Plan for today

- Simplex Method in Matrix Form
  - Example
  - Matrix Form vs Tabular Form
  - Dual Opt

- Sensitivity Analysis
  - Motivation
  - Change in obj
  - Change in RHS

Announcements:

- Next HW posted in Gradescope
- All Quiz and Exam dates posted in Canvas (tentative)
- Exam 1 Review Problems posted in Canvas

LP AND SIMPLEX: MATRIX FORM

… where we see how the simplex method is performed by computers

Algebraic form: The math behind Simplex
Tabular form: Simplex on paper with pen (cil and eraser!)
Matrix form: Simplex by computers

LP IN MATRIX FORM

Recall: Linear equations in matrix form

$\boldsymbol{Ax} = \boldsymbol{b}$ ← Equations are coordinate-wise

LP in matrix form

How to express a LP in matrix form? More generally,

Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
subject to:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \\
x_1 \ge 0, x_2 \ge 0, \dots, x_n &\ge 0
\end{aligned}
$$

Suppose that we let

Variable vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ Constraint matrix $\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ RHS vector $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

then $\boldsymbol{Ax} =$ LHS of constraints

We also let Obj vector $\boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \Rightarrow \boldsymbol{c}^T = (c_1 \ \ \dots \ \ c_n)$

then $\boldsymbol{c}^T\boldsymbol{x} =$ obj

and our problem becomes

$$
\begin{aligned}
\max\ Z &= \boldsymbol{c}^T\boldsymbol{x} \\
\boldsymbol{Ax} &\le \boldsymbol{b} \\
\boldsymbol{x} &\ge \boldsymbol{0}
\end{aligned}
$$

← Inequalities are coordinate-wise

Augmented LP in matrix form

More generally, after we introduce slack variables to formulate the augmented LP

$$
\begin{aligned}
\max Z &= c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} &= b_m \\
x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+m} &\geq 0
\end{aligned}
$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Variable vector **Constraint matrix** **RHS vector** **Obj vector**

Correspondingly, we introduce $\boldsymbol{x_s} = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix}$ $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and obtain $(\boldsymbol{A} \quad \boldsymbol{I})$

$m \times m$ identity matrix

Augmented constraint matrix

Then our problem becomes

$$
\begin{aligned}
&\max Z = \boldsymbol{c}^T\boldsymbol{x} \\
&(\boldsymbol{A} \quad \boldsymbol{I})\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \boldsymbol{b} \\
&\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} \geq \boldsymbol{0}
\end{aligned}
$$

Inequalities and equations are coordinate-wise

SIMPLEX METHOD IN MATRIX FORM

Simplex Method (Matrix Form): algorithm

$$\max Z = \boldsymbol{c}^T \boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{pmatrix} \geq \boldsymbol{0}$$

Initialization

- transform the original LP into the augmented LP, say the matrix form is
- determine basic and non-basic variables similar to algebraic form
- Rewrite constraints and objective in proper format:
  1. Identify basis matrix $\boldsymbol{B}$, compute $\boldsymbol{B}^{-1}$, identify $\boldsymbol{c_B}$
  2. Constraints in proper format is obtained by moving non-basic variables to the RHS using
  $$(\boldsymbol{B}^{-1}\boldsymbol{A} \quad \boldsymbol{B}^{-1}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b}$$
  3. Objective in proper format is: $Z = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + \left(\boldsymbol{c}^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}\right)\boldsymbol{x} - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{x}_s$

Iteration

1. Select the entering basic variable: the variable with the largest positive coefficient in the obj. function
2. Select the leaving basic variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering basic variable increases
3. Update the basis matrix and rewrite constraints and objective in proper format

use this method

Termination: Optimality Test

- if no entering variable can be found, then **optimal solution has been found**
- if no leaving variable can be found, **the problem is unbounded**

Simplex Method (Matrix Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \leq 4$$
$$3x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

$$\boldsymbol{Z = c_B^T B^{-1} b + (c^T - c_B^T B^{-1} A)x - c_B^T B^{-1} x_s}$$
$$\boldsymbol{(B^{-1}A \quad B^{-1}) \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b}$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \geq 0$$

matrix representation

$$\boldsymbol{c}^T = (6 \quad 5) \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{x_s} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$(\boldsymbol{A} \quad \boldsymbol{I}) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\max Z = \boldsymbol{c}^T \boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} \geq \boldsymbol{0}$$

Initialization Basic variables: x_3, x_4

$$\boldsymbol{x_B} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{B^{-1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{B^{-1}A} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\boldsymbol{c_B^T} = (0 \quad 0) \qquad \boldsymbol{c_B^T B^{-1}} = (0 \quad 0)$$

$$Z = (0 \quad 0)\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \left((6 \quad 5) - (0 \quad 0)\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (0 \quad 0)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 0 + 6x_1 + 5x_2 + 0x_3 + 0x_4$$

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$Z = 6x_1 + 5x_2$$
$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Basic variables: x_3, x_4

matrix representation

$$c^T = (6 \quad 5) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Simplex Method (Matrix Form): Iteration

Iteration 1:

Basic variables: x_3, x_4

$$Z = 6x_1 + 5x_2$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$Z = c_B^T B^{-1} b + (c^T - c_B^T B^{-1} A) x - c_B^T B^{-1} x_s$$

$$(B^{-1}A \quad B^{-1}) \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

Entering variable: x_1
Leaving variable: x_4

Entering variable: Non-basic Variable x_i with the largest positive value in obj

Leaving basic variable: Min ratio test

New basic variables: x_1, x_3

$$x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \quad B^{-1}A = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$c_B^T = (6 \quad 0) \qquad c_B^T B^{-1} = (0 \quad 2) \qquad c_B^T B^{-1} A = (6 \quad 2)$$

$$\Rightarrow \quad Z = (0 \quad 2)\begin{pmatrix} 4 \\ 5 \end{pmatrix} + ((6 \quad 5) - (6 \quad 2))\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (0 \quad 2)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 10 + 0x_1 + 3x_2 + 0x_3 - 2x_4$$

$$\Rightarrow \quad \begin{pmatrix} 1 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\Rightarrow \quad Z = 10 + 3x_2 - 2x_4$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Basic variables: x_1, x_3

Simplex Method (Matrix Form): Iteration

matrix representation

$$c^T = (6 \quad 5) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Iteration 2:

