 0 | 0 | $-\frac{1}{6}$ | $-\frac{1}{3}$ | 1 | $-4$ |

Simplex cannot continue from here

No entering variable!
Terminate?

Solution is
$x_1 = 2, x_2 = 6, x_3 = 2, x_6 = -4$
Solution is infeasible!

What do we do?

But dual simplex can continue from here!
The tableau is in the format needed to apply dual simplex:

- non-negative $Z$-row
- Equations corresponding to all rows have exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
- (Some RHS entries could be negative)

So reoptimize using dual simplex…

# sensitivity analysis: adding a new constraint

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | | Original | | Slack | | | | |
| $Z$ | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 0 | 36 |
| $x_1$ | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 2 |
| $x_2$ | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 0 | 6 |
| $x_3$ | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 0 | 2 |
| $x_6$ | 0 | 0 | 0 | 0 | $-\frac{1}{6}$ | $-\frac{1}{3}$ | 1 | $-4$ |
| ratio | | | | | $\frac{3/2}{1/6}$ | $\frac{1}{1/3}$ | | |

# sensitivity analysis: adding a new constraint

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | | Original | | Slack | | | | |
| $Z$ | $1$ | $0$ | $0$ | $0$ | $1$ | $0$ | $3$ | $24$ |
| $x_1$ | $0$ | $1$ | $0$ | $0$ | $-\frac{1}{2}$ | $0$ | $1$ | $-2$ |
| $x_2$ | $0$ | $0$ | $1$ | $0$ | $\frac{1}{2}$ | $0$ | $0$ | $6$ |
| $x_3$ | $0$ | $0$ | $0$ | $1$ | $\frac{1}{2}$ | $0$ | $-1$ | $6$ |
| $x_5$ | $0$ | $0$ | $0$ | $0$ | $\frac{1}{2}$ | $1$ | $-3$ | $12$ |
| ratio | | | | | $\frac{1}{1/2}$ | | | |

After dual simplex iteration

# sensitivity analysis: adding a new constraint

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | RHS |
|---|---|---|---|---|---|---|---|---|
| | | Original | | Slack | | | | |
| $Z$ | 1 | 2 | 0 | 0 | 0 | 0 | 5 | 20 |
| $x_4$ | 0 | $-2$ | 0 | 0 | 1 | 0 | $-2$ | 4 |
| $x_2$ | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 4 |
| $x_3$ | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 4 |
| $x_5$ | 0 | 1 | 0 | 0 | 0 | 1 | $-2$ | 10 |
| ratio | | | | | | | | |

After another dual simplex iteration

No leaving variable, so feasibility is attained

Tableau is optimal

Optimal profit reduces to $Z^* = 20$

Optimal solution becomes $x_2^* = 4, x_3^* = 4, x_4^* = 4, x_5^* = 10$

# SENSITIVITY ANALYSIS

4. Addition of a new variable

# sensitivity analysis: adding a new variable

$$\begin{gathered}\max\ Z = 3x_1 + 5x_2 \\ x_1 \leq 4 \\ 2x_2 \leq 12 \\ 3x_1 + 2x_2 \leq 18 \\ x_1 \geq 0, x_2 \geq 0\end{gathered}$$

$$\Rightarrow$$

$$\begin{gathered}\max\ Z = 3x_1 + 5x_2 + 8x_{new} \\ x_1 + 2x_{new} \leq 4 \\ 2x_2 + x_{new} \leq 12 \\ 3x_1 + 2x_2 + x_{new} \leq 18 \\ x_1, x_2, x_{new} \geq 0\end{gathered}$$

Suppose that in Example 2, SIMPLEX method is applied, leading to $x_1^* = 2, x_2^* = 6$

| Basic Var. | $Z$ | $x_1$ (Original) | $x_2$ (Original) | $x_3$ (Slack) | $x_4$ (Slack) | $x_5$ (Slack) | $RHS$ | ratio |
|---|---|---|---|---|---|---|---|---|
| $Z$ | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 | |
| $x_1$ | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 | |
| $x_2$ | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 | |
| $x_3$ | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 | |

Suppose we add a new variable $x_{new}$ to the system with coefficients as above

Qn: What is the new optimal solution?

