Abstract—We consider the Hypergraph-$k$-Cut problem. The input consists of a hypergraph $G = (V, E)$ with non-negative hyperedge-costs $c : E \to \mathbb{R}_+$ and a positive integer $k$. The objective is to find a least-cost subset $F \subseteq E$ such that the number of connected components in $G - F$ is at least $k$. An alternative formulation of the objective is to find a partition $G$ of vertices and a finite set $E$ and runs in a divide and conquer approach. The first algorithm is simpler for fixed constant. The input to this problem consists of a...
Submodular Partition Problems: Graph and hypergraph cut functions are submodular and one can view GRAPH-

-\text{k-CUT} and HYPERGRAPH-\text{k-CUT} as special cases of a more general problem called \text{SUBMODULAR-}\text{-k-PARTITION}

(abbreviated to \text{SUBMOD-}\text{-k-PART}) that we define now. We recall that a real-valued set function \( f : 2^V \to \mathbb{R} \) is submodular iff \( f(A \cap B) + f(A \cup B) \leq f(A) + f(B) \) for all \( A, B \subseteq V \). Zhao, Nagamochi, and Ibaraki [13] defined \text{SUBMOD-k-PART} as follows: given \( f \) specified via a value oracle and a positive integer \( k \), the goal is to partition \( V \) into non-empty sets \( V_1, V_2, \ldots, V_k \) so as to minimize \( \sum_{i=1}^{k} f(V_i) \). A special case of \text{SUBMOD-k-PART} is \text{SYM-SUBMOD-k-PART} when \( f \) is symmetric (that is \( f(A) = f(V \setminus A) \) for all \( A \subseteq V \)). It is not hard to see that GRAPH-\text{k-CUT} is a special case of \text{SYM-SUBMOD-k-PART}. However, HYPERGRAPH-\text{k-CUT} is not a special case of \text{SYM-SUBMOD-k-PART} even though the hypergraph cut function is itself symmetric;\(^1\) as observed in [14], one can reduce HYPERGRAPH-\text{k-CUT} to SUBMOD-

-\text{k-PART}. SUBMOD-\text{-k-PART} and \text{SYM-SUBMOD-}\text{-k-PART} are very general problems. For \( k = 2 \), they can be solved in polynomial-time via submodular function minimization. It is a very interesting open problem to decide whether they admit polynomial-time algorithms for all fixed \( k \). Okumoto, Fukunaga, and Nagamochi [14] showed that \text{SUBMOD-}\text{-k-PART} is polynomial-time solvable for \( k = 3 \). They generalized the work of Xiao [15] who showed that HYPERGRAPH-

-\text{k-CUT} is polynomial-time solvable for \( k = 3 \). Queyranne claimed, in 1999, a polynomial-time algorithm for \text{SYM-SUBMOD-}\text{-k-PART} when \( k \) is fixed [16], however the claim was retracted subsequently. This is reported in [17] where it is also shown that \text{SYM-SUBMOD-}\text{-k-PART} has a polynomial-

-time algorithm for \( k \leq 4 \).

Multiterminal variants: We also mention that GRAPH-\text{k-CUT}, HYPERGRAPH-\text{k-CUT}, and SUBMOD-\text{-k-PART} have natural variants involving separating specified terminal vertices \( s_1, s_2, \ldots, s_k \). These versions are NP-hard for \( k \geq 3 \). We discuss approximation algorithms for these problems in the related work section.

HYPERGRAPH-\text{k-CUT} and main result: The complexity of HYPERGRAPH-\text{k-CUT} for fixed \( k \) has been open since

\(^1\) \text{SYM-SUBMOD-k-PART} when the input function \( f \) is the cut function of a hypergraph is known as \text{HYPERGRAPH-k-PARTITION} in the literature [13], [14]. We emphasize that the objective in \text{HYPERGRAPH-k-PARTITION} is different from the objective in HYPERGRAPH-\text{k-CUT}.

the work of Goldschmidt and Hochbaum for graphs (1988) [1]. For \( k = 2 \), this is the HYPERGRAPH-MINCUT problem and can be solved via reduction to min \( s-t \) cuts in directed graphs [18] or via other approaches that take advantage of the submodularity structure of the hypergraph cut function (see [19] and references therein). For \( k \geq 3 \) and bounded rank hypergraphs, Fukunaga [20] generalized Thorup’s tree packing approach [11] to solve HYPERGRAPH-k-CUT for fixed \( k \) — the run-time depends exponentially in the rank (rank is the maximum cardinality of a hyperedge in the input hypergraph). It was also observed that Karger’s random contraction approach for graphs easily extends to give a randomized algorithm for bounded rank hypergraphs. As we noted earlier, Xiao [15] obtained a polynomial-time algorithm for HYPERGRAPH-k-CUT when \( k = 3 \). In fairly recent work, Chandrasekaran, Xu, and Yu [3] obtained the first randomized polynomial-time algorithm for HYPERGRAPH-

-k-CUT for any fixed \( k \); their Monte Carlo algorithm runs in \( \widetilde{O}(p n^2 k^{-1}) \) time where \( p = \sum_{e \in E} |e| \) is the representation size of the input hypergraph. Subsequently, Fox, Panigrahi, and Zhang [21] improved the randomized run-time to \( \widetilde{O}(mn n^2 k^{-2}) \), where \( m \) is the number of hyperedges in the input hypergraph. Both these randomized algorithms are based on random contraction of hyperedges and are inspired partly by earlier work in [22] for HYPERGRAPH-MINCUT.