Basic variables: x_1, x_3

$$Z = 10 + 3x_2 - 2x_4$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

$$Z = c_B^T B^{-1} b + \left(c^T - c_B^T B^{-1} A\right) x - c_B^T B^{-1} x_s$$

$$\left(B^{-1}A \quad B^{-1}\right) \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

Entering variable: x_2
Leaving variable: x_3

New basic variables: x_1, x_2

Entering variable: Non-basic Variable x_i with the largest positive value in obj

Leaving basic variable: Min ratio test

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \quad B^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$c_B^T = (6 \quad 5) \qquad c_B^T B^{-1} = (9 \quad -4) \qquad c_B^T B^{-1} A = (6 \quad 5)$$

⇨ $$Z = (9 \quad -4)\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \left((6 \quad 5) - (6 \quad 5)\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (9 \quad -4)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 16 + 0x_1 + 0x_2 - 9x_3 + 4x_4$$

⇨ $$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

⇨

$$Z = 16 - 9x_3 + 4x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Basic variables: x_1, x_2

Simplex Method (Matrix Form): Iteration

matrix representation

$c^T = (6 \quad 5) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$

$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Iteration 3:

Basic variables: x_1, x_2

$$Z = 16 - 9x_3 + 4x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$Z = c_B^T B^{-1} b + (c^T - c_B^T B^{-1} A) x - c_B^T B^{-1} x_s$$

$$(B^{-1}A \quad B^{-1}) \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

Entering variable: x_4
Leaving variable: x_1

New basic variables: x_2, x_4

Entering variable: Non-basic Variable x_i with the largest positive value in obj

Leaving basic variable: Min ratio test

$$x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad B^{-1}A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$c_B^T = (5 \quad 0) \qquad c_B^T B^{-1} = (5 \quad 0) \qquad c_B^T B^{-1} A = (10 \quad 5)$$

$$\Rightarrow \quad Z = (5 \quad 0) \begin{pmatrix} 4 \\ 5 \end{pmatrix} + ((6 \quad 5) - (10 \quad 5)) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (5 \quad 0) \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 20 - 4x_1 + 0x_2 - 5x_3 + 0x_4$$

$$\Rightarrow \quad \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \Rightarrow$$

$$Z = 20 - 4x_1 - 5x_3$$

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

Basic variables: x_2, x_4

matrix representation

$$c^T = (6 \quad 5) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Simplex Method (Matrix Form): Iteration

~~Iteration 4:~~ Basic variables: x_2, x_4

$$Z = 20 - 4x_1 - 5x_3$$

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

No entering variable ⇒ optimal solution has been found
Terminate!

$$Z = c_B^T B^{-1} b + \left(c^T - c_B^T B^{-1} A\right) x - c_B^T B^{-1} x_s$$

$$(B^{-1}A \quad B^{-1}) \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

Entering variable: Non-basic Variable x_i with the largest positive value in obj

Leaving basic variable: Min ratio test

Simplex Method (Matrix Form): algorithm

$$\max Z = \boldsymbol{c}^T \boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} \geq \boldsymbol{0}$$

Initialization

- transform the original LP into the augmented LP, say the matrix form is
- determine basic and non-basic variables similar to algebraic form
- Rewrite constraints and objective in proper format:
  1. Identify basis matrix $\boldsymbol{B}$, compute $\boldsymbol{B}^{-1}$, identify $\boldsymbol{c_B}$
  2. Constraints in proper format is obtained by moving non-basic variables to the RHS using
  $$(\boldsymbol{B}^{-1}\boldsymbol{A} \quad \boldsymbol{B}^{-1}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b}$$
  3. Objective in proper format is: $Z = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} + \left(\boldsymbol{c}^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{A}\right)\boldsymbol{x} - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{x_s}$

Iteration

1. Select the entering basic variable: the variable with the largest positive coefficient in the obj. function
2. Select the leaving basic variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering basic variable increases
3. Update the basis matrix and rewrite constraints and objective in proper format

use this method

Termination: Optimality Test

- if no entering variable can be found, then **optimal solution has been found**
- if no leaving variable can be found, **the problem is unbounded**

MATRIX FORM AND TABULAR FORM

… where we see the formulae for entries in the table of the tabular form

Simplex Method (Tabular Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \leq 4$$
$$3x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Initialization

$$Z = 6x_1 + 5x_2$$
$$x_3 = 4 - 2x_1 - x_2$$
$$x_4 = 5 - 3x_1 - x_2$$

Basic variables: $x_3 = 4, x_4 = 5$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

$$Z - 6x_1 - 5x_2 = 0$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$

		Original vars		Slack vars		
Basic Var.	Z	x_1	x_2	x_3	x_4	RHS
Z	1	-6	-5	0	0	0
x_3	0	2	1	1	0	4
x_4	0	3	1	0	1	5

Simplex Method (Matrix Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

$$\boldsymbol{Z} = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A})\boldsymbol{x} - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{x}_s$$
$$(\boldsymbol{B}^{-1}\boldsymbol{A} \quad \boldsymbol{B}^{-1})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b}$$

matrix representation

$$\boldsymbol{c}^T = (6 \quad 5) \qquad \boldsymbol{x} = \begin{pmatrix}x_1\\ x_2\end{pmatrix}, \boldsymbol{x}_s = \begin{pmatrix}x_3\\ x_4\end{pmatrix}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) = \begin{pmatrix}2 & 1 & 1 & 0\\ 3 & 1 & 0 & 1\end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix}4\\ 5\end{pmatrix}$$

$$\max Z = \boldsymbol{c}^T\boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} \ge \boldsymbol{0}$$

$$\boldsymbol{x}_B = \begin{pmatrix}x_3\\ x_4\end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \boldsymbol{B}^{-1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \boldsymbol{B}^{-1}\boldsymbol{A} = \begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}$$

$$\boldsymbol{c}_B^T = (0 \quad 0) \qquad \boldsymbol{c}_B^T\boldsymbol{B}^{-1} = (0 \quad 0)$$

Initialization Basic variables: x_3, x_4

$$Z = (0 \quad 0)\begin{pmatrix}4\\ 5\end{pmatrix} + \left((6 \quad 5) - (0 \quad 0)\begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}\right)\begin{pmatrix}x_1\\ x_2\end{pmatrix} - (0 \quad 0)\begin{pmatrix}x_3\\ x_4\end{pmatrix} = 0 + 6x_1 + 5x_2 + 0x_3 + 0x_4$$

$$\begin{pmatrix}2 & 1 & 1 & 0\\ 3 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}4\\ 5\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix}$$

$$Z = 6x_1 + 5x_2$$
$$\begin{pmatrix}x_3\\ x_4\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix} - \begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Basic variables: x_3, x_4

