Plan for today

- Dual Simplex Method

- Matrix Form
  - LP
  - Simplex Method

Obtaining the dual problem (general case)

Primal LP

$$\text{Max } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \quad \rightarrow y_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \quad \rightarrow y_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \quad \rightarrow y_m$$
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

Dual LP

$$\text{Min } b_1y_1 + b_2y_2 + \cdots + b_my_m$$
$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \quad \rightarrow x_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2 \quad \rightarrow x_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n \quad \rightarrow x_n$$
$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0$$

n variables, *m* constraints

maximization problem

m variables, *n* constraints,

minimization problem

Symmetry Property: The dual of the dual LP is the primal LP

Simplex: An alternative viewpoint

- Simplex Algorithm begins at a primal basic feasible solution
- Explores adjacent basic feasible solutions until all Z-row coefficients are non-negative

The Z-row coefficients give the complementary dual solution!

- The complementary dual solution becomes feasible when all Z-row coefficients are non-negative
- At that point, the algorithm has achieved an optimal solution due to the complementary slackness property and so it terminates

Another way to view Simplex

- Begin at a primal basic feasible solution, continue until the complementary dual solution becomes feasible

Can we do the opposite?

- Dual Simplex Method: Begin at a dual basic feasible solution, continue until the complementary primal solution becomes feasible

DUAL SIMPLEX METHOD

… where we see a variant of the simplex method motivated by Duality Theory

Dual Simplex Method

- Starting tableau should have the following properties:
  - All Z-row entries are non-negative
  - Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
  - (Some RHS entries could be negative)

- Iteration:
  1. Determine the leaving basic variable: variable with the most negative RHS (should be strictly negative)
  2. Determine the entering variable (new min-ratio test):
     Choices: non-basic variables with a negative entry in the row of the leaving basic variable
     Selection: pick the one with the smallest absolute value of the ratio between its Z-row entry and its entry in the row of the leaving basic variable
  3. Update tableau similar to simplex method

- Termination: Feasibility test
  - If all RHS values are non-negative, STOP
  - The basic solution in this tableau is primal feasible and hence optimal

Dual Simplex Method: Initialization

$$\min Z = 6x_1 + 8x_2 + 5x_3$$
$$x_1 + x_2 + x_3 \geq 20$$
$$x_1 + x_3 \geq 4$$
$$x_2 + x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

LP

⇨

$$\max Z = -6x_1 - 8x_2 - 5x_3$$
$$-x_1 - x_2 - x_3 \leq -20$$
$$-x_1 - x_3 \leq -4$$
$$-x_2 - x_3 \leq -8$$
$$x_1, x_2, x_3 \geq 0$$

LP in standard form

Dual Simplex Method: Initialization

- All Z-row entries are non-negative
- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
- (Some RHS entries could be negative)

Original LP

$$\max Z = -6x_1 - 8x_2 - 5x_3$$
$$-x_1 - x_2 - x_3 \le -20$$
$$-x_1 - x_3 \le -4$$
$$-x_2 - x_3 \le -8$$
$$x_1, x_2, x_3 \ge 0$$

Augmented LP

$$\max Z = -6x_1 - 8x_2 - 5x_3$$
$$-x_1 - x_2 - x_3 + s_1 = -20$$
$$-x_1 - x_3 + s_2 = -4$$
$$-x_2 - x_3 + s_3 = -8$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

Initialization

$$\max Z = -6x_1 - 8x_2 - 5x_3$$
$$s_1 = -20 + x_1 + x_2 + x_3$$
$$s_2 = -4 + x_1 + x_3$$
$$s_3 = -8 + x_2 + x_3$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

Basic variables: $s_1 = -20, s_2 = -4, s_3 = -8$
Non-basic variables: $x_1 = x_2 = x_3 = 0$
$Z = 0$

$$Z + 6x_1 + 8x_2 + 5x_3 = 0$$
$$-x_1 - x_2 - x_3 + s_1 = -20$$
$$-x_1 - x_3 + s_2 = -4$$
$$-x_2 - x_3 + s_3 = -8$$

Basic Var.	Z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
Z	1	6	8	5	0	0	0	0
s_1	0	−1	−1	−1	1	0	0	−20
s_2	0	−1	0	−1	0	1	0	−4
s_3	0	0	−1	−1	0	0	1	−8

