IE 310: Deterministic Models in Optimization

Plan for today:

- Course Logistics
- Introduction to Optimization/Operations Research (OR)
- Formulations
  - Examples
- Linear Programming Formulations
  - Definition

Announcements:

- You have the control on mute/unmute button in Zoom
- HW 1 posted in Gradescope after lecture

INTRODUCTION

… where we see the ORigins and an outline of the Optimization methodology

What is the course about?

- Optimization/Operations Research (OR)
  - scientific approach to decision making
  - seek to determine how best to design and operate a system
  - under conditions requiring the allocation of scarce resources

- Provides a set of algorithms that act as tools for effective problem solving and decision making

- Extensive applications
  - Engineering
  - Business
  - Public systems
  - Manufacturing
  - Service industries

‘OR’igin

- The field started during WW II
  - e.g., airplane patrol, logistics,….

- Application of mathematics and scientific methods to military was called “Operations Research”

- Early 1950s: OR introduced into business, industry, and government

- Today OR is synonymous with Optimization

Operations Research Today

- Large professional society (INFORMS, https://www.informs.org)
- Many publications

  Management Science, Operations Research, Annals of Operations Research, European Journal of Operational Research, Mathematics of Operations Research, OR Today, Interface, IIE Transactions….

- Applications
  - Transportation - How can an airline quickly and efficiently get its flight crews in place to fly following a major disruption to operations?
    Continental airlines: http://www.orms-today.org/orms-6-02/edelman.html
  - Telecommunication – How to reroute traffic in case of link failures? How to design networks with sufficient restoration capacity for rerouting?
    AT&T Network: https://www.informs.org/Sites/Getting-Started-With-Analytics/Analytics-Success-Stories/Case-Studies/AT-T-Network
  - Services – Logistical nightmares of package delivery: Fast, Limited by crew, weight, dimensions, packing capacity
    Federal Express: http://fedexlegends.info/earlyit/absolutely.pdf
  - Finance – How to invest to maximize return?
    Operations Research and Bank Planning: http://www.sciencedirect.com/science/article/pii/0024630175900928
  - Health Care
    Operations Research for Surgical Services: http://www.franklindexter.net/PDF%20Files/SurgicalServicesCourse.pdf
  - Sports – Scheduling baseball games: want good games in primetime, want primetime to be equally distributed, …
    http://mat.gsia.cmu.edu/trick/tourn_final.pdf

OPTIMIZE WHAT/WHY?

… where we see some concrete scenarios in which optimization could help

Optimize what/why?