Simplex Method (Tabular Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Initialization

$$Z = 6x_1 + 5x_2$$
$$x_3 = 4 - 2x_1 - x_2$$
$$x_4 = 5 - 3x_1 - x_2$$

Basic variables: $x_3 = 4, x_4 = 5$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

$$Z - 6x_1 - 5x_2 = 0$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$

		Original vars		Slack vars		
Basic Var.	Z	x_1	x_2	x_3	x_4	RHS
Z	1	−6	−5	0	0	0
x_3	0	2	1	1	0	4
x_4	0	3	1	0	1	5

Simplex Method (Matrix Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

$$Z = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A})\boldsymbol{x} - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{x}_s$$
$$(\boldsymbol{B}^{-1}\boldsymbol{A} \quad \boldsymbol{B}^{-1})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b}$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

matrix representation

$$\boldsymbol{c}^T = (6 \quad 5) \qquad \boldsymbol{x} = \begin{pmatrix}x_1\\ x_2\end{pmatrix}, \boldsymbol{x}_s = \begin{pmatrix}x_3\\ x_4\end{pmatrix}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) = \begin{pmatrix}2 & 1 & 1 & 0\\ 3 & 1 & 0 & 1\end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix}4\\ 5\end{pmatrix}$$

$$\max Z = \boldsymbol{c}^T\boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} \ge \boldsymbol{0}$$

$$\boldsymbol{x}_B = \begin{pmatrix}x_3\\ x_4\end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \boldsymbol{B}^{-1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \boldsymbol{B}^{-1}\boldsymbol{A} = \begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}$$

$$\boldsymbol{c}_B^T = (0 \quad 0) \qquad \boldsymbol{c}_B^T\boldsymbol{B}^{-1} = (0 \quad 0)$$

Initialization

Basic variables: x_3, x_4

$$Z = (0 \quad 0)\begin{pmatrix}4\\ 5\end{pmatrix} + \left((6 \quad 5) - (0 \quad 0)\begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}\right)\begin{pmatrix}x_1\\ x_2\end{pmatrix} - (0 \quad 0)\begin{pmatrix}x_3\\ x_4\end{pmatrix} = 0 + 6x_1 + 5x_2 + 0x_3 + 0x_4$$

$$\begin{pmatrix}2 & 1 & 1 & 0\\ 3 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}4\\ 5\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix}$$

$$\Rightarrow$$

$$Z = 6x_1 + 5x_2$$
$$\begin{pmatrix}x_3\\ x_4\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix} - \begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Basic variables: x_3, x_4

Simplex Method (Tabular Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \leq 4$$
$$3x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Initialization

$$Z = 6x_1 + 5x_2$$
$$x_3 = 4 - 2x_1 - x_2$$
$$x_4 = 5 - 3x_1 - x_2$$

Basic variables: $x_3 = 4, x_4 = 5$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

$$Z - 6x_1 - 5x_2 = 0$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$

		Original vars		Slack vars		
Basic Var.	Z	x_1	x_2	x_3	x_4	RHS
Z	1	-6	-5	0	0	0
x_3	0	2	1	1	0	4
x_4	0	3	1	0	1	5

Simplex Method (Matrix Form): Initialization

$$Z = c_B^T B^{-1} b + (c^T - c_B^T B^{-1} A)x - c_B^T B^{-1} x_s$$
$$\begin{pmatrix} B^{-1}A & B^{-1} \end{pmatrix} \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

matrix representation

$c^T = (6 \quad 5)$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$

$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$$\max Z = c^T x$$
$$(A \quad I)\begin{pmatrix} x \\ x_s \end{pmatrix} = b$$
$$\begin{pmatrix} x \\ x_s \end{pmatrix} \ge 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

$x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad B^{-1}A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$

$c_B^T = (0 \quad 0)$ $\quad c_B^T B^{-1} = (0 \quad 0)$

Initialization Basic variables: x_3, x_4

$$Z = (0 \quad 0)\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \left((6 \quad 5) - (0 \quad 0)\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (0 \quad 0)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 0 + 6x_1 + 5x_2 + 0x_3 + 0x_4$$

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\Rightarrow \quad Z = 6x_1 + 5x_2$$
$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Basic variables: x_3, x_4

Simplex Method (Tabular Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

⇩

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

⇩

Initialization

$$Z = 6x_1 + 5x_2$$
$$x_3 = 4 - 2x_1 - x_2$$
$$x_4 = 5 - 3x_1 - x_2$$

Basic variables: $x_3 = 4, x_4 = 5$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

⇨

$$Z - 6x_1 - 5x_2 = 0$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$

⇧

		Original vars		Slack vars		
Basic Var.	Z	x_1	x_2	x_3	x_4	RHS
Z	1	-6	-5	0	0	0
x_3	0	2	1	1	0	4
x_4	0	3	1	0	1	5

Simplex Method (Matrix Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

$$Z = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A})\boldsymbol{x} - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{x}_s$$
$$(\boldsymbol{B}^{-1}\boldsymbol{A} \quad \boldsymbol{B}^{-1})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b}$$

matrix representation

$$\boldsymbol{c}^T = (6 \quad 5) \qquad \boldsymbol{x} = \begin{pmatrix}x_1\\ x_2\end{pmatrix}, \boldsymbol{x}_s = \begin{pmatrix}x_3\\ x_4\end{pmatrix}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) = \begin{pmatrix}2 & 1 & 1 & 0\\ 3 & 1 & 0 & 1\end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix}4\\ 5\end{pmatrix}$$

$$\max Z = \boldsymbol{c}^T\boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} \ge \boldsymbol{0}$$

Initialization

Basic variables: x_3, x_4

$$\boldsymbol{x}_B = \begin{pmatrix}x_3\\ x_4\end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \boldsymbol{B}^{-1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \boldsymbol{B}^{-1}\boldsymbol{A} = \begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}$$

$$\boldsymbol{c}_B^T = (0 \quad 0) \qquad \boldsymbol{c}_B^T\boldsymbol{B}^{-1} = (0 \quad 0)$$

$$Z = (0 \quad 0)\begin{pmatrix}4\\ 5\end{pmatrix} + \left((6 \quad 5) - (0 \quad 0)\begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}\right)\begin{pmatrix}x_1\\ x_2\end{pmatrix} - (0 \quad 0)\begin{pmatrix}x_3\\ x_4\end{pmatrix} = 0 + 6x_1 + 5x_2 + 0x_3 + 0x_4$$

$$\left(\begin{matrix}2 & 1\\ 3 & 1\end{matrix} \;\; \begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right)\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}4\\ 5\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix}$$

$$Z = 6x_1 + 5x_2$$
$$\begin{pmatrix}x_3\\ x_4\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix} - \begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Basic variables: x_3, x_4