Dual Simplex Method: Iteration

1. Select leaving variable **(pivot row)**: the variable with the most **negative** entry in the RHS col
2. Select entering variable **(pivot col) by min-ratio test**:
    1. Divide each Z-row entry by the absolute value of the pivot row entry if the pivot row entry is strictly negative
    2. Select the col with the smallest of these ratios

   The number in the intersection of the pivot row and the pivot col is the **pivot number**

Basic Var.	Z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
Z	1	6	8	5	0	0	0	0
s_1	0	-1	-1	-1	1	0	0	-20
s_2	0	-1	0	-1	0	1	0	-4
s_3	0	0	-1	-1	0	0	1	-8
ratio		6	8	5				

Dual Simplex Method: Iteration

3. Replace the pivot row variable by the variable in the pivot col
4. Divide the pivot row by the pivot number to obtain the new pivot row

   For every other Row_i:

      Multiply the new pivot row by the entry in the pivot col of Row_i and subtract from Row_i

Basic Var.	Z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
Z	1	6	8	5	0	0	0	0
s_1	0	−1	−1	−1	1	0	0	−20
s_2	0	−1	0	−1	0	1	0	−4
s_3	0	0	−1	−1	0	0	1	−8
ratio		6	8	5				

	Basic Var.	Z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
$\text{Row}_1 - 5 \times \text{Row}_2^{\text{new}}$	Z	1	1	3	0	5	0	0	−100
$\text{Row}_2^{\text{new}} \leftarrow \text{Row}_2/(-1)$	x_3	0	1	1	1	−1	0	0	20
$\text{Row}_3 - (-1) \times \text{Row}_2^{\text{new}}$	s_2	0	0	1	0	−1	1	0	16
$\text{Row}_4 - (-1) \times \text{Row}_2^{\text{new}}$	s_3	0	1	0	0	−1	0	1	12

Dual Simplex Method: Termination

1. Select leaving variable **(pivot row)**: the variable with the most **negative** entry in the RHS col
2. Select entering variable **(pivot col) by min-ratio test**:
    1. Divide each Z-row entry by the absolute value of the pivot row entry if the pivot row entry is strictly negative
    2. Select the col with the smallest of these ratios

    The number in the intersection of the pivot row and the pivot col is the **pivot number**

Basic Var.	Z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
Z	1	1	3	0	5	0	0	-100
x_3	0	1	1	1	-1	0	0	20
s_2	0	0	1	0	-1	1	0	16
s_3	0	1	0	0	-1	0	1	12

RHS is non-negative ⇒ no leaving variable ⇒ Terminate!

Optimum solution is $x_1^* = 0, x_2^* = 0, x_3^* = 20$, Opt obj is $Z^* = -100$.
Recall that we multiplied the objective by -1 earlier to convert to maximization.
We convert back to the minimization objective by multiplying by -1 again.
So, optimal obj value of the original LP is 100.

Dual Simplex Method: algorithm

Initialization

- Transform original LP into augmented LP
- Find a basic solution such that $Z = c_0 + c_1 x_1 + \cdots c_n x_n$ in proper format (i.e., no basic variables) has non-positive coefficients. Basic solution may not be feasible.
- Rewrite constraints with all variables on the LHS: The constant on the RHS *could be negative*
- Rewrite objective with all variables on the LHS
- Setup dual simplex tableau:
  - One column for each variable plus additional RHS column
  - One row for each constraint plus additional Z-row
  - Enter the coefficients of the corresponding variable in each constraint and enter the RHS
  - Specify the basic variables

Iteration

1. Select the **pivot row**: the variable with the most **negative** entry in the RHS
2. Select the **pivot col** by **min-ratio test**:
   1. Divide each Z-row entry by the absolute value of the pivot row entry if the pivot row entry is strictly negative
   2. Select the col with the smallest of these ratios

   The entry at the intersection of the pivot row and the pivot col is the **pivot number**
3. Replace the basic variable in the pivot row by the variable in the pivot col
4. Divide the pivot row by the pivot number. For every other Row_i :
   1. Multiply the new pivot row by the entry in the pivot col of Row_i and subtract from Row_i