The existence of a randomized algorithm for HYPERGRAPH-k-CUT raises the question of the existence of a deterministic algorithm. Random contraction based algorithms do not lend themselves naturally to derandomization. Perhaps, more pertinent is our interest in addressing the complexity of \text{SUBMOD-}\text{-k-PART}. There is no natural random contraction approach for this more general problem. For GRAPH-k-CUT, two distinct approaches lead to deterministic algorithms and among these, the tree packing approach, like the random contraction approach, does not appear to apply to \text{SUBMOD-}\text{-k-PART}. This leaves the divide and conquer approach initiated in the paper of Goldschmidt and Hochbaum [1], [2]. Is there a variant of this approach that works for HYPERGRAPH-k-CUT and \text{SUBMOD-}\text{-k-PART}? We discovered certain structural properties of HYPERGRAPH-k-CUT (that do not hold for other submodular functions) to prove our main result stated below.

Theorem 1. There is a deterministic polynomial-time algorithm for HYPERGRAPH-k-CUT for any fixed \( k \).

Our work raises the hope for a polynomial-time algorithm for \text{SUBMOD-}\text{-k-PART} when \( k \) is fixed.

A. Technical overview and structural results

We focus on the unit-cost variant of the problem in the rest of this work for the sake of notational simplicity. Note that we allow multigraphs and hence this is without loss
of generality. All our algorithms extend in a straightforward manner to arbitrary hyperedge costs. They rely only on minimum \((s,t)\)-cut computations and hence, they are strongly polynomial.

A key algorithmic tool will be the use of terminal cuts. We need some notation. Let \(G = (V, E)\) be a hypergraph. For a subset \(U\) of vertices, we will use \(\overline{U}\) to denote the set of hyperedges crossing \(U\), and \(d(U) := |\delta(U)|\) to denote the value of \(U\). More generally, given a partition \((V_1, V_2, \ldots, V_h)\), we denote the number of hyperedges crossing the partition by \(\text{cost}(V_1, V_2, \ldots, V_h)\).

Let \(S, T\) be disjoint subsets of vertices. A \(2\)-partition \((U, \overline{U})\) is an \((S, T)\)-terminal cut if \(S \subseteq U \subseteq V \setminus T\). Here, the set \(U\) is known as the source set and the set \(\overline{U}\) is known as the sink set. A minimum valued \((S, T)\)-terminal cut is known as a minimum \((S, T)\)-terminal cut. Since there could be multiple minimum \((S, T)\)-terminal cuts, we will be interested in source maximal minimum \((S, T)\)-terminal cuts and source minimal minimum \((S, T)\)-terminal cuts. These cuts are unique and can be found in polynomial-time via standard maxflow algorithms. In fact, these definitions extend to general submodular functions. Given \(f : 2^V \rightarrow \mathbb{R}\) and disjoint sets \(S, T \subseteq V\), we can define a minimum \((S, T)\)-terminal cut for \(f\) as \(\min_{U, \overline{S}, \overline{T} \subseteq \overline{V}} f(U)\). Uniqueness of source-maximal and source-minimal \((S, T)\)-terminal cuts follow from submodularity and one can also find these in polynomial-time via submodular function minimization.

Our algorithm follows the divide-and-conquer approach that was first used by Goldschmidt and Hochbaum [1], [2] for \textsc{Graph}-\(k\)-\textsc{Cut}, and in a more general fashion by Kamidoi, Yoshida, and Nagamochi [8] to improve the running time for \textsc{Graph}-\(k\)-\textsc{Cut}. The goal in this approach is to identify one part of some fixed optimum \(k\)-partition \((V_1, V_2, \ldots, V_h)\), say \(V_i\) without loss of generality, and then recursively find a \((k-1)\)-partition of \(V \setminus V_i\). How do we find such a part? Goldschmidt and Hochbaum proved a key structural lemma for \textsc{Graph}-\(k\)-\textsc{Cut}:

Suppose \((V_1, V_2, \ldots, V_h)\) is an optimum \(k\)-partition such that \(V_i\) is the part with the smallest cut value (i.e., \(|\delta(V_i)| \leq |\delta(V_j)|\) for all \(i \in [k]\) and \(V_i\) is maximal subject to this condition. Then, either \(|V_i| \leq k - 2\) or there exist disjoint sets \(S, T\) such that \(S \subseteq V_i, T \subseteq \overline{V_i}\) with \(|S| \leq k - 1\) and \(|T \cap V_j| = 1\) for every \(j \in \{2, \ldots, k\}\) so that the source maximal minimum \((S, T)\)-terminal cut is \((V_i, \overline{V_i})\). One can guess/enumerate all small-sized \((S, T)\)-pairs to find an \(O(n^{2k-2})\)-sized collection of sets containing \(V_i\) and recursively find an optimum \((k-1)\)-partition of \(V \setminus U\) for each \(U\) in the collection. This leads to an \(O(n^{O(k^2)})\)-time algorithm for \textsc{Graph}-\(k\)-\textsc{Cut}.