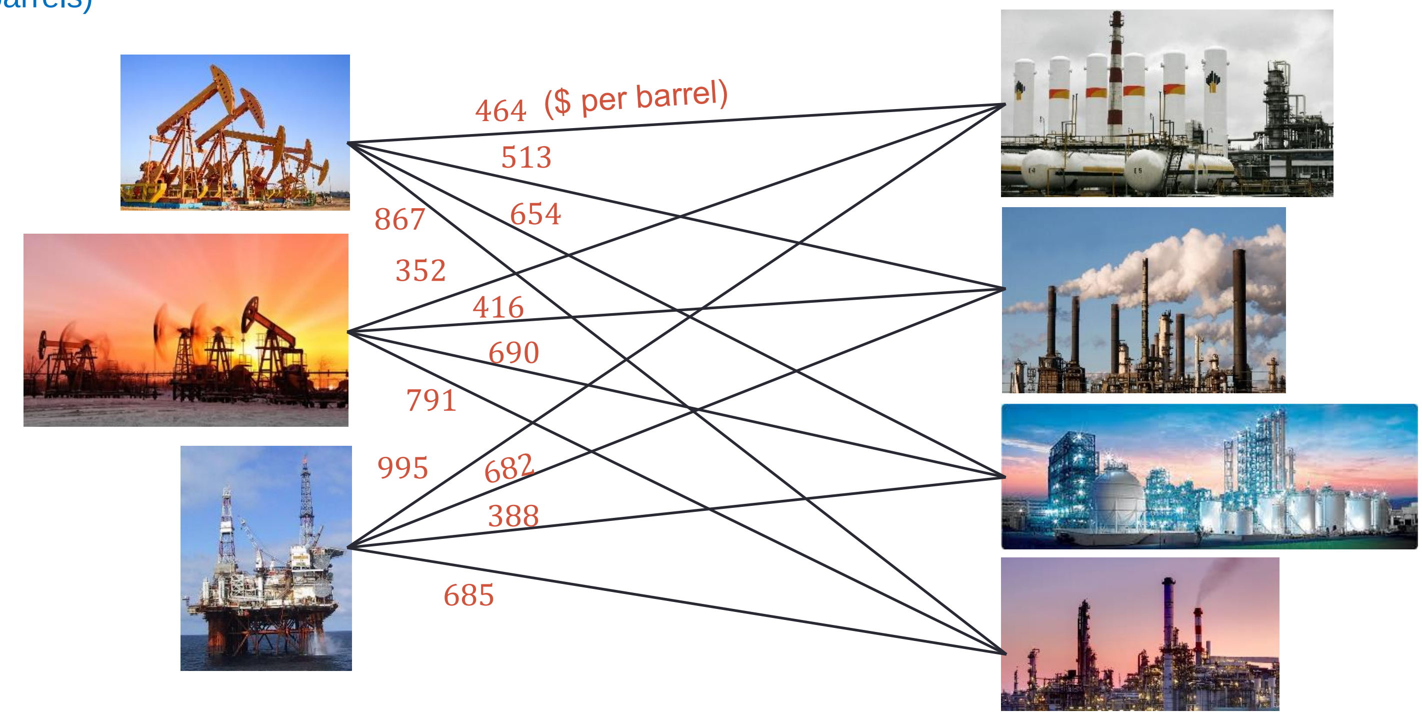
- Cancer treatment machine sends ionized radiation beam through a patient's body
- Use two beams

	Fraction of Entry Dose Absorbed by area		
Area	Beam 1	Beam 2	Restriction on Total Average Dosage
Healthy Tissues	0.4	0.5	Minimize
Critical Tissues	0.3	0.1	≤2.7
Tumor Tissues	0.5	0.5	=6
Tumor Center Tissues	0.6	0.4	≥6

Question: What is the dosage of beams 1 and 2 that minimizes the damage to healthy tissues, does not over-damage critical tissues, kills tumor and tumor center tissues?

Optimize what/why?

Supply (1000s barrels) | Oil rigs | Factories | Demand (1000s barrels)



Supply: 75, 125, 100

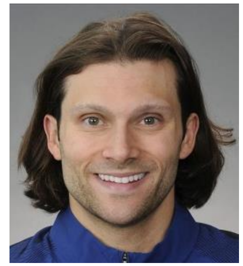
Demand: 80, 65, 70, 85

Question: How much to ship from each supply point to each demand point to meet the demands while minimizing shipping cost?

