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Online Supplement: Improving the Integrality Gap for Multiway Cut

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In this online supplement, we prove Lemma 2 of the main article, i.e., a non-fragmenting non-opposite cut in $\Delta_{3,n}$ that has lot of nodes labeled as 4 has large cost. We restate it below for convenience.

Lemma 2 *Let $Q : \Delta_{3,n} \to [4]$ be a non-opposite cut with αn^2 nodes labeled as 4 for some $\alpha \in [0, 1/2]$. If Q is a non-fragmenting cut and $n \geq 10$, then the cost of Q on J is at least $1.2 + 0.4\alpha - \frac{1}{n}$.*

Proof. We first show that the labeling Q may be assumed to indicate reachability in the graph $\mathcal{G} - \delta(Q)$.

Claim 1 *For every non-opposite non-fragmenting cut $Q : \Delta_{3,n} \to [4]$, there exists a labeling $Q' : \Delta_{3,n} \to [4]$ such that*

1. *a node $v \in \Delta_{3,n}$ is reachable from s_i in $\mathcal{G} - \delta(Q')$ iff $Q'(v) = i$,*
2. *$Cost_J(\delta(Q')) \leq Cost_J(\delta(Q))$,*
3. *the number of nodes in $\Delta_{3,n}$ that are labeled as 4 by Q is at most the number of nodes in $\Delta_{3,n}$ that are labeled as 4 by Q', and*
4. *Q' is a non-opposite non-fragmenting cut.*

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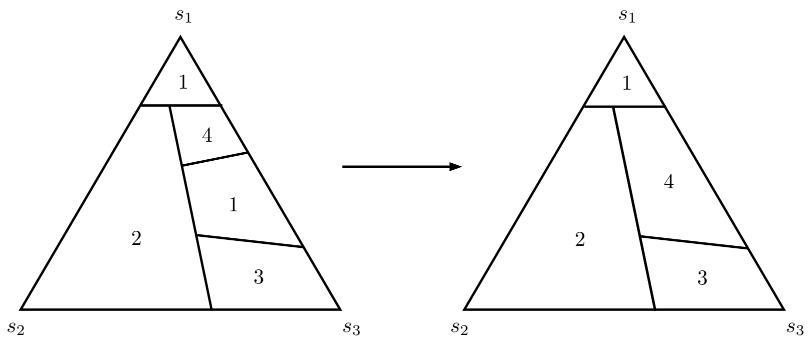

Fig. 1 An example of a cut Q and the cut Q' obtained in the proof of Claim 1.

Proof. For $i \in [3]$, let S_i be the set of nodes that can be reached from s_i in $\mathcal{G} - \delta(Q)$. Consider a labeling Q' defined by

$$Q'(v) := \begin{cases} i & \text{if } v \in S_i, \text{ and} \\ 4 & \text{if } v \in \Delta_{3,n} \setminus (S_1 \cup S_2 \cup S_3). \end{cases}$$

See Figure 1 for an example of a cut Q and the cut Q' obtained as above. We prove the required properties for the labeling Q' below.

1. By definition, $Q'(v) = i$ iff v is reachable from s_i in $\mathcal{G} \setminus \delta(Q')$.
2. Since $\delta(Q') = (\delta(S_1) \cup \delta(S_2) \cup \delta(S_3)) \cap \delta(Q)$, we have that $\delta(Q') \subseteq \delta(Q)$. Hence, $Cost_J(\delta(Q')) \leq Cost_J(\delta(Q))$.
3. Let $i \in [3]$. Since all nodes of S_i are labeled as i by Q, the nodes labeled as i by Q' is a subset of the set of nodes labeled as i by Q. This implies that Q' is also a non-opposite cut and that the number of nodes in $\Delta_{3,n}$ that are labeled as 4 by Q is at most the number of nodes in $\Delta_{3,n}$ that are labeled as 4 by Q'.
4. Since Q is a non-fragmenting cut, there exist distinct $i, j \in [3]$ such that $|\delta(Q) \cap L_{ij}| = 1$. Since $\delta(Q') \subseteq \delta(Q)$, we have that $|\delta(Q') \cap L_{ij}| \leq 1$. On the other hand, Q' labels s_i by i and s_j by j and hence, $|\delta(Q') \cap L_{ij}| \geq 1$. Combining the two, we have that $|\delta(Q') \cap L_{ij}| = 1$ and hence Q' is a non-fragmenting cut.

□

Let $\mathcal{G}' := \mathcal{G} - \delta(Q)$. By Claim 1, we may henceforth assume that

$$\text{For every node } v \in V,\ v \text{ is reachable from } s_i \text{ in } \mathcal{G}' \text{ iff } Q(v) = i. \tag{1}$$

In order to show a lower bound on the cost of Q, we will modify Q to obtain a non-opposite cut while reducing its cost by at least 0.4α. [1] showed that the cost of every non-opposite cut on J is at least $1.2 - \frac{1}{n}$. Therefore, the cost of Q on J must be at least $1.2 - \frac{1}{n} + 0.4\alpha$.