Queyranne [16] claimed that a natural generalization of the preceding structural lemma holds in the more general setting of \textsc{Sym-Submod}-\(k\)-\textsc{Part}. However, as reported in [17], the claimed proof was incorrect and it was only proved for \(k = 3, 4\). More importantly, as also noted in [17], this structural lemma (even if true for arbitrary \(k\)) is not useful for \textsc{Sym-Submod}-\(k\)-\textsc{Part} because one cannot recurse on \(V \setminus V_i\); the function \(f\) restricted to \(V \setminus V_i\) is no longer symmetric! The reader might now wonder how the approach works for \textsc{Graph}-\(k\)-\textsc{Cut}? Interestingly, \textsc{Graph}-\(k\)-\textsc{Cut} has the very nice property that the graph cut function restricted to \(V \setminus V_i\) is still symmetric!

However, \textsc{Hypergraph}-\(k\)-\textsc{Cut}, the problem of interest here, is not a special case of \textsc{Sym-Submod}-\(k\)-\textsc{Part}. Nevertheless, we are able to prove a strong structural characterization. We state the structural characterization now. We consider the partition viewpoint of \textsc{Hypergraph}-\(k\)-\textsc{Cut}. We will denote a \(k\)-partition by an ordered tuple. A \(k\)-partition is a minimum \(k\)-partition if it has the minimum number of crossing hyperedges among all possible \(k\)-partitions. Since there could be multiple minimum \(k\)-partitions, we will be interested in the \(k\)-partition \((V_1, \ldots, V_h)\) for which \(V_i\) is maximal: formally, we define a minimum \(k\)-partition \((V_1, \ldots, V_h)\) to be a \(k\)-minimal minimum \(k\)-partition if there is no other minimum \(k\)-partition \((V'_1, \ldots, V'_h)\) such that \(V_i\) is strictly contained in \(V'_i\). The following is our main structural result.

**Theorem 2.** Let \(G = (V, E)\) be a hypergraph and let \((V_1, \ldots, V_k)\) be a maximal minimum \(k\)-partition in \(G\) for an integer \(k \geq 2\). Suppose \(|V_i| \geq 2k - 2\). Then, for every subset \(T \subseteq V_i\) such that \(T\) intersects \(V_j\) for every \(j \in \{2, \ldots, k\}\), there exists a subset \(S \subseteq V_i\) of size \(2k - 2\) such that \((V_i, V_i')\) is the source maximal minimum \((S, T)\)-terminal cut.

Some important remarks regarding the preceding theorem are in order. Firstly, this is surprising: for instance, if the optimum \(k\)-partition is unique, then the theorem allows us to find any part \(V_i\) of the optimum \(k\)-partition \((V_1, \ldots, V_h)\) by solving minimum \((S, T)\)-terminal cuts for \(S\) and \(T\) of bounded sizes (by noting that the reordered \(k\)-partition \((V_i, V_i', \ldots, V_i'_{-1}, V_i'_{+1}, \ldots, V_h)\) is also a maximum minimum \(k\)-partition due to uniqueness and by applying Theorem 2 to this reordered \(k\)-partition). Such a result was not known even for graphs. Secondly, our structural theorem differs crucially from the structural lemma of Goldschmidt and Hochbaum [2] for \textsc{Graph}-\(k\)-\textsc{Cut} in that it does not rely on \(V_i\) being the part with the smallest cut value. This also explains why we need \(S\) to be of size \(2k - 2\) instead of \(k - 1\): one can show that \(2k - 2\) is tight for our structural theorem if we want to identify an arbitrary part even when considering \textsc{Graph}-\(k\)-\textsc{Cut}. Thirdly, our structural theorem does not hold for general submodular functions. The theorem statement was partly inspired by experiments on small sized instances and the proof is partly inspired by a structural theorem in [8] for graphs.

Theorem 2 implies, relatively easily, an \(O^{O(k)}\)-time algorithm for \textsc{Hypergraph}-\(k\)-\textsc{Cut}. We improve the running time to \(O^{O(k)}\) using a similar but more involved structural result that allows us to recover the union of \(k/2\) parts of an optimum \(k\)-partition. This high-level approach of
recovering the union of \( k/2 \) parts of an optimum \( k \)-partition was developed in [8] for \( \text{GRAPH-}k\text{-CUT} \). As we already mentioned in the preceding paragraph, a proof of a key structural lemma in [8] was an inspiration for our proofs though the precise statement of our structural theorem is different from the structural lemma of [8] and more subtle. We clarify this subtlety: the key structural lemma in [8] is different from the structural lemma of [8] and more subtle. The structural lemma in [8] was an inspiration for our proofs mentioned in the preceding paragraph, a proof of a key structural theorem (Theorem 2). We prove an important un-crossing property of the hypergraph cut function in Section III and use it to prove Theorem 2 in Section IV. Due to page limits, we defer the details of the faster \( n^{O(k)} \)-time algorithm based on divide-and-conquer to the full version of this work. All missing proofs also appear in the full version.