Example 3: Sports

Swimming: 4x100 relay medley

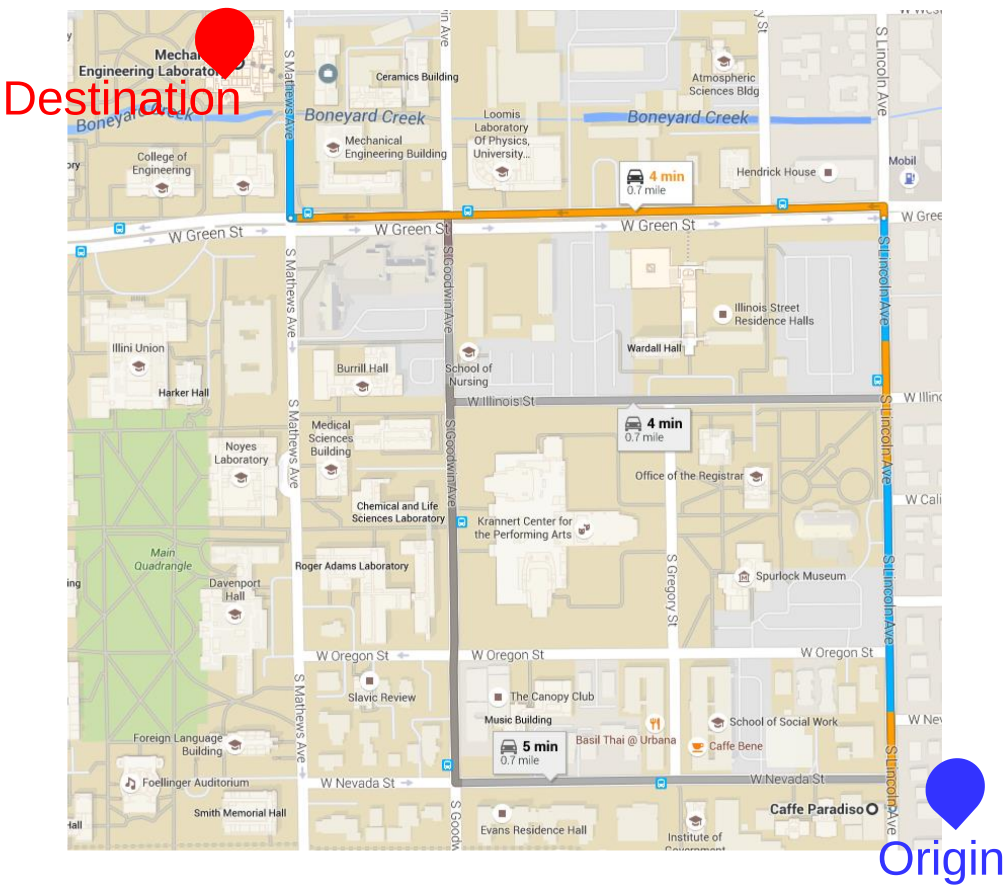
Optimize what/why?

	Adrian	Miller	Phelps	Murphy
Backstroke	$51.77s$	$51.99s$	$52.33s$	$51.85s$
Breaststroke	$58.86s$	$58.87s$	$58.91s$	$58.95s$
Butterfly	$51.59s$	$51.17s$	$51.14s$	$51.83s$
Freestyle	$47.85s$	$48.93s$	$48.01s$	$49.31s$