Since Q is a non-fragmenting cut, there exist distinct $i, j \in [3]$ such that $|\delta(Q) \cap L_{ij}| = 1$. Without loss of generality, suppose that $i = 1$ and $j = 3$.

For $i \in [3]$, let $S_i := \{v \in \Delta_{3,n} \mid Q(v) = i\}$, i.e. S_i is the set of nodes that can be reached from s_i in $\mathcal{G}'$. Let $B := \{v \in \Delta_{3,n} \mid Q(v) = 4\}$ be the set of nodes labeled as 4 by Q. Then, $|B| = \alpha n^2$. We note that S_1, S_2, and S_3 are components of $\mathcal{G}'$, and the set B is the union of the remaining components.

We recall that V_{ij} is the set of end nodes of edges in L_{ij}. We say that a node $v \in \Delta_{3,n}$ can reach V_{ij} in $\mathcal{G}'$ if there exists a path from v to some node $w \in V_{ij}$ in $\mathcal{G}'$. We observe that all nodes in V_{13} are reachable from either s_1 or s_3 in $\mathcal{G}'$. In particular, this means that no node of B can reach V_{13} in $\mathcal{G}'$. We partition the node set B based on reachability as follows (see Figure 2):

$$\begin{aligned}
B_1 &:= \{v \in B \mid \ v \text{ cannot reach } V_{12} \text{ and } V_{23} \text{ in } \mathcal{G}'\},\\
B_2 &:= \{v \in B \mid \ v \text{ can reach } V_{12} \text{ but not } V_{23} \text{ in } \mathcal{G}'\},\\
B_3 &:= \{v \in B \mid \ v \text{ can reach } V_{23} \text{ but not } V_{12} \text{ in } \mathcal{G}'\}, \text{ and}\\
B_4 &:= \{v \in B \mid v \text{ can reach } V_{12} \text{ and } V_{23} \text{ in } \mathcal{G}'\}.
\end{aligned}$$

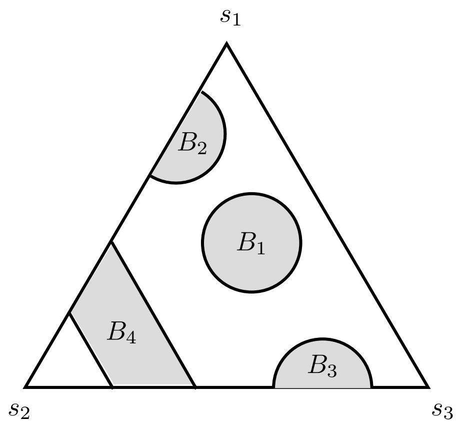

Fig. 2 Partition of B into B_1, B_2, B_3, B_4.

For $r \in [4]$, let $\beta_r := |B_r|/n^2$. We next summarize the properties of the sets defined above.

Proposition 1 *The sets B_1, B_2, B_3, B_4 defined above satisfy the following properties:*

(i) For every distinct $r, p \in [4]$, we have $B_r \cap B_p = \emptyset$.
(ii) For every $r \in [4]$, we have $\delta(B_r) \subseteq \delta(Q)$, i.e. B_r is the union of some components of $\mathcal{G}'$.
(iii) For every $r \in [4]$ and every edge $e \in \delta(B_r)$, one end node of e is in B_r and the other one is in $S_1 \cup S_2 \cup S_3$.
(iv) For every distinct $r, p \in [4]$, we have $\delta(B_r) \cap \delta(B_p) = \emptyset$.
(v) $B = \cup_{r=1}^4 B_r, \sum_{r=1}^4 \beta_r = \alpha$, and $\beta_r \leq 0.66$ for every $r \in [4]$.

Proof. (i) The disjointness property follows from the definition of the sets.

(ii) Suppose $\delta(B_r)$ is not a subset of $\delta(Q)$ for some $r \in [4]$. Without loss of generality, let $r = 1$ (the proof is similar for the other cases). Then, there exists an edge $uv \in E_{3,n} \setminus \delta(Q)$ with $u \in B_1, v \in B \setminus B_1$. Since v is in $B \setminus B_1$, it follows that the node v can reach either V_{12} or V_{13} in $\mathcal{G}'$. Moreover, since the edge uv is in $\mathcal{G}'$, it follows that the node u can also reach either V_{12} or V_{13} in $\mathcal{G}'$, and hence $u \notin B_1$. This contradicts the assumption that $u \in B_1$.

(iii) Let $uv \in \delta(B_r)$ with $u \in B_r$ and $v \notin B_r$. Since $Q(u) = 4$, the node u is not reachable from any of the terminals in $\mathcal{G}'$. Suppose that the node v is also not reachable from any of the terminals in $\mathcal{G}'$. Then, by the reachability assumption, it follows that $Q(v) = 4$. Hence, the edge uv has both end-nodes labeled as 4 by Q and therefore $uv \notin \delta(Q)$. Thus, we have an edge $uv \in \delta(B_i) \setminus \delta(Q)$ contradicting part ii.