B. Other related work

Our main focus is on \( \text{HYPERGRAPH-}k\text{-CUT} \) and \( \text{GRAPH-}k\text{-CUT} \) when \( k \) is fixed. As we mentioned already, \( \text{GRAPH-}k\text{-CUT} \) is \( \text{NP-Hard} \) when \( k \) is part of the input [1]. A 2(1−1/k)-approximation is known for \( \text{GRAPH-}k\text{-CUT} \) [23]; several other approaches also give a 2-approximation (see [12], [24] and references therein). Manurangsi [25] showed that there is no polynomial-time \((2−\epsilon)\)-approximation for any constant \( \epsilon > 0 \) assuming the Small Set Expansion Hypothesis [26]. In contrast, \( \text{HYPERGRAPH-}k\text{-CUT} \) was recently shown [27] to be at least as hard as the densest \( k \)-subgraph problem. Combined with results in [28], this shows that \( \text{HYPERGRAPH-}k\text{-CUT} \) is unlikely to have a sub-polynomial factor approximation ratio and illustrates that \( \text{HYPERGRAPH-}k\text{-CUT} \) differs significantly from \( \text{GRAPH-}k\text{-CUT} \) when \( k \) is part of the input.

As we mentioned earlier, terminal versions of \( \text{SUBMOD-}k\text{-PART} \) and its special cases such as \( \text{Multiway-Cut} \) in graphs have been extensively studied. The most general version here is the following: given a submodular function \( f : 2^V \rightarrow \mathbb{R} \) (by value oracle) and terminals \( \{s_1, s_2, \ldots, s_k\} \subset V \) the goal is to find a partition \( \{V_1, \ldots, V_k\} \) to minimize \( \sum_i f(V_i) \) subject to the constraint that \( s_i \in V_i \) for \( 1 \leq i \leq k \). These problems are \( \text{NP-Hard} \) even for \( k = 3 \) and the main focus has been on approximation algorithms. We refer the reader to [13], [29]–[31] for further references. We mention that for non-negative \( f \) and fixed \( k \), the best approximation algorithms for \( \text{SUBMOD-}k\text{-PART} \) and \( \text{SYM-SUBMOD-}k\text{-PART} \) are via the terminal versions; a \((1.5−1/k)\) for \( \text{SYM-SUBMOD-}k\text{-PART} \) and a \( 2(1−1/k)\)-approximation for \( \text{SUBMOD-}k\text{-PART} \) [30], [31].

Fixed parameter tractability of \( \text{GRAPH-}k\text{-CUT} \) has also been investigated. It is known that \( \text{GRAPH-}k\text{-CUT} \) is \( \text{W}[1]\)-hard (and hence not likely to be \( \text{FPT} \)) parameterized by \( k \) [32] while it is \( \text{FPT} \) when parameterized by \( k \) and the solution size [33]. We observed, via a simple reduction from a result of Marx on vertex separators [34], that \( \text{HYPERGRAPH-}k\text{-CUT} \) is \( \text{W}[1] \) hard even when parameterized by \( k \) and the solution size. This also demonstrates that \( \text{HYPERGRAPH-}k\text{-CUT} \) differs in complexity from \( \text{GRAPH-}k\text{-CUT} \).

Another problem closely related to \( \text{HYPERGRAPH-}k\text{-CUT} \) is the \( \text{HYPERGRAPH-}k\text{-PARTITION} \) problem. The input to \( \text{HYPERGRAPH-}k\text{-PARTITION} \) is a hypergraph \( G = (V, E) \) and a positive integer \( k \) and the goal is to partition \( V \) into \( k \) non-empty sets \( V_1, \ldots, V_k \) but the objective is to minimize \( \sum_{i=1}^{k} \text{opt}(V_i) \); this means that a hyperedge \( e \) that crosses \( h \geq 2 \) parts pays \( h \) instead of only once (as is the case in \( \text{HYPERGRAPH-}k\text{-CUT} \)). \( \text{HYPERGRAPH-}k\text{-PARTITION} \) is a special case of \( \text{SYM-SUBMOD-}k\text{-PART} \) and its complexity status for fixed \( k \geq 5 \) is open. \( \text{HYPERGRAPH-}k\text{-PARTITION} \) in constant rank hypergraphs is solvable in polynomial-time by relying on the fact that the number of constant-approximate minimum \( k \)-cuts in a constant rank hypergraph is polynomial.

II. Recursive Algorithm

Theorem 2 allows us to design a recursive algorithm for hypergraph \( k \)-cut that we describe now. For a hypergraph \( G = (V, E) \) and for a subset \( U \) of vertices, let \( G[U] \) denote the hypergraph obtained from \( G \) by discarding the vertices in \( U \) and by discarding all hyperedges \( e \in E \) that intersect \( U \). We describe the formal algorithm in Figure 1. It follows the high-level outline given in the technical overview. It enumerates \( n^{O(k)} \) minimum \((S, T)\)-terminal cuts, one of which is guaranteed to identify one part of an optimum \( k \)-partition, and then recursively finds an optimum \((k−1)\)-partition after removing the found part. The run-time guarantee is given in Theorem 3.

Theorem 3. Let \( G = (V, E) \) be a \( n \)-vertex hypergraph of size \( p \) and let \( k \) be an integer. Then, algorithm \( \text{CUT}(G, k) \) returns a partition corresponding to a minimum \( k \)-cut in \( G \) and it can be implemented to run in \( n^{O(k^2)}T(n, p) \) time, where \( T(n, p) \) denotes the time complexity for computing the source maximal \((s, t)\)-terminal cut in a \( n \)-vertex hypergraph of size \( p \).