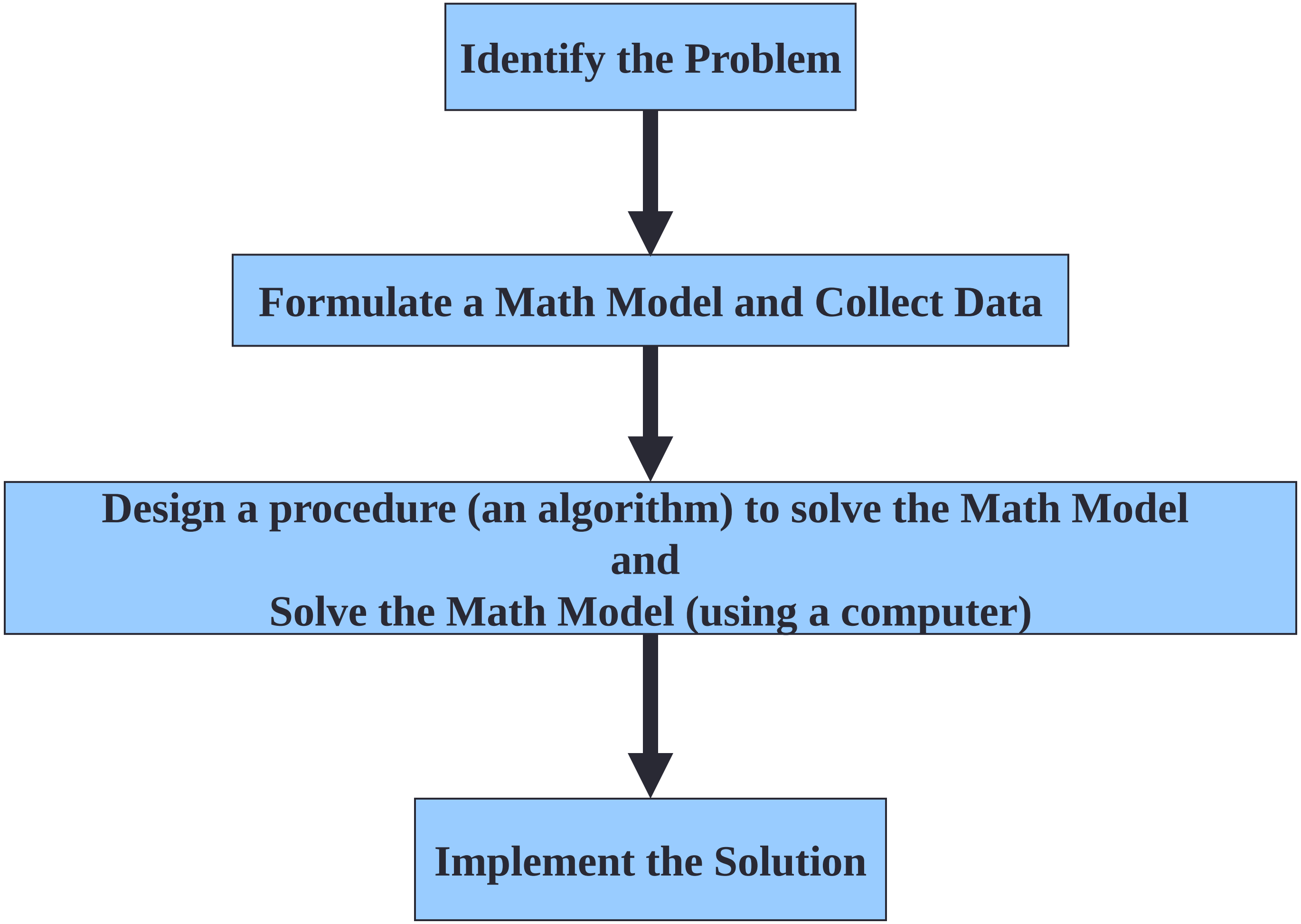
Question: Who swims which leg of the relay to minimize total time?

Optimize what/why?



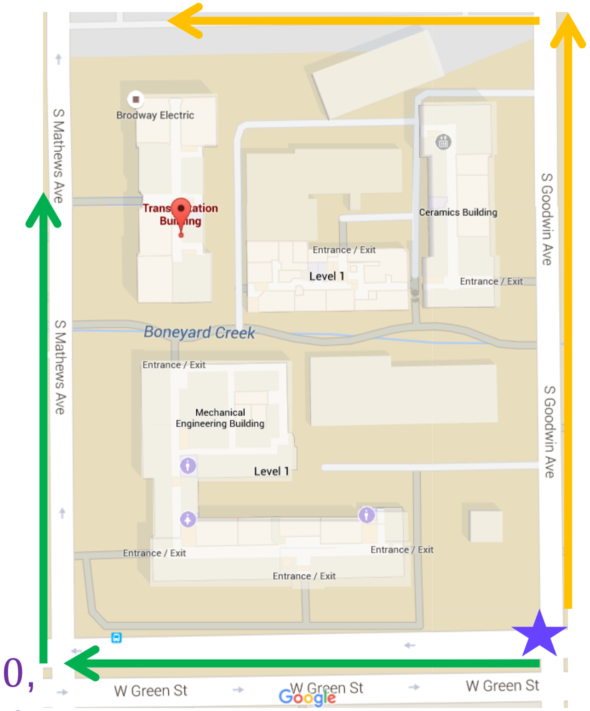
Question: What is the shortest route to the destination?

Optimize how?