(iv) Follows from parts i and iii.

(v) By definition, we have that $B = \cup_{r=1}^{4} B_r$. Since the sets B_1, B_2, B_3, B_4 are pair-wise disjoint, they induce a partition of B and hence $|B| = \sum_{r=1}^{4} |B_r|$. Consequently, $\sum_{r=1}^{4} \beta_r n^2 = \alpha n^2$ and thus, $\sum_{r=1}^{4} \beta_r = \alpha$. Next, we note that $|\Delta_{3,n}| = (n+1)(n+2)/2$. Since $B_r \subseteq B \subseteq \Delta_{3,n}$, we have that $\beta_r = |B_r|/n^2 \leq |\Delta_{3,n}|/n^2 \leq (1+1/n)(1+2/n)/2 \leq 0.66$ since $n \geq 10$.

□

By Proposition 2(i) of the main article, the cut-set $\delta(Q)$ is a non-opposite cut-set. The following claim shows a way to modify $\delta(Q)$ to obtain a non-opposite cut-set with strictly smaller cost if $\beta_r > 0$.

Claim 2 *For every $r \in [4]$, there exists $E_r \subseteq \delta(B_r), E'_r \subseteq \mathcal{G}[B_r]$ such that*

1. *$E_r \subseteq \delta(S_i)$ for some $i \in [3]$,*
2. *$(\delta(Q) \setminus E_r) \cup E'_r$ is a non-opposite cut-set and*
3. *$Cost_J(E_r) - Cost_J(E'_r) \geq 0.4\beta_r$.*

Proof. We consider the cases $r = 1, 2, 4$ individually as the proofs are different for each of them. The case of $r = 3$ is similar to the case of $r = 2$. We begin with a few notations that will be used in the proof. For distinct $i, j \in [3]$, and for $t \in \{0, 1, \ldots, 2n/3\}$, let $V_{ij}^t := \{u \in \Delta_{3,n} : u_k = t/n \text{ for } \{k\} = [3] \setminus \{i, j\}\}$. Thus, V_{ij}^t denotes the set of nodes that are on the line parallel to V_{ij} and at distance t/n from it. We will call the sets V_{ij}^t as lines for convenience. Let L_{ij}^t denote the edges of $E_{3,n}$ whose end-nodes are in V_{ij}^t. Thus, the edges in L_{ij}^t are parallel to L_{ij} (see Figure 3).

1. **Suppose $r = 1$.** We partition the set $\delta(B_1)$ of edges into three sets $X_i := \delta(B_1) \cap \delta(S_i)$ for $i \in [3]$ (see Figure 4).
   By Proposition 1(iii), we have that (X_1, X_2, X_3) is a partition of B_1. Let

$$E_1 := \arg\max\{Cost_J(F) : F \in \{X_1, X_2, X_3\}\} \text{ and}$$
$$E'_1 := \emptyset.$$

We now show the required properties for this choice of E_1 and E'_1.

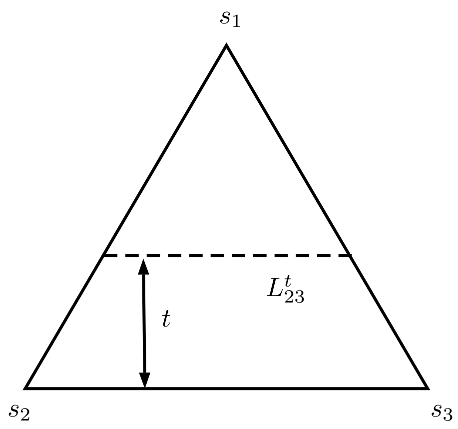

Fig. 3 The set of edges L_{23}^t.

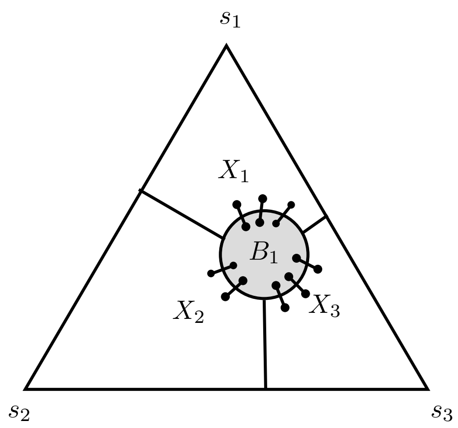

Fig. 4 Partition of $\delta(B_1)$ into X_i's.