Proof: We first show the correctness of the algorithm. All candidates considered by the algorithm correspond to a \( k \)-partition, so we only have to show that the algorithm returns a \( k \)-partition corresponding to a minimum \( k \)-cut. We
Hence, terminal cuts and hence it is \( \text{OCT}(N_k) \). Show this by induction on Theorem 2. The reader may want to skip the rather long proof of this uncrossing theorem will be clearer in the proof of Theorem 2. See Figure 2 for an illustration of the sets that appear in Theorem 4 and Lemma 2.

Figure 1. Algorithm to compute minimum \( k \)-cut in hypergraphs.

Among all \( k \)-partitions in \( \mathcal{R} \), pick the one with minimum cost and return it.

Figure 2. Illustration of the sets that appear in Theorem 4 and Lemma 2.

Theorem 4. Let \( G = (V,E) \) be a hypergraph, \( k \geq 2 \) be an integer, and \( \emptyset \neq R \subseteq U \subseteq V \). Let \( S = \{u_1, \ldots, u_p\} \subseteq U \setminus R \) for \( p \geq k+1 \). Let \( \mathcal{A}, A_i \) be a minimum \((S \cup R) \setminus \{u_i\} \cup \overline{U}\)-terminal cut. Suppose that \( u_i \in \mathcal{A} \setminus \bigcup_{j \in [p] \setminus \{i\}} A_j \) for every \( i \in [p] \). Then, there exists a \( k \)-partition \( \{P_1, \ldots, P_k\} \) of \( V \) with \( U \subseteq P_i \) such that

\[
\text{cost}(P_1, \ldots, P_k) \leq \frac{1}{2} \min\{d(A_i) + d(A_j) : i, j \in [p], i \neq j \}.
\]

Figure 3. Illustration of the sets that appear in Theorem 4 and Lemma 2.

The rest of the section is devoted to the proof of Theorem 4. We begin with some background on the hypergraph cut function. Let \( G = (V,E) \) be a hypergraph. For a subset \( A \) of vertices, we recall that \( d(A) \) denotes the number of hyperedges that intersect both \( A \) and \( \overline{A} \). The function \( d : 2^V \rightarrow \mathbb{R}_+ \) is known as the hypergraph cut function. The hypergraph cut function is symmetric, i.e.,

\[
d(A) = d(\overline{A}) \quad \text{for all} \quad A \subseteq V,
\]

and submodular, i.e.,

\[
d(A) + d(B) \geq d(A \cap B) + d(A \cup B) \quad \text{for all subsets} \quad A, B \subseteq V.
\]

III. UNCROSSING PROPERTIES OF THE HYPERGRAPH CUT FUNCTION

In this section, we show the following uncrossing theorem which will be useful to prove the main structural theorem. See Figure 2 for an illustration of the sets that appear in the statement of Theorem 4. The motivation for the statement of this uncrossing theorem will be clearer in the proof of Theorem 2. The reader may want to skip the rather long and technical proof of the uncrossing theorem in the first reading and come back to it after seeing its use in the proof of Theorem 2.

**Theorem 4.** Let \( G = (V,E) \) be a hypergraph, \( k \geq 2 \) be an integer, and \( \emptyset \neq R \subseteq U \subseteq V \). Let \( S = \{u_1, \ldots, u_p\} \subseteq U \setminus R \) for \( p \geq k+1 \). Let \( \mathcal{A}, A_i \) be a minimum \((S \cup R) \setminus \{u_i\} \cup \overline{U}\)-terminal cut. Suppose that \( u_i \in \mathcal{A} \setminus \bigcup_{j \in [p] \setminus \{i\}} A_j \) for every \( i \in [p] \). Then, there exists a \( k \)-partition \( \{P_1, \ldots, P_k\} \) of \( V \) with \( U \subseteq P_i \) such that

\[
\text{cost}(P_1, \ldots, P_k) \leq \frac{1}{2} \min\{d(A_i) + d(A_j) : i, j \in [p], i \neq j \}.
\]
For our purposes, it will help to count the hyperedges more accurately than employ the submodularity inequality. We define some notation that will help in more accurate counting. Let \((Y_1, \ldots, Y_p, W, Z)\) be a partition of \(V\). We recall that \(\text{cost}(Y_1, \ldots, Y_p, W, Z)\) denotes the number of hyperedges that cross the partition. We note that when considering these hyperedges it is convenient to visualize each part of the partition as a single vertex obtained by contracting the part. We define the following quantities:

1. Let \(\text{cost}(W, Z) = \{\{e \in W \cup Z, e \cap W \neq \emptyset, e \cap Z \neq \emptyset\}\}\) be the number of hyperedges contained in \(W \cup Z\) that intersect both \(W \) and \(Z\).
2. Let \(\alpha(Y_1, \ldots, Y_p, W, Z)\) be the number of hyperedges that intersect \(Z\) and at least two of the sets in \(\{Y_1, \ldots, Y_p, W\}\).
3. Let \(\beta(Y_1, \ldots, Y_p, Z)\) be the number of hyperedges that are disjoint from \(Z\) but intersect at least two of the sets in \(\{Y_1, \ldots, Y_p\}\).