Termination: Feasibility Test

- if no pivot row can be found, then **optimal solution has been found**
- if no pivot col can be found, then **the primal problem is infeasible** (i.e., the dual problem is unbounded)

Dual Simplex Method applications

- The dual simplex method can be applied from any simplex tableau satisfying:
  - All Z-row coefficients non-negative
  - Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
  - (Some RHS entries could be negative)

- The dual simplex method operates on the dual of the LP, beginning with (and maintaining) a dual feasible solution and a primal complementary pair
  and iterating until it finds a primal complementary pair that is feasible

LP AND SIMPLEX: MATRIX FORM

… where we see how the simplex method is performed by computers

Algebraic form: The math behind Simplex
Tabular form: Simplex on paper with pen (cil and eraser!)
Matrix form: Simplex by computers

MATRIX: BACKGROUND

Matrix Algebra

- *A matrix* is a rectangular array of numbers

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3m} \\ \dots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix}$$

- The matrix is $n \times m$ if it has n rows and m columns
- For matrix $\boldsymbol{A}$, the entry in row *i* and column *j* is a_{ij}
- Matrix $\boldsymbol{A}$ is sometimes expressed as $\boldsymbol{A} = \{a_{ij}\}$, the collection of its indexed elements
- Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are identical if every element is the same (i.e., $a_{ij} = b_{ij}$ for all *i,j*)

Matrix Operations

(i) Matrix Addition: $\boldsymbol{A} + \boldsymbol{B} = \{a_{ij} + b_{ij}\}$

(add elements individually, **A** and **B** must have the same dimensions)

$$\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 7 & 10 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 12 & 16 \end{pmatrix}$$

(ii) Scalar Multiplication: $k\boldsymbol{A} = \{ka_{ij}\}$ (multiply each element by k)

$$3\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 15 & 18 \end{pmatrix}$$

(iii) Matrix Multiplication: $\boldsymbol{AB} = \{i, j$'th entry is $\sum_k a_{ik}\, b_{kj}\}$

(i,j'th entry is the dot product of row *i* of **A** and column *j* of **B**)

If $\boldsymbol{A}$ is an $n \times m$ matrix and $\boldsymbol{B}$ is an $r \times s$ matrix, then

1. For $\boldsymbol{AB}$ to exist, it must be that $m = r$
2. If it exists, $\boldsymbol{AB}$ is an $n \times s$ matrix

$$\begin{pmatrix} 1 & 3 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 3 \times 7 & 1 \times 3 + 3 \times 5 \\ 7 \times 2 + 2 \times 7 & 7 \times 3 + 2 \times 5 \end{pmatrix} = \begin{pmatrix} 23 & 18 \\ 28 & 31 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ 7x_1 + 2x_2 \end{pmatrix}$$

Matrix Operations

(iv) Transpose: $\boldsymbol{A}^T = \{a_{ji}\}$

(v) The identity matrix I: $\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$

(vi) The zero matrix 0: $\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{A}$

$$\boldsymbol{A0} = \boldsymbol{0A} = \boldsymbol{0}$$

All matrices can be decomposed into *submatrices*

Vectors are single column matrices

A set of linear equations can be represented in matrix form:

$$\boldsymbol{Ax} = \boldsymbol{b}$$

Notation:
BOLDFACE UPPERCASE = matrix
boldface lowercase = vector

Matrix Operations: Inverse

- A must be square
- A^{-1} exists if the matrix has full row or column rank

  (i.e, rank = number of rows = number of columns)
- $\boldsymbol{AA^{-1} = A^{-1}A = I}$
- If $\boldsymbol{AB} = \boldsymbol{I}$ or $\boldsymbol{BA} = \boldsymbol{I}$ (with $\boldsymbol{A}$ and $\boldsymbol{B}$ having the same dimensions), then $\boldsymbol{B} = \boldsymbol{A^{-1}}$

Example: Let $A = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} 5 & -1 \\ -4 & 1 \end{pmatrix}$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc}$

LP IN MATRIX FORM

Recall: Linear equations in matrix form

$\boldsymbol{Ax} = \boldsymbol{b}$ ⟵ Equations are coordinate-wise

LP in matrix form

How to express a LP in matrix form?