(a) Since $E_1' = \emptyset$, we need to show that $\delta(Q) \setminus E_1$ is a non-opposite cut-set. Let $\mathcal{G}'' := \mathcal{G} - (\delta(Q) \setminus E_1)$. For each edge $e \in E_1$, the end node of e in $\Delta_{3,n} \setminus B_1$ is reachable from a terminal s_i in $\mathcal{G}'$ iff it is reachable from s_i in $\mathcal{G}''$. Therefore, for each node $v \in \Delta_{3,n} \setminus B_1$ and a terminal s_i for $i \in [3]$, we have that v is reachable from s_i in $\mathcal{G}'$ iff v is reachable from s_i in $\mathcal{G}''$. Since $\delta(Q)$ is a non-opposite cut-set, it follows that s_i cannot reach V_{jk} in $\mathcal{G}'$ for $\{i, j, k\} = [3]$. Since $B_1 \cap (V_{12} \cup V_{23} \cup V_{13}) = \emptyset$, the terminal s_i cannot reach V_{jk} in $\mathcal{G}''$ for $\{i, j, k\} = [3]$. Hence, $\delta(Q) \setminus E_1$ is a non-opposite cut-set.

(b) We note that none of the nodes in B_1 can reach V_{12}, V_{23} and V_{13} in $\mathcal{G}'$. Therefore, if there exists a node from B_1 in V_{ij}^t for some $t \in \{1, \ldots, n\}$, then at least two edges in L_{ij}^t should be in $\delta(B_1)$ (see Figure 5). Therefore, if $V_{ij}^t \cap B_1 \neq \emptyset$, then $|\delta(B_1) \cap L_{ij}^t| \geq 2$.

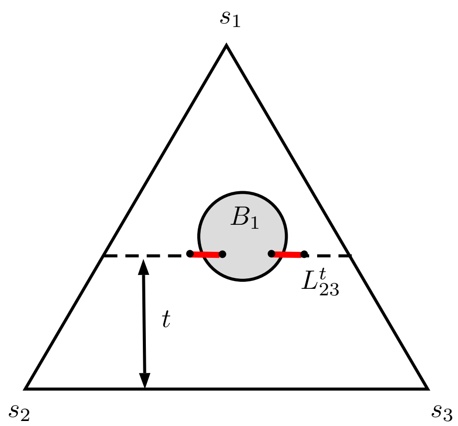

Fig. 5 $B_1 \cap V^t_{23} \neq \emptyset$ implies that $|\delta(B_1) \cap L^t_{23}| \geq 2$.

Every node $v \in B_1$ is in at least two lines among V^t_{ij} for distinct $i, j \in [3]$ and $t \in \{1, \dots, 2n/3\}$. Each line V^t_{ij} for $t \in \{1, \dots, 2n/3\}$ has at most n nodes. Hence, the number of lines with non-empty intersection with B_1 is at least $2|B_1|/n$. For each line that has a non-empty intersection with B_1, we have at least two edges in $\delta(B_1)$. Hence,

$$\left|\delta(B_1) \cap \left(\cup_{i,j\in[3],t\in\{1,\dots,2n/3\}} L^t_{ij}\right)\right| \geq 4 \cdot \frac{|B_1|}{n}.$$

The cost of each edge in $\cup_{i,j\in[3],t\in\{1,\dots,2n/3\}} L^t_{ij}$ is $3/(5n)$. So,

$$\begin{aligned} Cost_J(\delta(B_1)) &\geq Cost_J\left(\delta(B_1) \cap \left(\cup_{i,j\in[3],t\in\{1,\dots,2n/3\}} L^t_{ij}\right)\right) \\ &\geq \frac{12}{5}\frac{|B_1|}{n^2} = \frac{12}{5}\beta_1. \end{aligned}$$

Since we set E_1 to be the X_i with maximum cost, we get that

$$Cost_J(E_1) \geq (4/5)\beta_1.$$

Moreover, $Cost_J(E'_1) = 0$ as $E'_1 = \emptyset$. Hence, $Cost_J(E_1) - Cost_J(E'_1) \geq (4/5)\beta_1 \geq 0.4\beta_1$.

2. **Suppose $r = 2$.** We assume that $B_2 \neq \emptyset$ as otherwise, the claim is trivial. Similar to the previous case, we partition the set $\delta(B_2)$ into three sets $X_i := \delta(B_2) \cap \delta(S_i)$ for $i \in [3]$ (see Figure 6).
We also define
$$Z := X_3 \cap \delta(B_2 \cap V_{12})$$
and let

$$\begin{aligned} &E_2 := X_1 \text{ and } E'_2 := \emptyset \text{ if } Cost_J(X_1) \geq 0.4\beta_2, \\ &E_2 := X_2 \text{ and } E'_2 := \emptyset \text{ if } Cost_J(X_1) < 0.4\beta_2 \text{ and } Cost_J(X_2) \geq 0.4\beta_2, \\ &E_2 := X_3 \setminus Z \text{ and } E'_2 := \delta_{\mathcal{G}}(B_2 \setminus V_{12}, B_2 \cap V_{12}) \text{ otherwise.} \end{aligned}$$

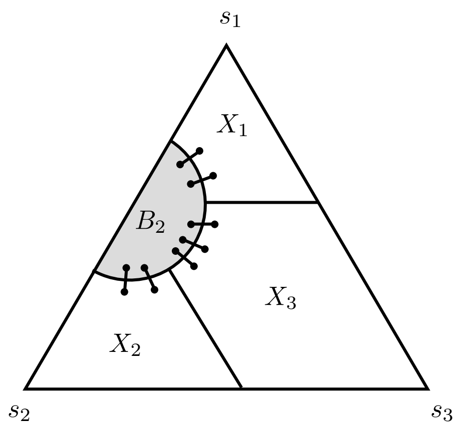

Fig. 6 Partition of $\delta(B_2)$ into X_i's.