For a partition \((Y_1, \ldots, Y_p, W, Z)\), we will be interested in the sum of \(\text{cost}(Y_1, \ldots, Y_p, W, Z)\) with the three quantities defined above which we denote as \(\sigma(Y_1, \ldots, Y_p, W, Z)\), i.e.,

\[
\sigma(Y_1, \ldots, Y_p, W, Z) := \text{cost}(Y_1, \ldots, Y_p, W, Z) + \alpha(Y_1, \ldots, Y_p, W, Z) + \beta(Y_1, \ldots, Y_p, Z).
\]

We note that \(\sigma(Y_1, \ldots, Y_p, W, Z)\) counts every hyperedge that crosses the partition twice except for those hyperedges that intersect exactly one of the sets in \(\{Y_1, \ldots, Y_p\}\) and exactly one of the sets in \(\{W, Z\}\) which are counted exactly once (see Figure 3).

**Figure 3.** Hyperedges counted by \(\sigma(Y_1, \ldots, Y_p, W, Z)\): The dashed hyperedges are counted only by \(\text{cost}(Y_1, \ldots, Y_p, W, Z)\). The rest of the hyperedges are counted twice in \(\sigma(Y_1, \ldots, Y_p, W, Z)\); once by the term \(\text{cost}(Y_1, \ldots, Y_p, W, Z)\) and once more by the indicated term.

The motivation behind considering the function \(\sigma(Y_1, \ldots, Y_p, W, Z)\) comes from Proposition 1. We emphasize that the interpretation for \(\sigma(Y_1, \ldots, Y_p, W, Z)\) given in the proposition holds only for \(p = 2\).

**Proposition 1.** Let \((Y_1, Y_2, W, Z)\) be a partition of \(V\) and let \(A_1 := Y_1 \cup W\) and \(A_2 := Y_2 \cup W\). Then,

\[
d(A_1) + d(A_2) = \sigma(Y_1, Y_2, W, Z).
\]

The next lemma will help in obtaining a \((p+3)\)-partition from a \((p+2)\)-partition while controlling the increase in \(\sigma\)-value. This will be used in a subsequent inductive argument. See Figure 4 for an illustration of the sets appearing in the statement of the lemma. Our proof of Lemma 1 is through case analysis. Currently we do not know how to prove this lemma without a somewhat laborious case analysis. We remark that this is partly due to the fact that hyperedges can have different cardinalities as well as due to the fact that we cannot rely only on submodularity of the hypergraph cut function.

**Lemma 1.** Let \(G = (V, E)\) be a hypergraph and let \((X_1, \ldots, X_p, W_0, Z_0)\) be a partition for some integer \(p \geq 1\). Let \(Q \subset V\) be a set such that

\[
Y_i := X_i - Q \neq \emptyset \quad \forall \ i \in [p],
Y_{p+1} := Q \cap Z_0 \neq \emptyset,
Z := Z_0 - Q \neq \emptyset, \quad \text{and}
W := W_0 \cup (Q \setminus Z_0) \neq \emptyset.
\]

Then, \((Y_1, \ldots, Y_p, Y_{p+1}, W, Z)\) is a partition of \(V\) such that

\[
\sigma(Y_1, \ldots, Y_p, Y_{p+1}, W, Z) \leq \sigma(X_1, \ldots, X_p, W_0, Z_0) + d(Q) - d(W_0 \cap Q).
\]

**Proof:** By definition \((Y_1, \ldots, Y_p, Y_{p+1}, W, Z)\) is a partition of \(V\).

We rewrite the required inequality in the following form as it becomes convenient to prove:

\[
\sigma(X_1, \ldots, X_p, W_0, Z_0) - \sigma(Y_1, \ldots, Y_p, Y_{p+1}, W, Z) \\
\geq d(W_0 \cap Q) - d(Q).
\]

For a hyperedge \(e \in E\), let \(\lambda_e^0 \in \{0, 1, 2\}\) and \(\lambda_e^1 \in \{0, 1, 2\}\) be the number of times that \(e\) is counted by
σ(X_1, \ldots, X_p, W_0, Z_0) and σ(Y_1, \ldots, Y_{p+1}, W, Z) respectively, and let λ_0^Q ∈ \{0, 1\} and λ_e^{W_0 \cap Q} ∈ \{0, 1\} be the number of times that e is counted by d(Q) and d(W_0 \cap Q) respectively.

Let ℓ_e := λ_e^Q - λ_e^1 and r_e := λ_e^{W_0 \cap Q} - λ_e^Q. Thus, ℓ_e and r_e denote the number of times the hyperedge e is counted in the LHS and RHS of inequality (1) respectively and moreover ℓ_e ∈ \{0, ±1, ±2\} and r_e ∈ \{0, ±1\} for every hyperedge e ∈ E. Let

\[
\text{Positives}(ℓ) := \sum_{e \in E, ℓ_e \geq 1} ℓ_e,
\]

\[
\text{Negatives}(ℓ) := \sum_{e \in E, ℓ_e \leq -1} ℓ_e,
\]

\[
\text{Positives}(r) := \sum_{e \in E, r_e = 1} r_e,
\]

\[
\text{Negatives}(r) := \sum_{e \in E, r_e = -1} r_e.
\]

Claims 1 and 2 complete the proof of the lemma. ■

Claim 1.