Take this familiar LP for example

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$
$$x_1 \ge 0, x_2 \ge 0$$

Variable vector | Constraint matrix | RHS vector

Suppose that we let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{pmatrix}$ $\quad \boldsymbol{b} = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix}$

then $\boldsymbol{Ax} = \begin{pmatrix} x_1 \\ 2x_2 \\ 3x_1 + 2x_2 \end{pmatrix}$

Obj vector

We also let $\boldsymbol{c} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \Rightarrow \boldsymbol{c}^T = (3 \quad 5)$

then $\boldsymbol{c}^T\boldsymbol{x} = 3x_1 + 5x_2$

and our problem becomes

$$\max\ Z = \boldsymbol{c}^T\boldsymbol{x}$$
$$\boldsymbol{Ax} \le \boldsymbol{b}$$
$$\boldsymbol{x} \ge \boldsymbol{0}$$

← Inequalities are coordinate-wise

LP in matrix form

How to express a LP in matrix form? More generally,

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \\ x_1 \ge 0, x_2 \ge 0, \dots, x_n &\ge 0 \end{aligned}$$

Suppose that we let

Variable vector $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ **Constraint matrix** $\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ **RHS vector** $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

then $\boldsymbol{Ax}$ = LHS of constraints

Obj vector

We also let $\boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \Rightarrow \boldsymbol{c}^T = (c_1 \;\; \dots \;\; c_n)$

then $\boldsymbol{c}^T\boldsymbol{x}$ = obj

and our problem becomes

$$\begin{aligned} \max\ Z &= \boldsymbol{c}^T\boldsymbol{x} \\ \boldsymbol{Ax} &\le \boldsymbol{b} \\ \boldsymbol{x} &\ge \boldsymbol{0} \end{aligned}$$

← Inequalities are coordinate-wise

Augmented LP in matrix form

Next we introduce slack variables to formulate the augmented LP

$$\begin{aligned} \max\ Z &= 3x_1 + 5x_2 \\ x_1 + x_3 &= 4 \\ 2x_2 + x_4 &= 12 \\ 3x_1 + 2x_2 + x_5 &= 18 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} \quad \boldsymbol{c} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Variable vector **Constraint matrix** **RHS vector** **Obj vector**

Correspondingly, we introduce $\boldsymbol{x_s} = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}$ $\quad \boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow (\boldsymbol{A} \quad \boldsymbol{I}) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix}$$

Augmented constraint matrix

then $(\boldsymbol{A} \quad \boldsymbol{I})\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ 2x_2 + x_4 \\ 3x_1 + 2x_2 + x_5 \end{pmatrix}$

so our problem becomes

$$\begin{aligned} &\max Z = \boldsymbol{c}^T \boldsymbol{x} \\ &(\boldsymbol{A} \quad \boldsymbol{I})\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \boldsymbol{b} \\ &\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} \geq \boldsymbol{0} \end{aligned}$$

← Inequalities and equations are coordinate-wise

Augmented LP in matrix form

More generally, after we introduce slack variables to formulate the augmented LP

$$\begin{aligned} \max Z &= c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + x_{n+2} &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + x_{n+m} &= b_m \\ x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m} &\geq 0 \end{aligned}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Variable vector

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Constraint matrix

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

RHS vector

$$\boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Obj vector

Correspondingly, we introduce $\boldsymbol{x_s} = \begin{pmatrix} x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix}$ $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and obtain $(\boldsymbol{A} \quad \boldsymbol{I})$

$m \times m$ identity matrix

Augmented constraint matrix

Then our problem becomes

$$\begin{aligned} &\max Z = \boldsymbol{c}^T \boldsymbol{x} \\ &(\boldsymbol{A} \quad \boldsymbol{I}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \boldsymbol{b} \\ &\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} \geq \boldsymbol{0} \end{aligned}$$

Inequalities and equations are coordinate-wise

Simplex Method (Algebraic Form): algorithm

- **Initialization**
  - transform the original LP into the augmented LP, determine basic and non-basic variables
  - Rewrite constraints in proper format:
    - one basic variable on the LHS with coefficient 1,
    - constants and non-basic variables on the RHS and
    - RHS constant should be non-negative
  - Rewrite objective function in proper format: contains only non-basic variables