A toy example

Problem:
choose the faster route to get to TB from where you are



Data:

route 1: Goodwin-Springfield, travel time t_1
route 2: Green-Mathew, travel time t_2

Model/Formulation:

Decisions: $x_1 = 1$ if route 1 is chosen, otherwise $x_1 = 0$,
$x_2 = 1$ if route 2 is chosen, otherwise $x_2 = 0$

Restrictions: x_1, x_2 are 1 or 0, $x_1 + x_2 = 1$ (choose exactly one route)

Objective: minimize $t_1 x_1 + t_2 x_2$

Solution Procedure: set $x_1 = 1$ if $t_1 < t_2$, otherwise set $x_2 = 1$

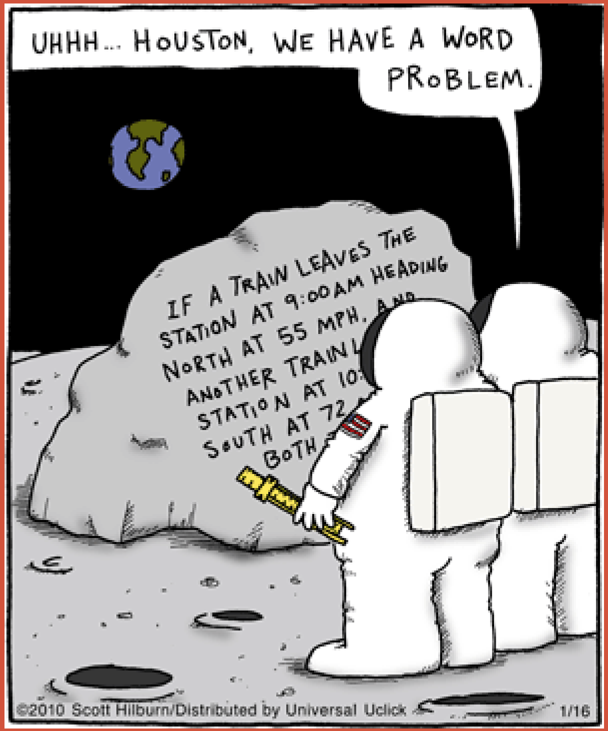
Implementation: start ~~your hoverboard~~ biking!

Elements of Formulating Optimization Models

- **Decision variables:** what we need to decide
  - example: which route to take
  - Represent numerical value(s)

- **Objective function:** goal to achieve
  - example: minimize travel time
  - Quantifiable, i.e., can be expressed as a function of decision variables

- **Constraints:** conditions to satisfy
  - example: only one route can be chosen
  - Quantifiable, i.e., can be expressed as functions of decision variables



FORMULATIONS

… where we model problems mathematically

- So that we can use standard solution techniques

Example 1

- Consider a small manufacturer making two commodities A and B
- Two resources R_1 and R_2 are required to make these commodities
- Each batch of commodity A requires 1 batch of R_1 and 3 batches of R_2
- Each batch of commodity B requires 1 batch of R_1 and 2 batches of R_2
- Manufacturer has 5 batches of R_1 and 12 batches of R_2
- Manufacturer makes a profit of
  - 6$ per batch of commodity A sold
  - 5$ per batch of commodity B sold

Question: What is the best production mix?

Example 1: Summarize data

- Consider a small manufacturer making two commodities A and B
- Two resources R_1 and R_2 are required to make these commodities
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Question: What is the best production mix?

	A	B	Availability
R_1	1	1	5
R_2	3	2	12
Profit	6	5	

Example 1: Formulation

	A	B	Availability
R_1	1	1	5
R_2	3	2	12
Profit	6	5	

Step 1: identify decision variables

x_1: number of batches of commodity A to make
x_2: number of batches of commodity B to make

Step 2: determine the objective function

$$\max Z(x_1, x_2) = 6x_1 + 5x_2$$

Step 3: identify constraints

- Resource constraints

$$x_1 + x_2 \leq 5 \quad \text{(Resource } R_1\text{)}$$
$$3x_1 + 2x_2 \leq 12 \quad \text{(Resource } R_2\text{)}$$

- “Common Sense” (non-negativity) constraints

$$x_1 \geq 0, x_2 \geq 0$$

Example 1: Formulation

	A	B	Availability
R_1	1	1	5
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Profit	6	5	

$$\max Z(x_1, x_2) = 6x_1 + 5x_2$$

$x_1 + x_2 \leq 5$ (Resource R_1)

$3x_1 + 2x_2 \leq 12$ (Resource R_2)

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- Manufacturer makes a profit of
  - 6$ per batch of commodity A sold
  - 5$ per batch of commodity B sold

Question: What is the best production mix?