\text{Positives}(ℓ) ≥ \text{Positives}(r).

Proof: Let e be a hyperedge such that r_e = 1. Then, e is counted by d(W_0 \cap Q) but not d(Q). This means that e \subseteq Q, e \cap (W_0 \cap Q) \neq \emptyset, and e \cap (Q \setminus W_0) \neq \emptyset. Thus, e intersects W_0 \cap Q and at least one of the sets in \{X_1 \cap Q, \ldots, X_p \cap Q, Z_0 \cap Q\}. It suffices to show that ℓ_e ≥ 1. We consider different cases for e below and show that ℓ_e ≥ 1 in all cases.

1) Suppose e intersects Z_0 \cap Q.

a) Suppose e is disjoint from X_1 \cap Q, \ldots, X_p \cap Q. Then, λ_e^0 = 2 since e is counted by both cost(X_1, \ldots, X_p, W_0, Z_0) and by cost(W_0, Z_0). Hence, ℓ_e = λ_e^0 - λ_e^1 = 1.

b) Suppose e intersects at least one of the sets in \{X_1 \cap Q, \ldots, X_p \cap Q\}. Then, λ_e^0 = 2 since e is counted by both cost(X_1, \ldots, X_p, W_0, Z_0) and by α(X_1, \ldots, X_p, W_0, Z_0). If e is disjoint from Z_0, then again λ_e^0 = 2 since e is counted by both cost(X_1, \ldots, X_p, W_0, Z_0) and β(X_1, \ldots, X_p, W_0, Z_0). In both cases, we have ℓ_e = λ_e^0 - λ_e^1 ≥ 1.

2) Suppose e is disjoint from Z_0 \cap Q. Then e has to intersect at least one of the sets in \{X_1 \cap Q, \ldots, X_p \cap Q\}.

a) Suppose e intersects exactly one of the sets in \{X_1 \cap Q, \ldots, X_p \cap Q\}. Then, λ_e^0 = 1 since e is counted only by cost(X_1, \ldots, X_p, W_0, Z_0). Hence, ℓ_e = λ_e^0 - λ_e^1 ≥ 1.

b) Suppose e intersects at least two of the sets in \{X_1 \cap Q, \ldots, X_p \cap Q\}. Then, λ_e^0 = 2 since e is counted by both cost(X_1, \ldots, X_p, W_0, Z_0) and by β(X_1, \ldots, X_p, Z_0). However, λ_e^1 = 0 since e does not cross the partition \{Y_1, \ldots, Y_{p+1}, W, Z\}. Hence, ℓ_e = λ_e^0 - λ_e^1 = 2 ≥ 1.

Claim 2.

\text{Negatives}(ℓ) ≥ \text{Negatives}(r).

Proof: Let e be a hyperedge such that ℓ_e ≤ -1, i.e., λ_e^1 ≥ λ_e^0 + 1. Then λ_e^1 ≥ 1 and hence, e crosses the partition \{Y_1, \ldots, Y_{p+1}, W, Z\}. It suffices to show that r_e ≤ ℓ_e, i.e., λ_e^Q ≥ λ_e^{W_0 \cap Q} + λ_e^1 - λ_e^0. We consider different cases for e below and for each case, we show that either λ_e^Q ≥ λ_e^{W_0 \cap Q} + λ_e^1 - λ_e^0 or the case is impossible.

1) Suppose e is disjoint from Z. Then, e intersects at least one of the sets in \{Y_1, \ldots, Y_{p+1}\} since e crosses the partition \{Y_1, \ldots, Y_{p+1}, W, Z\}.

a) Suppose e intersects exactly one of the sets in \{Y_1, \ldots, Y_{p+1}\}, say Y_i \subseteq [p] + 1. Then, e intersects W and consequently, λ_e^1 = 1 since e is counted by cost(Y_1, \ldots, Y_{p+1}, W, Z). Since 1 = λ_e^1 ≥ λ_e^0 + 1, it follows that λ_e^0 = 0. This implies that e does not cross the partition \{X_1, \ldots, X_p, W_0, Z_0\}. Therefore, e \supseteq [p] \setminus Y_i with e intersecting X_i \cap Q and Y_i = X_i \setminus Q. Consequently, λ_e^Q = 1 and λ_e^{W_0 \cap Q} = 0. Hence λ_e^Q ≥ λ_e^{W_0 \cap Q} + λ_e^1 - λ_e^0.

b) Suppose e intersects at least two of the sets in \{Y_1, \ldots, Y_{p+1}\}. Then, λ_e^1 = 2 since e is counted by both cost(Y_1, \ldots, Y_{p+1}, W, Z) as well as β(Y_1, \ldots, Y_{p+1}, Z).

i) Suppose e intersects at least two of the sets in \{Y_1, \ldots, Y_p\}. If e intersects Z_0, then λ_e^0 = 2 since e is counted by both cost(X_1, \ldots, X_p, W_0, Z_0) and by α(X_1, \ldots, X_p, W_0, Z_0). If e is disjoint from Z_0, then again λ_e^0 = 2 since e is counted by both cost(X_1, \ldots, X_p, W_0, Z_0) and β(X_1, \ldots, X_p, W_0, Z_0). In both cases, we have 2 = λ_e^1 ≥ λ_e^0 + 1 = 3, a contradiction.