These two steps are done to obtain obj and constraints in proper format after we know the basic variables

- **Iteration**
  1. select the entering variable: the variable with the largest positive coefficient in the obj. function
  2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
  3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
  4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

- **Termination: Optimality Test**
  - if every variable in the objective function is with a negative coefficient, then no entering variable can be found ⇒ **the optimal solution has been found**
  - if the entering variable can be increased to infinity without driving any other basic variable to below zero, then no leaving variable can be found ⇒ **the problem is unbounded**

OBTAINING SIMPLEX PROPER FORMAT IN MATRIX FORM

… once we know the collection of basic variables

SIMPLEX proper format in matrix representation: Constraints

Augmented LP

$$\max\ Z = 3x_1 + 5x_2$$
$$x_1 + x_3 = 4$$
$$2x_2 + x_4 = 12$$
$$3x_1 + 2x_2 + x_5 = 18$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Matrix representation

$$\max \boldsymbol{c}^T \boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} \geq \boldsymbol{0}$$

Matrix representation

$$\max Z = (3 \quad 5) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} \text{ and } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \geq 0$$

Simplex: selects basic variables and non-basic variables, and transforms constraints into proper format

If x_1, x_2, x_3 are basic vars,
then the constraints in proper format are

$$x_1 = 2 + \frac{1}{3}x_4 - \frac{1}{3}x_5$$
$$x_2 = 6 - \frac{1}{2}x_4$$
$$x_3 = 2 - \frac{1}{3}x_4 + \frac{1}{3}x_5$$

Matrix representation: the process is equivalent to

1. Identify **basis matrix $\boldsymbol{B}$**: column vectors corresponding to basic variables in the augmented constraint matrix
2. Compute $\boldsymbol{B}^{-1}$

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 2 & 0 \end{pmatrix}$$

$$\boldsymbol{B}^{-1} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

SIMPLEX proper format in matrix representation: Constraints

Matrix representation

$$\max \boldsymbol{c}^T\boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} \geq \boldsymbol{0}$$

Matrix form (contd…)

3. multiply both sides of the equality constraints

$$(\boldsymbol{A} \quad \boldsymbol{I})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{b}$$

by $\boldsymbol{B}^{-1}$ to obtain

$$(\boldsymbol{B}^{-1}\boldsymbol{A} \quad \boldsymbol{B}^{-1})\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b}$$

4. move non-basic variables to the right

$$\boldsymbol{B}^{-1}\begin{pmatrix}1 & 0 & 1 & 0 & 0\\ 0 & 2 & 0 & 1 & 0\\ 3 & 2 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix} = \mathbf{B}^{-1}\begin{pmatrix}4\\ 12\\ 18\end{pmatrix}$$

⇓

$$\begin{pmatrix}1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3}\\ 0 & 1 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{pmatrix} = \begin{pmatrix}2\\ 6\\ 2\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}2\\ 6\\ 2\end{pmatrix} - \begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{2} & 0\\ \frac{1}{3} & -\frac{1}{3}\end{pmatrix}\begin{pmatrix}x_4\\ x_5\end{pmatrix}$$

which is the same as $\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}2\\ 6\\ 2\end{pmatrix} + \begin{pmatrix}\frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{2} & 0\\ -\frac{1}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}x_4\\ x_5\end{pmatrix}$

SIMPLEX proper format in matrix representation: Objective

Simplex: substitute for the basic variables in the objective function

$$x_1 = 2 + \frac{1}{3}x_4 - \frac{1}{3}x_5$$
$$x_2 = 6 - \frac{1}{2}x_4$$
$$x_3 = 2 - \frac{1}{3}x_4 + \frac{1}{3}x_5$$

Basic vars: x_1, x_2, x_3

$$\begin{aligned} Z &= 3x_1 + 5x_2 \\ &= 3\left(2 + \frac{1}{3}x_4 - \frac{1}{3}x_5\right) + 5\left(6 - \frac{1}{2}x_4\right) \\ &= 36 - \frac{3}{2}x_4 - x_5 \end{aligned}$$