Simplex Method (Tabular Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Initialization

$$Z = 6x_1 + 5x_2$$
$$x_3 = 4 - 2x_1 - x_2$$
$$x_4 = 5 - 3x_1 - x_2$$

Basic variables: $x_3 = 4, x_4 = 5$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

$$Z - 6x_1 - 5x_2 = 0$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$

		Original vars		Slack vars		
Basic Var.	Z	x_1	x_2	x_3	x_4	RHS
Z	1	-6	-5	0	0	0
x_3	0	2	1	1	0	4
x_4	0	3	1	0	1	5

Simplex Method (Matrix Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

$$Z = c_B^T B^{-1} b + (c^T - c_B^T B^{-1} A)x - c_B^T B^{-1} x_s$$
$$(B^{-1}A \quad B^{-1}) \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

matrix representation

$$c^T = (6 \quad 5) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$
$$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\max Z = c^T x$$
$$(A \quad I)\begin{pmatrix} x \\ x_s \end{pmatrix} = b$$
$$\begin{pmatrix} x \\ x_s \end{pmatrix} \ge 0$$

Initialization

Basic variables: x_3, x_4

$$x_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B^{-1}A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$$

$$c_B^T = (0 \quad 0) \qquad c_B^T B^{-1} = (0 \quad 0)$$

$$Z = (0 \quad 0)\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \left((6 \quad 5) - (0 \quad 0)\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (0 \quad 0)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 0 + 6x_1 + 5x_2 + 0x_3 + 0x_4$$

$$\left(\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$Z = 6x_1 + 5x_2$$
$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Basic variables: x_3, x_4

Simplex Method (Tabular Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Initialization

$$Z = 6x_1 + 5x_2$$
$$x_3 = 4 - 2x_1 - x_2$$
$$x_4 = 5 - 3x_1 - x_2$$

Basic variables: $x_3 = 4, x_4 = 5$
Non-basic variables: $x_1 = x_2 = 0$
$Z = 0$

$$Z - 6x_1 - 5x_2 = 0$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$

		Original vars		Slack vars		
Basic Var.	Z	x_1	x_2	x_3	x_4	RHS
Z	1	−6	−5	0	0	0
x_3	0	2	1	1	0	4
x_4	0	3	1	0	1	5

Simplex Method (Matrix Form): Initialization

Original LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 \le 4$$
$$3x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

Augmented LP

$$\max Z = 6x_1 + 5x_2$$
$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + x_2 + x_4 = 5$$
$$x_1, x_2, x_3, x_4 \ge 0$$

$$Z = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + (\boldsymbol{c}^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A})\boldsymbol{x} - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{x}_s$$
$$(\boldsymbol{B}^{-1}\boldsymbol{A} \quad \boldsymbol{B}^{-1})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b}$$

matrix representation

$$\boldsymbol{c}^T = (6 \quad 5) \qquad \boldsymbol{x} = \begin{pmatrix}x_1\\ x_2\end{pmatrix}, \boldsymbol{x}_s = \begin{pmatrix}x_3\\ x_4\end{pmatrix}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) = \begin{pmatrix}2 & 1 & 1 & 0\\ 3 & 1 & 0 & 1\end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix}4\\ 5\end{pmatrix}$$

$$\max Z = \boldsymbol{c}^T\boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} \ge \boldsymbol{0}$$

$$\boldsymbol{x}_B = \begin{pmatrix}x_3\\ x_4\end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \boldsymbol{B}^{-1} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \boldsymbol{B}^{-1}\boldsymbol{A} = \begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}$$

$$\boldsymbol{c}_B^T = (0 \quad 0) \qquad \boldsymbol{c}_B^T\boldsymbol{B}^{-1} = (0 \quad 0)$$

Initialization

Basic variables: x_3, x_4

$$Z = (0 \quad 0)\begin{pmatrix}4\\ 5\end{pmatrix} + \left((6 \quad 5) - (0 \quad 0)\begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}\right)\begin{pmatrix}x_1\\ x_2\end{pmatrix} - (0 \quad 0)\begin{pmatrix}x_3\\ x_4\end{pmatrix} = 0 + 6x_1 + 5x_2 + 0x_3 + 0x_4$$

$$\left(\begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}4\\ 5\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix}$$

$$Z = 6x_1 + 5x_2$$
$$\begin{pmatrix}x_3\\ x_4\end{pmatrix} = \begin{pmatrix}4\\ 5\end{pmatrix} - \begin{pmatrix}2 & 1\\ 3 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

Basic variables: x_3, x_4

Simplex Method (Tabular Form): Iteration

			Original vars		Slack vars			
Iteration	**Basic Var.**	Z	x_1	x_2	x_3	x_4	RHS	**ratio**
0	Z	1	-6	-5	0	0	0	
	x_3	0	2	1	1	0	4	2
	x_4	0	3	1	0	1	5	$\frac{5}{3}$
1	Z	1	0	-3	0	2	10	
	x_3	0	0	$\frac{1}{3}$	1	$-\frac{2}{3}$	$\frac{2}{3}$	2
	x_1	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$	5

Simplex Method (Matrix Form): Iteration

matrix representation

$$c^T = (6 \quad 5) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$Z = c_B^T B^{-1} b + (c^T - c_B^T B^{-1} A) x - c_B^T B^{-1} x_s$$

$$(B^{-1}A \quad B^{-1}) \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

iteration 1:

Basic variables: x_3, x_4

$$Z = 6x_1 + 5x_2$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Entering variable: x_1
Leaving variable: x_4

Entering variable: Non-basic Variable x_i with the largest positive value in obj

Leaving basic variable: Min ratio test

New basic variables: x_1, x_3

$$x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \qquad B^{-1}A = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$c_B^T = (6 \quad 0) \qquad c_B^T B^{-1} = (0 \quad 2)$$

$$Z = (0 \quad 2)\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \left((6 \quad 5) - (0 \quad 2)\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (0 \quad 2)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 10 + 0x_1 + 3x_2 + 0x_3 - 2x_4$$

$$\begin{pmatrix} 1 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$Z = 10 + 3x_2 - 2x_4$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Basic variables: x_1, x_3