We emphasize that the last case is the only situation where we use a non-empty set for E_2'. We now show the required properties for this choice of E_2 and E_2'.

(a) Let $\mathcal{G}'' := \mathcal{G} - ((\delta(Q) \setminus E_2) \cup E_2')$. For each edge $e \in E_2$, the end node of e in $\Delta_{3,n} \setminus B_2$ is reachable from a terminal s_i in $\mathcal{G}'$ iff it is reachable from s_i in $\mathcal{G}''$. Therefore, for each node $v \in \Delta_{3,n} \setminus B_2$ and a terminal s_i for $i \in [3]$, we have that v is reachable from s_i in $\mathcal{G}'$ iff v is reachable from s_i in $\mathcal{G}''$. Since $B_2 \cap V_{13} = \emptyset$ and s_2 cannot reach V_{13} in $\mathcal{G}'$, we have that s_2 cannot reach V_{13} in $\mathcal{G}''$. Similarly, s_1 cannot reach V_{23} in $\mathcal{G}''$. It remains to argue that s_3 cannot reach V_{12} in $\mathcal{G}''$. We have two cases.

   i. Suppose $E_2 = X_1$ or $E_2 = X_2$. We note that X_1 (and X_2 respectively) is the set of edges in $\delta(B_2)$ whose end nodes outside B_2 are reachable from s_1 (and s_2 respectively) in $\mathcal{G}'$. So, if $E_2 = X_1$ or if $E_2 = X_2$, then the set of nodes reachable by s_3 in $\mathcal{G}'$ and $\mathcal{G}''$ remains the same. Since s_3 cannot reach V_{12} in $\mathcal{G}'$, we have that s_3 cannot reach V_{12} in $\mathcal{G}''$.

   ii. Suppose $E_2 = X_3$. We will show that $\delta(B_2 \cap V_{12}) \subseteq (\delta(Q) \setminus E_2) \cup E_2'$. Consequently, the nodes of $B_2 \cap V_{12}$ are not reachable from s_3 in $\mathcal{G}''$. Since nodes of $V_{12} \setminus B_2$ are not reachable from s_3 in $\mathcal{G}'$, we have that s_3 cannot reach V_{12}.
   We now show that $\delta(B_2 \cap V_{12}) \subseteq (\delta(Q) \setminus E_2) \cup E_2'$. Let $uv \in \delta(B_2 \cap V_{12})$ with $u \in B_2 \cap V_{12}$ and $v \notin B_2 \cap V_{12}$. If $v \in S_1 \cup S_2$, then $uv \in \delta(B_2) \subseteq \delta(Q)$ and $uv \notin X_3 \supseteq E_2$. Hence, $uv \in (\delta(Q) \setminus E_2) \cup E_2'$. If $v \in S_3$, then $uv \in Z$ and hence $uv \notin E_2$. Moreover, $uv \in \delta(B_2) \subseteq \delta(Q)$, hence $uv \in (\delta(Q) \setminus E_2) \cup E_2'$. If $v \in B$, then $v \in B_2$ by Proposition 1(iv) and hence $e \in E_2' \subseteq (\delta(Q) \setminus E_2) \cup E_2'$.

(b) If $Cost_J(X_1)$ or $Cost_J(X_2)$ is at least $0.4\beta_2$, then we are done. So, let us assume that $Cost_J(X_1), Cost_J(X_2) \leq 0.4\beta_2$. Let Y_1, Y_2 and Y_3 be the

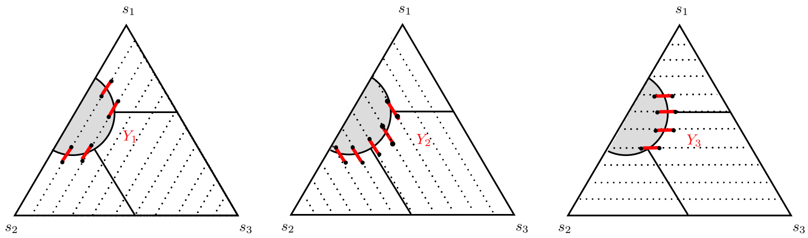

Fig. 7 Partition of $\delta(B_2)$ into Y_i's. The shaded region is B_2.

sets of edges in $\delta(B_2)$ that are parallel to L_{12}, L_{13} and L_{23} respectively (see Figure 7). Formally,

$$Y_1 := \delta(B_2) \cap \left(\cup_{t\in\{0,1,\dots,n\}} L_{12}^t\right)$$
$$Y_2 := \delta(B_2) \cap \left(\cup_{t\in\{0,1,\dots,n\}} L_{13}^t\right)$$
$$Y_3 := \delta(B_2) \cap \left(\cup_{t\in\{0,1,\dots,n\}} L_{23}^t\right)$$

Claims 3 and 4 will help us derive the required inequality on the cost.