Matrix representation: the process is equivalent to

1. Identify $\boldsymbol{c_B}$: column vector of coefficients of basic variables in the objective

$$\boldsymbol{c_B} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$$

2. Multiply both sides of the constraints

$$\begin{pmatrix} \boldsymbol{B^{-1}A} & \boldsymbol{B^{-1}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \boldsymbol{B^{-1}b}$$

by $\boldsymbol{c_B^T}$ to obtain

$$\begin{pmatrix} \boldsymbol{c_B^T B^{-1} A} & \boldsymbol{c_B^T B^{-1}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x_s} \end{pmatrix} = \boldsymbol{c_B^T B^{-1} b}$$

$$\boldsymbol{c_B^T B^{-1} b - c_B^T B^{-1} A x - c_B^T B^{-1} x_s = 0}$$

$$\begin{pmatrix} 3 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = 36$$

$$36 - 3x_1 - 5x_2 - \frac{3}{2}x_4 - x_5 = 0$$

3. Add the above to the objective

$$Z = \boldsymbol{c^T x + c_B^T B^{-1} b - c_B^T B^{-1} A x - c_B^T B^{-1} x_s}$$

$$Z = \boldsymbol{c_B^T B^{-1} b} + \left(\boldsymbol{c^T - c_B^T B^{-1} A}\right)\boldsymbol{x} - \boldsymbol{c_B^T B^{-1} x_s}$$

$$Z = 3x_1 + 5x_2 + \left(36 - 3x_1 - 5x_2 - \frac{3}{2}x_4 - x_5\right)$$

$$Z = 36 - \frac{3}{2}x_4 - x_5$$

SIMPLEX METHOD IN MATRIX FORM

Simplex Method (Algebraic Form): algorithm

- **Initialization**
  - transform the original LP into the augmented LP, determine basic and non-basic variables
  - Rewrite constraints in proper format:
    - one basic variable on the LHS with coefficient 1,
    - constants and non-basic variables on the RHS and
    - RHS constant should be non-negative
  - Rewrite objective function in proper format: contains only non-basic variables
- **Iteration**
  1. select the entering variable: the variable with the largest positive coefficient in the obj. function
  2. select the leaving variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering variable increases
  3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
  4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints
- **Termination: Optimality Test**
  - if every variable in the objective function is with a negative coefficient, then no entering variable can be found ⇒ **the optimal solution has been found**
  - if the entering variable can be increased to infinity without driving any other basic variable to below zero, then no leaving variable can be found ⇒ **the problem is unbounded**

Simplex Method (Matrix Form): algorithm

$$\max Z = \boldsymbol{c}^T \boldsymbol{x}$$
$$(\boldsymbol{A} \quad \boldsymbol{I}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{pmatrix} = \boldsymbol{b}$$
$$\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{pmatrix} \geq \boldsymbol{0}$$

Initialization

- transform the original LP into the augmented LP, say the matrix form is
- determine basic and non-basic variables similar to algebraic form
- Rewrite constraints and objective in proper format:
  1. Identify basis matrix $\boldsymbol{B}$, compute $\boldsymbol{B}^{-1}$, identify $\boldsymbol{c}_B$
  2. Constraints in proper format is obtained by moving non-basic variables to the RHS using
     $$(\boldsymbol{B}^{-1}\boldsymbol{A} \quad \boldsymbol{B}^{-1}) \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_s \end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b}$$
  3. Objective in proper format is: $Z = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{b} + \left(\boldsymbol{c}^T - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{A}\right)\boldsymbol{x} - \boldsymbol{c}_B^T \boldsymbol{B}^{-1}\boldsymbol{x}_s$

Iteration

1. Select the entering basic variable: the variable with the largest positive coefficient in the obj. function
2. Select the leaving basic variable **(min-ratio test)**: the first basic variable that reaches 0 as the value of the entering basic variable increases
3. Update the basis matrix and rewrite constraints and objective in proper format

use this method

Termination: Optimality Test

- if no entering variable can be found, then **optimal solution has been found**
- if no leaving variable can be found, **the problem is unbounded**

Simplex Method (Matrix Form): Initialization

Original LP

$$\begin{aligned} &\max Z = 6x_1 + 5x_2 \\ &2x_1 + x_2 \le 4 \\ &3x_1 + x_2 \le 5 \\ &x_1, x_2 \ge 0 \end{aligned}$$