Simplex Method (Tabular Form): Iteration

Iteration	Basic Var.	Z	x_1	x_2	x_3	x_4	RHS	ratio
			Original vars		Slack vars			
0	Z	1	-6	-5	0	0	0	
	x_3	0	2	1	1	0	4	2
	x_4	0	3	1	0	1	5	$\frac{5}{3}$
1	Z	1	0	-3	0	2	10	
	x_3	0	0	$\frac{1}{3}$	1	$-\frac{2}{3}$	$\frac{2}{3}$	2
	x_1	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$	5
2	Z	1	0	0	9	-4	16	
	x_2	0	0	1	3	-2	2	
	x_1	0	1	0	-1	1	1	1

Simplex Method (Matrix Form): Iteration

matrix representation

$c^T = (6 \quad 5) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$

$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

iteration 2:

Basic variables: x_1, x_3

$$Z = 10 + 3x_2 - 2x_4$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Entering variable: x_2
Leaving variable: x_3

New basic variables: x_1, x_2

$$Z = c_B^T B^{-1} b + \left(c^T - c_B^T B^{-1} A\right) x - c_B^T B^{-1} x_s$$

$$B^{-1}A \, B^{-1} \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

Entering variable: Non-basic Variable x_i with the largest positive value in obj

Leaving basic variable: Min ratio test

$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \quad B^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad c_B^T B^{-1} = (9 \quad 4)$

$c_B^T = (6 \quad 5)$

$$Z = (9 \quad 4)\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \left((6 \quad 5) - (9 \quad 4)\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (9 \quad 4)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 16 + 0x_1 + 0x_2 - 9x_3 + 4x_4$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$Z = 16 - 9x_3 + 4x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Basic variables: x_1, x_2

Simplex Method (Tabular Form): Iteration

Iteration	Basic Var.	Z	x_1 (Original vars)	x_2 (Original vars)	x_3 (Slack vars)	x_4 (Slack vars)	RHS	ratio
1	Z	1	0	-3	0	2	10	
	x_3	0	0	$\frac{1}{3}$	1	$-\frac{2}{3}$	$\frac{2}{3}$	2
	x_1	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$	5
2	Z	1	0	0	9	-4	16	
	x_2	0	0	1	3	-2	2	
	x_1	0	1	0	-1	1	1	1
3	Z	1	4	0	5	0	20	
	x_2	0	2	1	1	0	4	
	x_4	0	1	0	-1	1	1	

Simplex Method (Matrix Form): Iteration

matrix representation

$$c^T = (6 \quad 5) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$Z = c_B^T B^{-1} b + \left(c^T - c_B^T B^{-1} A\right) x - c_B^T B^{-1} x_s$$

$$\left(B^{-1}A \quad B^{-1}\right) \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

iteration 3:

Basic variables: x_1, x_2

$$Z = 16 - 9x_3 + 4x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Entering variable: x_4
Leaving variable: x_1

New basic variables: x_2, x_4

Entering variable: Non-basic Variable x_i with the largest positive value in obj

Leaving basic variable: Min ratio test

$$x_B = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad B^{-1}A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad c_B^T B^{-1} = (5 \quad 0)$$

$$c_B^T = (5 \quad 0)$$

$$\Rightarrow \quad Z = (5 \quad 0)\begin{pmatrix} 4 \\ 5 \end{pmatrix} + \left((6 \quad 5) - (5 \quad 0)\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (5 \quad 0)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 20 - 4x_1 + 0x_2 - 5x_3 + 0x_4$$

$$\Rightarrow \quad \left(\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \Rightarrow$$

$$Z = 20 - 4x_1 - 5x_3$$

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

Basic variables: x_2, x_4

Simplex Method: Matrix form in Tableau

$$\max Z = \boldsymbol{c}^T \boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{pmatrix} \geq \boldsymbol{0}$$

Iteration	Basic Var.	Z	Original Vars	Slack Vars	RHS
0	Z	1	$-\boldsymbol{c}^T$	$\boldsymbol{0}$	$\boldsymbol{0}$
	$\vert$				
	$\boldsymbol{x}_B$	$\boldsymbol{0}$	$\boldsymbol{A}$	$\boldsymbol{I}$	$\boldsymbol{b}$
	$\vert$				
i	Z	1	$\boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A} - \boldsymbol{c}^T$	$\boldsymbol{c}_B^T \boldsymbol{B}^{-1}$	$\boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b}$
	$\vert$				
	$\boldsymbol{x}_B$	$\boldsymbol{0}$	$\boldsymbol{B}^{-1} \boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1} \boldsymbol{b}$
	$\vert$				

MATRIX FORM: SHADOW PRICES/DUAL OPT

Recreating the final simplex tableau

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{A} - \boldsymbol{c}^T$	$\boldsymbol{c}_B^T \boldsymbol{B}^{-1}$	$\boldsymbol{c}_B^T \boldsymbol{B}^{-1} \boldsymbol{b}$
$\boldsymbol{x}_B$	$\boldsymbol{0}$	$\boldsymbol{B}^{-1} \boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1} \boldsymbol{b}$

Recall that shadow prices are coefficients of slack variables in the Z-row
We denote the shadow prices by $\boldsymbol{y}$ where $\boldsymbol{y}^T = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}$

Then we can rewrite the tableau using shadow prices $\boldsymbol{y}$ as follows:

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y}^T \boldsymbol{A} - \boldsymbol{c}^T$	$\boldsymbol{y}^T$	$\boldsymbol{y}^T \boldsymbol{b}$
$\boldsymbol{x}_B$	$\boldsymbol{0}$	$\boldsymbol{B}^{-1} \boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1} \boldsymbol{b}$

Recreating the final simplex tableau

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y}^T\boldsymbol{A} - \boldsymbol{c}^T$	$\boldsymbol{y}^T$	$\boldsymbol{y}^T\boldsymbol{b}$
x_B	$\mathbf{0}$	$\boldsymbol{B}^{-1}\boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

Note that this tableau uses only the following data:

$\boldsymbol{y}$: shadow prices

$\boldsymbol{B}^{-1}$: matrix of constraint coefficients of slack variables when we reach the final tableau

$\boldsymbol{A}$: constraint matrix (of original variables)

$\boldsymbol{b}$: RHS vector

$\boldsymbol{c}$: objective vector (of original variables)