Claim 3

$$Cost_J(E_2') \le Cost_J(Y_2) + Cost_J(Y_3) - Cost_J(Z).$$

Proof. We proceed in two steps: (1) we will show a one-to-one mapping f from edges in E_2' to edges in $(Y_2 \cup Y_3) \setminus Z$ such that the cost of every edge $e \in E_2'$ is the same as the cost of the mapped edge $f(e)$ in the instance J, i.e., $w(e) = w(f(e))$ for every $e \in E_2'$, and (2) we will show that $Z \subseteq Y_2 \cup Y_3$. Now, by observing that the sets Y_2 and Y_3 are disjoint, we get that $Cost_J(E_2') \le Cost_J(Y_2) + Cost_J(Y_3) - Cost_J(Z)$.
We now define the one-to-one mapping $f : E_2' \to (Y_2 \cup Y_3) \setminus Z$. Let $e = uv \in E_2'$ such that $u \in B_2 \cap V_{12}, v \in B_2 \setminus V_{12}$. Since E_2' only contains edges between $B_2 \cap V_{12}$ and $B_2 \setminus V_{12}$, it does not contain an edge parallel to L_{12}. Therefore, $e \in L_{13}^t$ or $e \in L_{23}^t$ for some $t \in \{1, \dots, n\}$. Suppose $e \in L_{13}^t$ for some $t \in \{1, \dots, n\}$. Since the nodes of B_2 cannot reach V_{23} in $\mathcal{G}'$, there exists an edge in $\delta(B_2) \cap L_{13}^t$. We map e to an arbitrary edge in $\delta(B_2) \cap L_{13}^t \subseteq Y_2$ (see Figure 8). We note that the set Z contains the set of edges incident to $B_2 \cap V_{12}$ whose other end node is in S_3. Since both u and v are in B_2, it follows that $L_{13}^t \cap Z = \emptyset$. So our mapping of e is indeed to an edge in $Y_2 \setminus Z$. Similarly, if $e \in L_{23}^t$ for some $t \in \{1, \dots, n\}$, then we map e to an arbitrary edge in $\delta(B_2) \cap L_{23}^t \subseteq Y_3 \setminus Z$. This mapping is a one-to-one mapping as E_2' contains at most one edge from L_{13}^t for each $t \in \{1, 2, \dots, n\}$ and at most one edge from L_{23}^t for each $t \in \{1, 2, \dots, n\}$. Moreover, for each $t \in \{1, 2, \dots, n\}$, the cost of all edges in L_{13}^t are identical and the cost of all edges in L_{23}^t are identical.
We now show that $Z \subseteq Y_2 \cup Y_3$. The set Z contains all edges whose one end node is in $B_2 \cap V_{12}$ and another end node is in S_3. Since $V_{12} \cap S_3 = \emptyset$, the set Z does not contain any edge between $B_2 \cap V_{12}$ and $V_{12} \setminus B_2$.

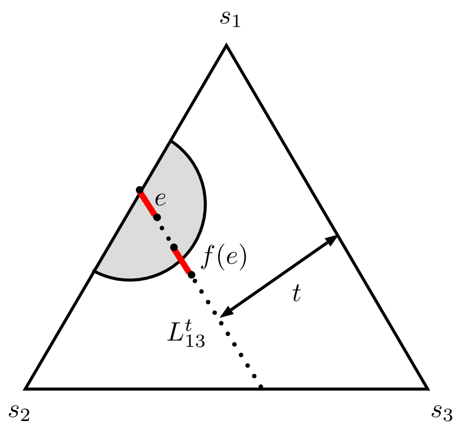

Fig. 8 Mapping from E_2' to $(Y_2 \cup Y_3) \setminus Z$. The shaded region is B_2.

Hence, $Y_1 \cap Z = \emptyset$. Since Z is a subset of X_3 which is a subset of $Y_1 \cup Y_2 \cup Y_3$, it follows that $Z \subseteq Y_2 \cup Y_3$. □

Claim 4

$$Cost_J(Y_1) \geq \frac{6}{5}\beta_2.$$

Proof. We first show a lower bound on the size of the set $W := \{t \in \{0, 1, \ldots, 2n/3\} : V_{12}^t \cap B_2 \neq \emptyset\}$. If $B_2 \cap V_{12}^t \neq \emptyset$ for some $t \in \{2n/3 + 1, \ldots, n\}$, then $B_2 \cap V_{12}^t \neq \emptyset$ for all $t \in \{0, 1, \ldots, 2n/3\}$ and hence, $|W| \geq 2n/3$. Otherwise, $B_2 \cap V_{12}^t = \emptyset$ for all $t \in \{2n/3 + 1, \ldots, n\}$. In this case, $B_2 \subseteq \cup_{t=0}^{2n/3} V_{12}^t$. For $t \geq 1$, each line V_{12}^t has at most n nodes. For $t = 0$, the set B_2 can contain at most $n - 1$ nodes from V_{12}^0 which are not s_1 or s_2. Hence, $|W| \geq |B_2|/n = \beta_2 n$. Thus, we have that $|W| \geq \min\{2n/3, \beta_2 n\} = \beta_2 n$ as $\beta_2 \leq 0.66$.
Since the nodes of B_2 cannot reach V_{23} and V_{13} in $\mathcal{G}'$, we have that $|\delta(B_2) \cap L_{12}^t| \geq 2$ if $B_2 \cap V_{12}^t \neq \emptyset$. Hence,

$$\left|\delta(B_2) \cap \left(\cup_{t=0}^{2n/3} L_{12}^t\right)\right| \geq 2|W| \geq 2\beta_2 n.$$