Augmented LP

$$\begin{aligned} &\max Z = 6x_1 + 5x_2 \\ &2x_1 + x_2 + x_3 = 4 \\ &3x_1 + x_2 + x_4 = 5 \\ &x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

$$Z = \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b} + \left(\boldsymbol{c}^T - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{A}\right)\boldsymbol{x} - \boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{x}_s$$

$$\begin{pmatrix}\boldsymbol{B}^{-1}\boldsymbol{A} & \boldsymbol{B}^{-1}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b}$$

matrix representation

$$\boldsymbol{c}^T = \begin{pmatrix}6 & 5\end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix}, \boldsymbol{x}_s = \begin{pmatrix}x_3\\x_4\end{pmatrix}$$

$$\begin{pmatrix}\boldsymbol{A} & \boldsymbol{I}\end{pmatrix} = \begin{pmatrix}2 & 1 & 1 & 0\\ 3 & 1 & 0 & 1\end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix}4\\5\end{pmatrix}$$

$$\begin{aligned} &\max Z = \boldsymbol{c}^T\boldsymbol{x} \\ &\begin{pmatrix}\boldsymbol{A} & \boldsymbol{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} = \boldsymbol{b} \\ &\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}_s\end{pmatrix} \ge \boldsymbol{0} \end{aligned}$$

Initialization Basic variables: x_3, x_4

$$\boldsymbol{x}_B = \begin{pmatrix}x_3\\x_4\end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \boldsymbol{B}^{-1} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \boldsymbol{B}^{-1}\boldsymbol{A} = \begin{pmatrix}2 & 1\\3 & 1\end{pmatrix}$$

$$\boldsymbol{c}_B^T = \begin{pmatrix}0 & 0\end{pmatrix} \qquad \boldsymbol{c}_B^T\boldsymbol{B}^{-1} = \begin{pmatrix}0 & 0\end{pmatrix}$$

$$Z = \begin{pmatrix}0 & 0\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} + \left(\begin{pmatrix}6 & 5\end{pmatrix} - \begin{pmatrix}0 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\3 & 1\end{pmatrix}\right)\begin{pmatrix}x_1\\x_2\end{pmatrix} - \begin{pmatrix}0 & 0\end{pmatrix}\begin{pmatrix}x_3\\x_4\end{pmatrix} = 0 + 6x_1 + 5x_2 + 0x_3 + 0x_4$$

$$\begin{pmatrix}2 & 1 & 1 & 0\\3 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$$

$$Z = 6x_1 + 5x_2$$

$$\begin{pmatrix}x_3\\x_4\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix} - \begin{pmatrix}2 & 1\\3 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

Basic variables:
x_3, x_4

Simplex Method (Matrix Form): Iteration

matrix representation

$$c^T = (6 \quad 5) \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x_s = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$(A \quad I) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Iteration 1:

Basic variables: x_3, x_4

$$Z = 6x_1 + 5x_2$$

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$Z = c_B^T B^{-1} b + (c^T - c_B^T B^{-1} A)x - c_B^T B^{-1} x_s$$
$$(B^{-1}A \quad B^{-1}) \begin{pmatrix} x \\ x_s \end{pmatrix} = B^{-1} b$$

Entering variable: x_1
Leaving variable: x_4

Entering variable: Non-basic Variable x_i with the largest positive value in obj

Leaving basic variable: Min ratio test

New basic variables: x_1, x_3

$$x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \quad B^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \quad B^{-1}A = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$c_B^T = (6 \quad 0) \qquad c_B^T B^{-1} = (0 \quad 2) \qquad c_B^T B^{-1} A = (6 \quad 2)$$

$$\Rightarrow \quad Z = (0 \quad 2)\begin{pmatrix} 4 \\ 5 \end{pmatrix} + ((6 \quad 5) - (6 \quad 2))\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (0 \quad 2)\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 10 + 0x_1 + 3x_2 + 0x_3 - 2x_4$$

$$\Rightarrow \quad \begin{pmatrix} 1 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\Rightarrow \quad Z = 10 + 3x_2 - 2x_4$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

Basic variables: x_1, x_3