Recreating the final simplex tableau

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y}^T\boldsymbol{A} - \boldsymbol{c}^T$	$\boldsymbol{y}^T$	$\boldsymbol{y}^T\boldsymbol{b}$
$\boldsymbol{x}_B$	$\boldsymbol{0}$	$\boldsymbol{B}^{-1}\boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

Suppose that the data from the original problem changed
i.e., $\boldsymbol{c}, \boldsymbol{A}$ or $\boldsymbol{b}$ changed values

- We can immediately construct a tableau for the new problem according to the final basis of the original problem by using the above formulae with the new values of $\boldsymbol{c}$, $\boldsymbol{A}$ and $\boldsymbol{b}$
- This tableau for the new problem could be non-optimal or infeasible, but we can proceed to iterate from this basis to optimize for the new problem!
- This observation will help us conduct sensitivity analysis on our problem

SENSITIVITY ANALYSIS

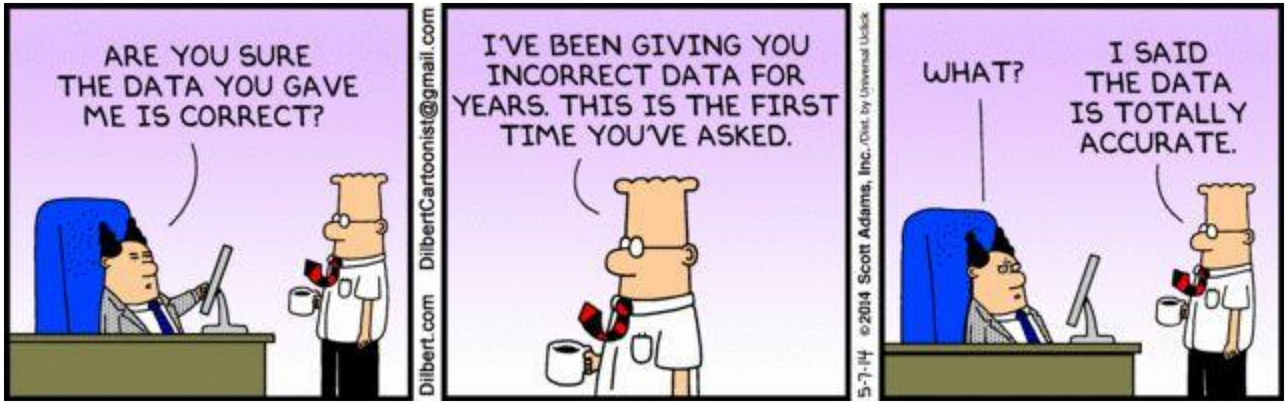
… where we see how to handle change in data after solving the LP

Here is where we will use two tools that we have learnt before:

1. the matrix form of the simplex method and
2. the dual simplex method

Sensitivity Analysis: Motivation

- Suppose that you have solved the LP via Simplex
- Now, there is some change in data
  - Perhaps, the profit for some item has changed
    or some resource is available more, etc.
  - Perhaps, your boss gave you some incorrect data to begin with!



How do we solve the changed LP?

- Obvious approach: Re-solve the LP from scratch
- Advantage of Simplex: Can reuse optimal tableau of the starting problem

SENSITIVITY ANALYSIS

1. Change in objective

sensitivity analysis: change in obj. function

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0, x_2 \ge 0$$

Suppose that in Example, SIMPLEX method is applied, leading to $x_1^* = 2, x_2^* = 6$

Basic Var.	Z	x_1 (Original)	x_2 (Original)	x_3 (Slack)	x_4 (Slack)	x_5 (Slack)	RHS	ratio
Z	1	0	0	0	$\frac{3}{2}$	1	36	
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
x_2	0	0	1	0	$\frac{1}{2}$	0	6	
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	

Qn: Now suppose price of Model I rises from 3 to c_1, does the optimal solution change?

sensitivity analysis: change in obj. function

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y^T A - c^T}$	$\boldsymbol{y^T}$	$\boldsymbol{y^T b}$
x_B	**0**	$\boldsymbol{B^{-1} A}$	$\boldsymbol{B^{-1}}$	$\boldsymbol{B^{-1} b}$

Qn: Now suppose price of Model I rises from 3 to c_1, does the optimal solution change?

Q1: which values in the above table change, which ones remain the same?

Q2: is the same solution still feasible?

Q3: are any optimality conditions violated?

sensitivity analysis: change in obj. function

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y^T A - c^T}$	$\boldsymbol{y^T}$	$\boldsymbol{y^T b}$
$\boldsymbol{x_B}$	$\boldsymbol{0}$	$\boldsymbol{B^{-1} A}$	$\boldsymbol{B^{-1}}$	$\boldsymbol{B^{-1} b}$

Example 1: say $c_1 = 6$

$$\boldsymbol{c_{new}^T} = (6 \quad 5)$$

Z-row changes while the rest of the entries stay the same

$$\boldsymbol{y^T A - c_{new}^T} = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{pmatrix} - (6 \quad 5) = (-3 \quad 0)$$

Basic Var.	Z	x_1 (Original)	x_2 (Original)	x_3 (Slack)	x_4 (Slack)	x_5 (Slack)	RHS	ratio
Z	1	-3	0	0	$\frac{3}{2}$	1	36	
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
x_2	0	0	1	0	$\frac{1}{2}$	0	6	
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	

sensitivity analysis: change in obj. function

Example 1: say $c_1 = 6$

Basic Var.	Z	x_1	x_2 Original	x_3	x_4 Slack	x_5	RHS	ratio
Z	1	-3	0	0	$\frac{3}{2}$	1	36	
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
x_2	0	0	1	0	$\frac{1}{2}$	0	6	
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	

$$Z - 3x_1 + \frac{3}{2}x_4 + x_5 = 36$$

$$x_1 - \frac{1}{3}x_4 + \frac{1}{3}x_5 = 2$$

Tableau is not in proper format!