Each edge in $\cup_{t=0}^{2n/3} L_{12}^t$ has cost at least $3/5n$. Hence,

$$Cost_J\left(\delta(B_2) \cap \left(\cup_{t=0}^{2n/3} L_{12}^t\right)\right) \geq \frac{6}{5}\beta_2.$$

Since $Y_1 = \delta(B_2) \cap (\cup_{t=0}^{n} L_{12}^t) \supseteq \delta(B_2) \cap \left(\cup_{t=0}^{2n/3} L_{12}^t\right)$, we get that $Cost_J(Y_1) \geq (6/5)\beta_2$. □

We now derive the required inequality on the cost as follows:

$$
\begin{aligned}
&Cost_J(E_2) - Cost_J(E_2') = Cost_J(X_3 \setminus Z) - Cost(E_2') \\
&\quad \geq Cost_J(X_3) - Cost_J(Z) - Cost_J(Y_2) - Cost_J(Y_3) + Cost_J(Z) \\
&\quad \geq Cost_J(X_3 \cap Y_1) + Cost_J(X_3 \cap Y_2) + Cost_J(X_3 \cap Y_3) \\
&\qquad\qquad - Cost_J(Y_2) - Cost_J(Y_3) \\
&\quad = Cost_J(X_3 \cap Y_1) - Cost_J((X_1 \cup X_2) \cap Y_2) \\
&\qquad\qquad - Cost_J((X_1 \cup X_2) \cap Y_3) \\
&\quad = Cost_J(Y_1) - Cost_J((X_1 \cup X_2) \cap Y_1) \\
&\qquad - Cost_J((X_1 \cup X_2) \cap Y_2) - Cost_J((X_1 \cup X_2) \cap Y_3) \\
&\quad = Cost_J(Y_1) - Cost_J(X_1 \cup X_2) \\
&\quad \geq \frac{6}{5}\beta_2 - 0.4\beta_2 - 0.4\beta_2 \\
&\quad = 0.4\beta_2.
\end{aligned}
$$

The first inequality follows from Claim 3 and the penultimate inequality follows Claim 4 and the fact that $Cost_J(X_1), Cost_J(X_2) \leq 0.4\beta_2$.

3. **Suppose** $r = 4$**.** We assume that $B_4 \neq \emptyset$, as otherwise the claim is trivial. We partition $\delta(B_4)$ into $X_1 := \delta(B_4) \cap \delta(S_2)$ and $X_2 := \delta(B_4) \setminus X_1$ (see Figure 9), and let $E_4 := X_1$ and $E_4' := \emptyset$.
   We now show the required properties for this choice of E_4 and E_4'. Let us fix a node $v \in B_4$ and a path $v, u_1, \ldots, u_t$ from v to L_{12} in $\mathcal{G}[B_4]$, and a path $v, w_1, \ldots, w_{t'}$ from v to L_{23} in $\mathcal{G}[B_4]$ (see Figure 9). Let $S := \{v, u_1, \ldots, u_t, w_1, \ldots, w_{t'}\}$. We note that $S \subseteq B_4$.

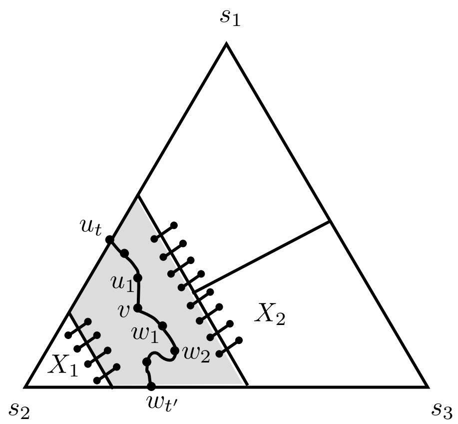

Fig. 9 Partition of $\delta(B_4)$ into X_1 and X_2. The shaded region is B_4.