Maximize $Z = 6x_1 + 5x_2$
subject to

$$x_1 + x_2 \leq 5$$
$$3x_1 + 2x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

Example 2

- Tesla makes two models of cars
  - Model I: makes a profit of $\$3$ million per batch
  - Model II: sells for $\$5$ million per batch
- Tesla has three plants with limited working hours
  - Plant 1: Frame I
    - at most 4 working hours per week
    - 1 hour to prepare a batch of Frame I
  - Plant 2: Frame II
    - at most 12 working hours per week
    - 2 hours to prepare a batch of Frame II
  - Plant 3: Assembly
    - at most 18 working hours per week
    - 3 hours to assemble a batch of Model I (using Frame I) and 2 hours to assemble a batch of Model II (using Frame II)

Question: What is the best product mix?

Example 2: Summarize data

- Tesla makes two models of cars
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	C_1	C_2	Availability
P_1	1		4
P_2		2	12
P_3	3	2	18
Profit	3	5	

Question: What is the best product mix?

Example 2: Formulation

	C_1	C_2	Availability
P_1	1		4
P_2		2	12
P_3	3	2	18
Profit	3	5	

Step 1: identify decision variables

x_1: number of batches of Model I to make per week
x_2: number of batches of Model II to make per week

Step 2: determine the objective function

$$\max Z(x_1, x_2) = 3x_1 + 5x_2$$

Step 3: identify constraints

- Hour constraints at each plant

$$x_1 \leq 4 \quad \text{(plant 1)}$$
$$2x_2 \leq 12 \quad \text{(plant 2)}$$
$$3x_1 + 2x_2 \leq 18 \quad \text{(plant 3)}$$

- “Common Sense” (non-negativity) constraints

$$x_1 \geq 0, x_2 \geq 0$$

Example 2: Formulation

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	C_1	C_2	Availability
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P_2		2	12
P_3	3	2	18
Profit	3	5	

Question: What is the best product mix?

$\max\ Z = 3x_1 + 5x_2$ (profit)

$x_1 \le 4$ (hour constraint for plant 1)

$2x_2 \le 12$ (hour constraint for plant 2)

$3x_1 + 2x_2 \le 18$ (hour constraint for plant 3)

$x_1 \ge 0$ (non-negative amount of commodity 1)

$x_2 \ge 0$ (non-negative amount of commodity 2)

	C_1	C_2	Availability
P_1	1		4
P_2		2	12
P_3	3	2	18
Profit	3	5	

Example 2: Formulation

Step 1: identify decision variables

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Step 2: determine the objective function

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- "Common Sense" (non-negativity) constraints

$$x_1 \geq 0, x_2 \geq 0$$

Formulating a Mathematical Model for a Problem

Step 1: identify decision variables

Step 2: determine the objective function

Step 3: identify constraints

Example 2: Formulation

$\max\ Z = 3x_1 + 5x_2$	(profit)
$x_1 \leq 4$	(hour constraint for plant 1)
$2x_2 \leq 12$	(hour constraint for plant 2)
$3x_1 + 2x_2 \leq 18$	(hour constraint for plant 3)
$x_1 \geq 0$	(non-negative amount of Frame 1)
$x_2 \geq 0$	(non-negative amount of Frame 2)

The objective function and the constraints are linear

Therefore, the model that we formulated is a linear programming formulation

SUDOKU

Sudoku

Given: a partially filled table

Each cell to be filled with a number from 1 to 9
Each number can appear exactly once
- in each row
- in each col
- in each subtable

Goal: Fill numbers in cells
to satisfy the above rules

Subtable

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

LINEAR PROGRAMMING FORMULATION

… where we formally define "Linear Programming Formulations" and see its standard form