… since obj. is expressed as a function of a basic variable

Use the constraint corresponding to the basic variable x_1 to substitute for x_1 in Z

$$Z + \frac{1}{2}x_4 + 2x_5 = 42$$

Update the tableau to get it to satisfy:

- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one

sensitivity analysis: change in obj. function

Example 1: say $c_1 = 6$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
		Original		Slack				
Z	1	0	0	0	$\frac{1}{2}$	2	42	
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
x_2	0	0	1	0	$\frac{1}{2}$	0	6	
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	

No entering variable, so optimality is attained

Objective has increased to $Z^* = 42$
Same solution is optimal
$x_1^* = 2, x_2^* = 6, x_3^* = 2$

Note: Shadow prices change from (0,3/2,1) to (0,1/2, 2)

sensitivity analysis: change in obj. function

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y}^T\boldsymbol{A}-\boldsymbol{c}^T$	$\boldsymbol{y}^T$	$\boldsymbol{y}^T\boldsymbol{b}$
$\boldsymbol{x}_B$	$\boldsymbol{0}$	$\boldsymbol{B}^{-1}\boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

Example 2: say $c_1 = 9$

$$\boldsymbol{c}_{new}^T = (9 \quad 5)$$

Z-row changes while the rest of the entries stay the same

$$\boldsymbol{y}^T\boldsymbol{A} - \boldsymbol{c}_{new}^T = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{pmatrix} - (9 \quad 5) = (-6 \quad 0)$$

Basic Var.	Z	x_1 (Original)	x_2 (Original)	x_3 (Slack)	x_4 (Slack)	x_5 (Slack)	RHS	ratio
Z	1	-6	0	0	$\frac{3}{2}$	1	36	
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
x_2	0	0	1	0	$\frac{1}{2}$	0	6	
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	

sensitivity analysis: change in obj. function

Example 2: say $c_1 = 9$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
		Original		Slack				
Z	1	-6	0	0	$\frac{3}{2}$	1	36	
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
x_2	0	0	1	0	$\frac{1}{2}$	0	6	
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	

$$Z - 6x_1 + \frac{3}{2}x_4 + x_5 = 36$$

Again tableau is not in proper format!
… since obj. is expressed as a function of a basic variable

Update the tableau to get it to satisfy:

- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one

sensitivity analysis: change in obj. function

Example 2: say $c_1 = 9$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
		Original		Slack				
Z	1	0	0	0	$-\frac{1}{2}$	3	48	
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
x_2	0	0	1	0	$\frac{1}{2}$	0	6	
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	

Tableau is not optimal! Reoptimize using the simplex algorithm

sensitivity analysis: change in obj. function

Example 2: say $c_1 = 9$

Basic Var.	Z	x_1 (Original)	x_2 (Original)	x_3 (Slack)	x_4 (Slack)	x_5 (Slack)	RHS	ratio
Z	1	0	0	0	$-\frac{1}{2}$	3	48	
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
x_2	0	0	1	0	$\frac{1}{2}$	0	6	$\frac{6}{1/2}$
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	$\frac{2}{1/3}$

Tableau is not optimal! Reoptimize using the simplex algorithm

Simplex iteration

	Basic Var.	Z	x_1 (Original)	x_2 (Original)	x_3 (Slack)	x_4 (Slack)	x_5 (Slack)	RHS	ratio
$\text{Row}_1 - \left(-\frac{1}{2}\right) \times \text{Row}_4^{\text{new}}$	Z								
$\text{Row}_2 - \left(-\frac{1}{3}\right) \times \text{Row}_4^{\text{new}}$	x_1								
$\text{Row}_3 - \left(\frac{1}{2}\right) \times \text{Row}_4^{\text{new}}$	x_2								
$\text{Row}_4^{\text{new}} \leftarrow \frac{\text{Row}_4}{1/3}$	x_4								

sensitivity analysis: change in obj. function

Example 2: say $c_1 = 9$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
		Original		Slack				
Z	1	0	0	0	$-\frac{1}{2}$	3	48	
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	
x_2	0	0	1	0	$\frac{1}{2}$	0	6	$\frac{6}{1/2}$
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2	$\frac{2}{1/3}$

Tableau is not optimal! Reoptimize using the simplex algorithm

Simplex iteration

	Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
			Original		Slack				
$\text{Row}_1 - \left(-\frac{1}{2}\right) \times \text{Row}_4^{\text{new}}$	Z	1	0	0	$\frac{3}{2}$	0	$\frac{5}{2}$	51	
$\text{Row}_2 - \left(-\frac{1}{3}\right) \times \text{Row}_4^{\text{new}}$	x_1	0	1	0	1	0	0	4	
$\text{Row}_3 - \left(\frac{1}{2}\right) \times \text{Row}_4^{\text{new}}$	x_2	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	
$\text{Row}_4^{\text{new}} \leftarrow \frac{\text{Row}_4}{1/3}$	x_4	0	0	0	3	1	-1	6	

sensitivity analysis: change in obj. function

Example 2: say $c_1 = 9$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS	ratio
		Original		Slack				
Z	1	0	0	$\frac{3}{2}$	0	$\frac{5}{2}$	51	
x_1	0	1	0	1	0	0	4	
x_2	0	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	3	
x_4	0	0	0	3	1	-1	6	

No entering variable, so optimality is attained

Objective has increased to $Z^* = 51$

Optimal solution is $x_1^* = 4, x_2^* = 3, x_4^* = 6$

SENSITIVITY ANALYSIS

2. Change in RHS

sensitivity analysis: change in RHS

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0, x_2 \ge 0$$

$$\begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 \\ b_2 \\ 18 \end{pmatrix}$$

Suppose that in Example, SIMPLEX method is applied, leading to $x_1^* = 2, x_2^* = 6$

Basic Var.	Z	x_1 (Original)	x_2 (Original)	x_3 (Slack)	x_4 (Slack)	x_5 (Slack)	RHS
Z	1	0	0	0	$\frac{3}{2}$	1	36
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
x_2	0	0	1	0	$\frac{1}{2}$	0	6
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2

Qn: Now suppose available working hours of Plant 2 changes from 12 to b_2, does the optimal solution change?

sensitivity analysis: change in RHS

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y^T A - c^T}$	$\boldsymbol{y^T}$	$\boldsymbol{y^T b}$
x_B	**0**	$\boldsymbol{B^{-1} A}$	$\boldsymbol{B^{-1}}$	$\boldsymbol{B^{-1} b}$

Qn: Now suppose available working hours of Plant 2 changes from 12 to b_2, does the optimal solution change?

Q1: which values in the above table change, which ones remain the same?

Q2: is the same solution still feasible?