   (a) Since $E_4' = \emptyset$, we need to show that $\delta(Q) \setminus E_4$ is a non-opposite cut-set. Let $\mathcal{G}'' := \mathcal{G} - (\delta(Q) \setminus E_4)$. We first observe that there are no paths between S and V_{13} in $\mathcal{G} - X_2$. Hence, there is no path from s_1 or s_3 to an end node of $E_4 = X_1$ in $\mathcal{G}'$. Moreover, there is no path from s_1 to V_{23} or from s_3 to V_{12} in $\mathcal{G}'$. So, there is no path from s_1 to V_{23} or from s_3 to V_{12} in $\mathcal{G}''$. Also, since $X_2 \subseteq \delta(Q) \setminus X_1$ and there is no path from s_2 to V_{13} in $\mathcal{G} - X_2$, it follows that there is no path from s_2 to V_{13} in $\mathcal{G}''$. Hence, $\delta(Q) \setminus E_4$ is a non-opposite cut-set.

(b) We note that there are no paths between s_2 and S in $\mathcal{G} - E_4$. Moreover, all paths in $\mathcal{G}$ between s_2 and V_{13} go through S. Hence, there are no paths between s_2 and V_{13} in $\mathcal{G} - E_4$. The cost of any such subset of nodes can be lower bounded by Lemma 4 of the main article. Thus, $Cost_J(E_4) - Cost_J(E_4') \geq 0.4 - (\frac{1}{n})/3 \geq 0.4\beta_4$. The last inequality is because $\beta_4 \leq 0.66$ by Proposition 1 and $n \geq 10$.

□

For $r \in [4]$, let E_r and E_r' be the sets given by Claim 2. We will show that

$$F := \left(\delta(Q) \setminus \left(\cup_{r=1}^4 E_r\right)\right) \cup \left(\cup_{r=1}^4 E_r'\right)$$

is a non-opposite cut-set and that $Cost_J(\delta(Q)) \geq Cost_J(F) + 0.4\alpha$. Then, we use Proposition 2(ii) of the main article to conclude that $Cost_J(\delta(Q)) \geq 1.2 - \frac{1}{n} + 0.4\alpha$.

Claim 5 *F is a non-opposite cut-set.*

Proof. Let $\mathcal{G}'' = \mathcal{G} - F$, and for $i \in [3]$, let S_i' be the set of nodes reachable from s_i in $\mathcal{G}''$. Since $E_r' \subseteq \mathcal{G}[B]$ for every r, S_i' is a superset of S_i, and $\mathcal{G}''[S_i] = \mathcal{G}'[S_i]$, which is connected. By the first property of Claim 2, for every $r \in [4]$ there exists $i \in [3]$ such that $E_r \subseteq \delta(B_r) \cap \delta(S_i)$. This implies, together with Proposition 1(ii), that the sets S_i' are disjoint. It also implies the following property:

($\star$) For every $r \in [4]$, there exists $i \in [3]$ such that $\delta_{\mathcal{G}''}(B_r) \subseteq \delta(S_i)$.

Suppose for contradiction that for some distinct $i, j, k \in [3]$, there exists a path P in $\mathcal{G}''$ from s_i to some $v \in V_{jk}$. Since $\delta(Q)$ is a non-opposite cut, the node v is not in S_i. Also, since $v \in S_i'$ and we have seen above that S_i' is disjoint from S_j' and S_k', it follows that $v \notin S_j' \cup S_k' \supseteq S_j \cup S_k$. Hence, $v \notin S_1 \cup S_2 \cup S_3$, and therefore $v \in B_r$ for some $r \in [4]$.

Let u be the last node of S_i on the path P. By property ($\star$), the end segment of P starting at the node after u is entirely in $\mathcal{G}[B_r] \setminus E_r'$. Since $\mathcal{G}''[S_i]$ is connected, we can replace the $s_i - u$ part of P by a path in $\mathcal{G}''[S_i]$, and obtain an $s_i - v$ path in $\mathcal{G}''$ that uses only edges in $\mathcal{G}'[S_i] \cup (\mathcal{G}'[B_r] \setminus E_r')$ and a single edge in $E_r \subseteq \delta(S_i) \cap \delta(B_r)$. Hence, this is also a path in $E \setminus ((\delta(Q) \setminus E_r) \cup E_r')$. But we have already seen in Claim 2 that $(\delta(Q) \setminus E_r) \cup E_r'$ is a non-opposite cut-set, so $v \notin V_{jk}$, a contradiction. □

To show that $Cost_J(\delta(Q)) \geq Cost_J(F) + 0.4\alpha$, we first observe that $E_i \subset \delta(B_i) \subset \delta(Q)$ for $i \in [4]$ and E_i's are mutually disjoint since $\delta(B_i)$'s are mutually disjoint by Proposition 1(iv). Therefore,

$$\begin{aligned}
Cost_J(F) &\leq Cost_J(\delta(Q) \setminus (\cup_{i=1}^4 E_i)) + Cost_J(\cup_{i=1}^4 E'_i) \\
&= Cost_J(\delta(Q)) - \sum_{i=1}^{4} (Cost_J(E_i) - Cost_J(E'_i)) \\
&\leq Cost_J(\delta(Q)) - \sum_{i=1}^{4} 0.4\beta_i \qquad \text{(By Claim 2)} \\
&\leq Cost_J(\delta(Q)) - 0.4\alpha \qquad \text{(By Proposition 1(v))}.
\end{aligned}$$

□

References

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