# sensitivity analysis: adding a new variable

$$\max\ Z = 3x_1 + 5x_2 + 8x_{new}$$
$$x_1 + 2x_{new} \leq 4$$
$$2x_2 + x_{new} \leq 12$$
$$3x_1 + 2x_2 + x_{new} \leq 18$$
$$x_1, x_2, x_{new} \geq 0$$

| Basic Var. | $Z$ | Original Variables | Slack Variables | RHS |
|---|---|---|---|---|
| $Z$ | 1 | $\boldsymbol{y^T A - c^T}$ | $\boldsymbol{y^T}$ | $\boldsymbol{y^T b}$ |
| $\boldsymbol{x_B}$ | **0** | $\boldsymbol{B^{-1}A}$ | $\boldsymbol{B^{-1}}$ | $\boldsymbol{B^{-1}b}$ |

Suppose we add a new variable $x_{new}$ to the system with coefficients as above

Qn: What is the new optimal solution?

Q1: which values in the above table change, which ones remain the same?

Q2: is the same solution still feasible?

Q3: are any optimality conditions violated?

# sensitivity analysis: adding a new variable

| Basic Var. | $Z$ | Original Variables | Slack Variables | RHS |
|---|---|---|---|---|
| $Z$ | 1 | $\boldsymbol{y}^T A - \boldsymbol{c}^T$ | $\boldsymbol{y}^T$ | $\boldsymbol{y}^T \boldsymbol{b}$ |
| $x_B$ | $\mathbf{0}$ | $\boldsymbol{B}^{-1} A$ | $\boldsymbol{B}^{-1}$ | $\boldsymbol{B}^{-1}\boldsymbol{b}$ |

New constraint matrix and objective vector $\boldsymbol{A}_{new} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$ $\boldsymbol{c}_{new}^T = \begin{pmatrix} 3 & 5 & 8 \end{pmatrix}$

From the final tableau of the original problem, we have

$$\boldsymbol{B}^{-1} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \qquad \boldsymbol{y}^T = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix}$$

So we can compute the entries of the tableau for the new parameters

$$\boldsymbol{y}^T \boldsymbol{A}_{new} - \boldsymbol{c}_{new}^T = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{11}{2} \end{pmatrix}$$

$$\boldsymbol{B}^{-1}\boldsymbol{A}_{new} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 2 \end{pmatrix}$$

# sensitivity analysis: adding a new variable

Resulting tableau is

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_{new}$ | $x_3$ | $x_4$ | $x_5$ | $RHS$ | ratio |
|---|---|---|---|---|---|---|---|---|---|
| | | Original | | | Slack | | | | |
| $Z$ | 1 | 0 | 0 | $-\frac{11}{2}$ | 0 | $\frac{3}{2}$ | 1 | 36 | |
| $x_1$ | 0 | 1 | 0 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 | |
| $x_2$ | 0 | 0 | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 6 | |
| $x_3$ | 0 | 0 | 0 | 2 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 | |

Non-optimal

So reoptimize using simplex…

# sensitivity analysis: adding a new variable

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_{new}$ | $x_3$ | $x_4$ | $x_5$ | $RHS$ | ratio |
|---|---|---|---|---|---|---|---|---|---|
| | | Original | | | Slack | | | | |
| $Z$ | 1 | 0 | 0 | $-\frac{11}{2}$ | 0 | $\frac{3}{2}$ | 1 | 36 | |
| $x_1$ | 0 | 1 | 0 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 | |
| $x_2$ | 0 | 0 | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 6 | 12 |
| $x_3$ | 0 | 0 | 0 | 2 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 | 1 |

Simplex Iteration

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_{new}$ | $x_3$ | $x_4$ | $x_5$ | $RHS$ | ratio |
|---|---|---|---|---|---|---|---|---|---|
| | | Original | | | Slack | | | | |
| $Z$ | | | | | | | | | |
| $x_1$ | | | | | | | | | |
| $x_2$ | | | | | | | | | |
| $x_{new}$ | | | | | | | | | |