Q3: are any optimality conditions violated?

sensitivity analysis: change in RHS

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y}^T\boldsymbol{A}-\boldsymbol{c}^T$	$\boldsymbol{y}^T$	$\boldsymbol{y}^T\boldsymbol{b}$
x_B	**0**	$\boldsymbol{B}^{-1}\boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

$$\boldsymbol{b} = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix}$$

$$\boldsymbol{b}_{new} = \begin{pmatrix} 4 \\ 6 \\ 18 \end{pmatrix}$$

Example 1: say $b_2 = 6$

RHS changes while the rest of the entries stay the same

From the final tableau of the original problem, we have

$$\boldsymbol{B}^{-1} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \qquad \boldsymbol{y}^T = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix}$$

So we can compute the entries of the tableau for the new parameters

$$\boldsymbol{y}^T\boldsymbol{b}_{new} = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \\ 18 \end{pmatrix} = 27 \quad \text{and } \boldsymbol{B}^{-1}\boldsymbol{b}_{new} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 4 \\ 6 \\ 18 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$$

sensitivity analysis: change in RHS

Example 1: say $b_2 = 6$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS
		Original		Slack			
Z	1	0	0	0	$\frac{3}{2}$	1	27
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	4
x_2	0	0	1	0	$\frac{1}{2}$	0	3
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	0

No entering variable, so optimality is attained

Tableau is optimal

Optimal profit reduces to $Z^* = 27$

Optimal solution becomes $x_1^* = 4, x_2^* = 3, x_3^* = 0$

sensitivity analysis: change in RHS

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$\boldsymbol{y}^T\boldsymbol{A} - \boldsymbol{c}^T$	$\boldsymbol{y}^T$	$\boldsymbol{y}^T\boldsymbol{b}$
x_B	**0**	$\boldsymbol{B}^{-1}\boldsymbol{A}$	$\boldsymbol{B}^{-1}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$

$$\boldsymbol{b} = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} \quad \boldsymbol{b}_{new} = \begin{pmatrix} 4 \\ 24 \\ 18 \end{pmatrix}$$

Example 2: say $b_2 = 24$

RHS changes while the rest of the entries stay the same

From the final tableau of the original problem, we have

$$\boldsymbol{B}^{-1} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \qquad \boldsymbol{y}^T = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix}$$

So we can compute the entries of the tableau for the new parameters

$$\boldsymbol{y}^T\boldsymbol{b}_{new} = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 24 \\ 18 \end{pmatrix} = 54 \quad \text{and } \boldsymbol{B}^{-1}\boldsymbol{b}_{new} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 4 \\ 24 \\ 18 \end{pmatrix} = \begin{pmatrix} -2 \\ 12 \\ 6 \end{pmatrix}$$

sensitivity analysis: change in RHS

Example 2: say $b_2 = 24$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS
		Original		Slack			
Z	1	0	0	0	$\frac{3}{2}$	1	54
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	-2
x_2	0	0	1	0	$\frac{1}{2}$	0	12
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6

No entering variable!
Terminate?

Solution is $x_1 = -2, x_2 = 12, x_3 = 6$ which is infeasible!

What do we do?

Simplex cannot continue from here

But dual simplex can continue from here!
The tableau is in the format needed to apply dual simplex:

- non-negative Z-row
- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
- (Some RHS entries could be negative)

So reoptimize using dual simplex method…

sensitivity analysis: change in RHS

Example 2: say $b_2 = 24$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS
		Original			Slack		
Z	1	0	0	0	$\frac{3}{2}$	1	54
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	-2
x_2	0	0	1	0	$\frac{1}{2}$	0	12
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
ratio					$\frac{3/2}{1/3}$		

Dual Simplex iteration

Row operation	Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS
$\text{Row}_1 - \left(\frac{3}{2}\right) \times \text{Row}_2^{\text{new}}$	Z							
$\text{Row}_2^{\text{new}} \leftarrow \text{Row}_2/(-1/3)$	x_4							
$\text{Row}_3 - \left(\frac{1}{2}\right) \times \text{Row}_2^{\text{new}}$	x_2							
$\text{Row}_4 - \left(\frac{1}{3}\right) \times \text{Row}_2^{\text{new}}$	x_3							

sensitivity analysis: change in RHS

Example 2: say $b_2 = 24$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS
		Original			Slack		
Z	1	0	0	0	$\frac{3}{2}$	1	54
x_1	0	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	-2
x_2	0	0	1	0	$\frac{1}{2}$	0	12
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	6
ratio					$\frac{3/2}{1/3}$		

Dual Simplex iteration

	Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS
$\text{Row}_1 - \left(\frac{3}{2}\right) \times \text{Row}_2^{\text{new}}$	Z	1	$\frac{9}{2}$	0	0	0	$\frac{3}{2}$	45
$\text{Row}_2^{\text{new}} \leftarrow \text{Row}_2/(-1/3)$	x_4	0	-3	0	0	1	-1	6
$\text{Row}_3 - \left(\frac{1}{2}\right) \times \text{Row}_2^{\text{new}}$	x_2	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
$\text{Row}_4 - \left(\frac{1}{3}\right) \times \text{Row}_2^{\text{new}}$	x_3	0	1	0	1	0	0	4

sensitivity analysis: change in RHS

Example 2: say $b_2 = 24$

Basic Var.	Z	x_1	x_2	x_3	x_4	x_5	RHS
Z	1	$\frac{9}{2}$	0	0	0	$\frac{3}{2}$	45
x_4	0	-3	0	0	1	-1	6
x_2	0	$\frac{3}{2}$	1	0	0	$\frac{1}{2}$	9
x_3	0	1	0	1	0	0	4
ratio							

No leaving variable, so feasibility is attained

Tableau is optimal

Optimal profit increases to
$Z^* = 45$
Optimal solution becomes
$x_2^* = 9, x_3^* = 4, x_4^* = 6$

Linear Programming – sensitivity analysis

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$y^T A - c^T$	y^T	$y^T b$
x_B	$\mathbf{0}$	$B^{-1}A$	B^{-1}	$B^{-1}b$

When there is a change in **coefficients of the objective function**, the current simplex tableau

- Will be feasible
- May not be optimal
    - If optimality conditions are satisfied, then terminate
    - If not optimal, then continue with simplex iterations

Before applying simplex, remember to ensure that the tableau satisfies:

- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one

Recap

Linear Programming – sensitivity analysis

Basic Var.	Z	Original Variables	Slack Variables	RHS
Z	1	$y^T A - c^T$	y^T	$y^T b$
x_B	0	$B^{-1}A$	B^{-1}	$B^{-1}b$

When there is a change in **RHS**, the current simplex tableau

- Will satisfy optimality conditions
- May not be feasible
  - If feasible, then terminate
  - If infeasible, then continue with dual simplex iterations

Before applying simplex, remember to ensure that the tableau satisfies:

- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
- When there is a change in RHS, this property will NOT be violated