Linear Programming Formulation

- A Linear Programming Formulation is an optimization problem formulation in which
  - the objective function and
  - all constraints

  are linear functions of decision variables, which are real variables

- More specifically, in a linear programming, the objective function and all constraints satisfy the following four conditions
  - Proportionality
  - Additivity
  - Divisibility
  - Certainty

Linear Programming: Proportionality

The contribution of each decision variable to the objective function and the LHS of each constraint is proportional to the value of the decision variable

- In Example 2, the proportionality constant
  - In the objective is the profit
  - In each constraint is the required number of hours at that plant

$$\begin{aligned}
\max\ Z = 3x_1 + 5x_2 &\quad \text{(profit)}\\
x_1 \le 4 &\quad \text{(hour constraint for plant 1)}\\
2x_2 \le 12 &\quad \text{(hour constraint for plant 2)}\\
3x_1 + 2x_2 \le 18 &\quad \text{(hour constraint for plant 3)}\\
x_1 \ge 0 &\quad \text{(non-negative amount of product 1)}\\
x_2 \ge 0 &\quad \text{(non-negative amount of product 2)}
\end{aligned}$$

$$\begin{aligned}
&\max_{x_1, x_2} \{x_1(3 - 0.1x_1) + x_2\}\\
&\text{subject to:} \quad x_1 \le 4\\
&\qquad 2x_2 \le 12\\
&\qquad 3x_1 + 2x_2 \le 18\\
&\qquad x_1 \ge 0,\ x_2 \ge 0
\end{aligned}$$

Linear Programming: Additivity

The objective function and the LHS of constraints are sums of individual contributions of decision variables

$$
\begin{aligned}
&\max_{x_1, x_2} \{x_1 + (1 + x_1)x_2\} \\
&\text{subject to:} \quad x_1 + x_2 \le 3 \\
&\qquad x_1 \ge 0,\ x_2 \ge 0
\end{aligned}
$$

$$
\begin{aligned}
&\max_{x_1, x_2} \{x_1 + 2x_2\} \\
&\text{subject to:} \quad x_1 + x_1 x_2 \le 3 \\
&\qquad x_1 \ge 0,\ x_2 \ge 0
\end{aligned}
$$

$$
\begin{aligned}
&\max_{x_1, x_2} \{x_1 - 3x_2\} \\
&\text{subject to:} \quad x_1 - 2x_2 \le 3 \\
&\qquad -x_1 + x_2 \ge 2 \\
&\qquad x_1 \ge 0,\ x_2 \ge 0
\end{aligned}
$$

Linear Programming: Divisibility

Each decision variable can take any real value as long as all constraints are satisfied

- Note: They need not be integers
- In Example 2, you are allowed to produce 1.340011343 batches of Model I and 3.43254432 batches of Model II
  (batches per week = production rate and it can be fractional)
- Our formulation of the route choice problem (toy example) is NOT a linear programming formulation
  … since x_1, x_2 are restricted to take binary values (0 or 1)
- Our formulation of Sodoku is also NOT a linear programming formulation for the same reason
- If the problem context requires some or all decision variables to take integer or binary values, then the formulation becomes an **Integer** Linear Programming Formulation, which will be discussed later

Linear Programming: Certainty

The data (objective coefficients, constraint coefficients, RHS of constraints) must have known values

Example: You have B dollars to invest in *n* different stocks and want to maximize the total return

r_i: return from stock i, $i = 1, \ldots, n$

x_i: Amount to invest in stock i, $i = 1, \ldots, n$

$$\max r_1 x_1 + r_2 x_2 + \cdots + r_n x_n$$
$$x_1 + x_2 + \cdots + x_n \leq B$$
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

What if, as is commonly the case, you do not know the exact values of return?

- Linear programming cannot generate an optimal solution for all possible values of $r_1, r_2, \ldots, r_n$
- Linear programming can only generate an optimal solution for a fixed choice of return values, e.g., best-case return values or worst-case return values

Linear Programming Formulation

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