$\mathrm{Row}_1 - \left(-\frac{11}{2}\right) \times \mathrm{Row}_4^{\mathrm{new}}$

$\mathrm{Row}_2 - 0 \times \mathrm{Row}_4^{\mathrm{new}}$

$\mathrm{Row}_3 - \left(\frac{1}{2}\right) \times \mathrm{Row}_4^{\mathrm{new}}$

$\mathrm{Row}_4^{\mathrm{new}} \leftarrow \mathrm{Row}_4/2$

# sensitivity analysis: adding a new variable

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_{new}$ | $x_3$ | $x_4$ | $x_5$ | $RHS$ | ratio |
|---|---|---|---|---|---|---|---|---|---|
| | | Original | | | Slack | | | | |
| $Z$ | 1 | 0 | 0 | $-\frac{11}{2}$ | 0 | $\frac{3}{2}$ | 1 | 36 | |
| $x_1$ | 0 | 1 | 0 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 | |
| $x_2$ | 0 | 0 | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 6 | 12 |
| $x_3$ | 0 | 0 | 0 | 2 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 | 1 |

Simplex Iteration

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_{new}$ | $x_3$ | $x_4$ | $x_5$ | $RHS$ | ratio |
|---|---|---|---|---|---|---|---|---|---|
| | | Original | | | Slack | | | | |
| $Z$ | 1 | 0 | 0 | 0 | $\frac{11}{4}$ | $\frac{7}{12}$ | $\frac{23}{12}$ | $\frac{83}{2}$ | |
| $x_1$ | 0 | 1 | 0 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 | |
| $x_2$ | 0 | 0 | 1 | 0 | $-\frac{1}{4}$ | $\frac{7}{12}$ | $-\frac{1}{12}$ | $\frac{11}{2}$ | |
| $x_{new}$ | 0 | 0 | 0 | 1 | $\frac{1}{2}$ | $\frac{1}{6}$ | $-\frac{1}{6}$ | 1 | |

$\mathrm{Row}_1 - \left(-\frac{11}{2}\right) \times \mathrm{Row}_4^{\mathrm{new}}$

$\mathrm{Row}_2 - 0 \times \mathrm{Row}_4^{\mathrm{new}}$

$\mathrm{Row}_3 - \left(\frac{1}{2}\right) \times \mathrm{Row}_4^{\mathrm{new}}$

$\mathrm{Row}_4^{\mathrm{new}} \leftarrow \mathrm{Row}_4/2$

# sensitivity analysis: adding a new variable

| Basic Var. | $Z$ | $x_1$ | $x_2$ | $x_{new}$ | $x_3$ | $x_4$ | $x_5$ | $RHS$ | ratio |
|---|---|---|---|---|---|---|---|---|---|
| | | Original | | | | Slack | | | |
| $Z$ | 1 | 0 | 0 | 0 | $\frac{11}{4}$ | $\frac{7}{12}$ | $\frac{23}{12}$ | $\frac{83}{2}$ | |
| $x_1$ | 0 | 1 | 0 | 0 | 0 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 | |
| $x_2$ | 0 | 0 | 1 | 0 | $-\frac{1}{4}$ | $\frac{7}{12}$ | $-\frac{1}{12}$ | $\frac{11}{2}$ | |
| $x_{new}$ | 0 | 0 | 0 | 1 | $\frac{1}{2}$ | $-\frac{1}{6}$ | $\frac{1}{6}$ | 1 | |

No entering variable, so optimality is attained

Tableau is optimal

Optimal profit increases to $Z^* = 41.5$

Optimal solution becomes $x_1^* = 2, x_2^* = 5.5, x_{new}^* = 1$

# sensitivity analysis: adding a new variable

When we added the variable, we computed $x_1$ and $x_2$ cols from scratch using matrix operations

- These cols did not change from their previous optimal basis!
- We could have taken a shortcut and only computed the $x_{new}$ col

Shortcut:

- $\boldsymbol{a_{new}}$ be the column of constraint coefficient for the new var
- $c_{new}$ be its objective coefficient
- Updated coefficients for $x_{new}$ are:

$Z$-row: $\boldsymbol{y^T a_{new}} - c_{new}$

Constraint rows: $\boldsymbol{B^{-1} a_{new}}$

# sensitivity analysis: adding a new variable

| Basic Var. | $Z$ | Original Variables | Slack Variables | RHS |
|---|---|---|---|---|
| $Z$ | 1 | $\boldsymbol{y}^T\boldsymbol{A}-\boldsymbol{c}^T$ | $\boldsymbol{y}^T$ | $\boldsymbol{y}^T\boldsymbol{b}$ |
| $\boldsymbol{x}_B$ | $\mathbf{0}$ | $\boldsymbol{B}^{-1}\boldsymbol{A}$ | $\boldsymbol{B}^{-1}$ | $\boldsymbol{B}^{-1}\boldsymbol{b}$ |

$$\boldsymbol{a}_{new} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$c_{new} = 8$$

The coefficients in the col of $x_{new}$ are

$$\boldsymbol{y}^T\boldsymbol{a}_{new} - c_{new} = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - 8 = -\frac{11}{2}$$

$$\boldsymbol{B}^{-1}\boldsymbol{a}_{new} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 2 \end{pmatrix}$$

# Change management: Beware

If lot of entries among $\boldsymbol{A}, \boldsymbol{c}, \boldsymbol{b}$ change, then the changes are not small …

# Summary of Linear Programming

1. Formulation

2. Solution methods
    - Graphic Method
    - SIMPLEX Method
        - Algebraic Form
        - Tabular Form

3. Duality Theory
    - Shadow Prices
    - Dual linear program
    - Duality Theorems, Weak and Strong duality, Complementary Slackness

4. Dual SIMPLEX Method

5. Matrix Form of Simplex Method

6. Sensitivity Analysis


# GAME THEORY

… where we see how to compute the optimal strategy for 2-player 0-sum games

# Significance

- Mathematics to understand competitive situations
- Commonly used in business, economics to identify best strategies and policies

Nobel Prizes in Economic Sciences

- Nash-Harsanyi-Selton (1994):
  Analysis of equilibria in non-cooperative games
- Aumann-Schelling (2005):
  Why do some countries/individuals promote cooperation while others suffer from conflict?

**Our topic:** 2-player 0-sum games

# WHAT IS A GAME?

… a mathematical framework for games

# Matching Pennies Game

- Two players, each has a penny
- Secretly turn the penny to head or tail in their hand
- Reveal their choices simultaneously
- If pennies match, then player A wins both pennies
- If not, then player B wins both pennies

Representation:

| $A$ \ $B$ |  |  |
|---|---|---|
|  | 1 | $-1$ |
|  | $-1$ | 1 |

Payoff table for Player $A$

$\Rightarrow$ zero-sum game

| $A$ \ $B$ |  |  |
|---|---|---|
|  | $-1$ | 1 |
|  | 1 | $-1$ |

Payoff table for Player $B$

# Two-player, zero-sum game

- A two-player zero-sum game is specified by
- $S_1, \dots, S_m$: **strategies** for Player $A$,
- $T_1, \dots, T_n$: **strategies** for Player $B$
- **Payoff table for $A$:** Shows the gain for Player $A$ for each combination of strategies for the two players

$B$

| | | $T_1$ | … | $T_j$ | … | $T_n$ |
|---|---|---|---|---|---|---|
| | $S_1$ | $p_{11}$ | … | $p_{1j}$ | … | $p_{1n}$ |
| | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $A$ | $S_i$ | $p_{i1}$ | … | $p_{ij}$ | … | $p_{in}$ |
| | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| | $S_m$ | $p_{m1}$ | … | $p_{mj}$ | … | $p_{mn}$ |

Payoff table for Player $A$

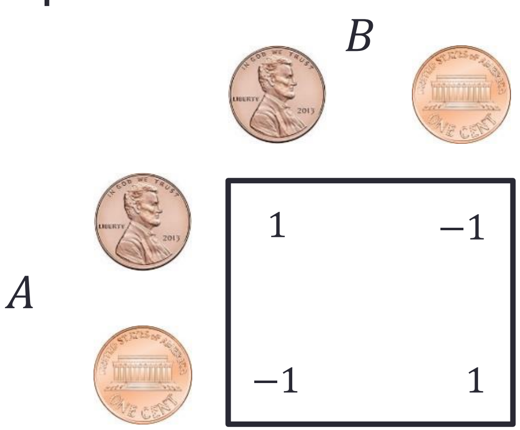
# Main assumption in Game Theory

- Both players are intelligent and rational!

# What would we like to understand?

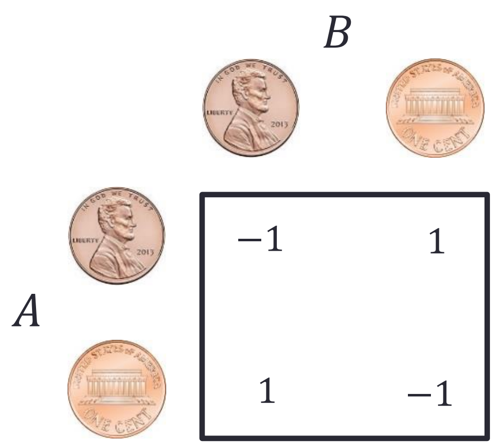
- Two players, each has a penny
- Secretly turn the penny to head or tail in their hand
- Reveal their choices simultaneously
- If pennies match, then player A wins both pennies
- If not, then player B wins both pennies

Representation:




Payoff table for Player $A$

$\Rightarrow$
zero-sum game




Payoff table for Player $B$

# What would we like to understand?

- Player $A$ has two strategies
- Assume that they play the game a large number of times
- What strategy would Player A play?

| | | $B$ | |
|---|---|---|---|
| | |  |  |
| $A$ |  | $1$ | $-1$ |
| |  | $-1$ | $1$ |

Payoff table for Player $A$

# What would we like to understand?

- Player $A$ has two strategies
- Assume that they play the game a large number of times
- What strategy would Player A play?

$B$

| | | $y$ | $1-y$ |
|---|---|---|---|
| | |  |  |
| $A$ | $x$  | $1$ | $-1$ |
| | $1-x$  | $-1$ | $1$ |

Payoff table for Player $A$

$\boxed{x = ??}$

$\boxed{y = ??}$

# What would we like to understand?

- Given: payoff table for $A$
- Assuming that players are intelligent and rational

Question:

- With what probability (i.e., proportion) $x_i$ should player A play each strategy $S_i$ and
- With what probability (i.e., proportion) $y_j$ should player B play each strategy $T_j$

so that $A$ maximizes her profit and B minimizes his loss

| | | | $y_1$ | … | $y_j$ | … | $y_n$ |
|---|---|---|---|---|---|---|---|
| | | | $T_1$ | … | $T_j$ | … | $T_n$ |
| | $x_1$ | $S_1$ | $p_{11}$ | … | $p_{1j}$ | … | $p_{1n}$ |
| | ⋮ | ⋮ | ⋮ | ⋱ | ⋮ | ⋱ | ⋮ |
| $A$ | $x_i$ | $S_i$ | $p_{i1}$ | … | $p_{ij}$ | … | $p_{in}$ |
| | ⋮ | ⋮ | ⋮ | ⋱ | ⋮ | ⋱ | ⋮ |
| | $x_m$ | $S_m$ | $p_{m1}$ | … | $p_{mj}$ | … | $p_{mn}$ |

($B$ labels the columns.)

Payoff table for Player $A$

- Player $A$ knows that Player $B$ is an intelligent player and so will not allow Player $A$ to get more and more profit
- So Player $A$'s objective will be to maximize the minimum profit that she can get
  - Player $A$: **Maximin criterion**
- Similarly, Player $B$'s objective will be to minimize the maximum loss
  - Player $B$: **Minimax criterion**
- **Value of the game** = Payoff to player $A$ when both players play optimally