ii) Suppose e intersects Y_{p+1} and exactly one of the sets in \{Y_1, \ldots, Y_p\}, say Y_i \subseteq [p]. Then, λ_e^0 ≥ 1 since e crosses the partition \{X_1, \ldots, X_p, W_0, Z_0\}. Since 2 = λ_e^1 ≥ λ_e^0 + 1, it follows that λ_e^0 = 1. This implies that none of cost(W_0, Z_0), α(X_1, \ldots, X_p, W_0, Z_0), and β(X_1, \ldots, X_p, Z_0) count e. Therefore, e is disjoint from W and e intersects Y_{p+1} = Z_0 \cap Q and Y_i = X_i \setminus Q. Thus, e is counted
by $d(Q)$ but not $d(W_0 \cap Q)$. Consequently, 
\[ \lambda^Q_2 = 1 \text{ and } \lambda^{W_0 \cap Q}_2 = 0. \] Hence, 
\[ \lambda^Q_2 \geq \lambda^{W_0 \cap Q}_2 + \lambda^e_2 - \lambda^e_0. \]

2) Suppose $e$ intersects $Z$. Then, $e$ intersects at least one of the sets in \{\(Y_1, \ldots, Y_{p+1}, W\)\} since $e$ crosses the partition \(Y_1, \ldots, Y_{p+1}, W, Z\).

a) Suppose $e$ intersects exactly one of the sets in \{\(Y_1, \ldots, Y_{p+1}, W\)\}. Then, $\lambda^e_0 = 1$ since $e$ is counted only by cost($Y_1, \ldots, Y_{p+1}, W$).

i) Suppose $e$ is disjoint from $W$. Then, $e$ intersects exactly one of the sets in \{\(Y_1, \ldots, Y_{p+1}\)\}. Since 
\[ 1 = \lambda^e_0 \geq \lambda^e_0 + 1, \] we have that $\lambda^e_0 = 0$. This implies that $e$ does not cross the partition \(X_1, \ldots, X_p, W_0, Z_0\). Hence, $e$ can only intersect $Y_{p+1}$. Thus, 
\[ e \subseteq Z_0 = Z \cup Y_{p+1} \] with $e$ intersecting 
\[ Z = Z_0 \setminus Q \text{ and } Y_{p+1} = Z_0 \cap Q. \] Thus, 
\[ e \text{ is counted by } d(Q) \text{ but not } d(W_0 \cap Q). \] Consequently, 
\[ \lambda^Q_2 = 1 \] and 
\[ \lambda^{W_0 \cap Q}_2 = 0. \] Hence, 
\[ \lambda^Q_2 \geq \lambda^{W_0 \cap Q}_2 + \lambda^e_2 - \lambda^e_0. \]

ii) Suppose $e$ intersects $W$. Then, $e$ has to cross the partition \(X_1, \ldots, X_p, W_0, Z_0\) and therefore, 
\[ \lambda^e_0 \geq 1. \] Thus, 
\[ 1 = \lambda^e_0 \geq \lambda^e_0 + 1 + 2, \] a contradiction.

b) Suppose $e$ intersects at least two of the sets in 
\{\(Y_1, \ldots, Y_{p+1}, W, Z\)\}. Then, $\lambda^e_0 = 2$ since $e$ is counted by both cost($Y_1, \ldots, Y_{p+1}, W, Z$) and 
\(\alpha(Y_1, \ldots, Y_{p+1}, W, Z)\).

i) Suppose $e$ intersects at least two of the sets in 
\{\(Y_1, \ldots, Y_p\)\}. Then, $\lambda^e_0 = 2$ since $e$ is counted by cost($X_1, \ldots, X_p, W_0, Z_0$) as well as 
\(\alpha(X_1, \ldots, X_p, W_0, Z_0)\). Thus, 
\[ 2 = \lambda^e_0 \geq \lambda^e_0 + 1 + 3, \] a contradiction.

ii) Suppose $e$ intersects exactly one of the sets in \{\(Y_1, \ldots, Y_p\)\}, say $Y_i$ for some 
\(i \in [p]\), and $e$ intersects $Y_{p+1}$ but is disjoint from $W$. Then, 
\[ \lambda^e_0 \geq 1 \] since $e$ crosses the partition \(X_1, \ldots, X_p, W_0, Z_0\). Since 
\[ 2 = \lambda^e_0 \geq \lambda^e_0 + 1, \] it follows that 
\[ \lambda^e_0 = 1. \] This implies that none of cost($W_0, Z_0$), \(\alpha(X_1, \ldots, X_p, W_0, Z_0)\), and 
\(\beta(X_1, \ldots, X_p, Z_0)\) count $e$ and hence, $e$ is contained in \(Y_i \cup Z_0 \subseteq X_i \cup Z_0\) with $e$ intersecting 
\(Y_{p+1} = Z_0 \cap Q \) and $Y_i = X_i \setminus Q$. Thus, 
\[ e \text{ is counted by } d(Q) \text{ but not } d(W_0 \cap Q). \] Consequently, 
\[ \lambda^Q_2 = 1 \] and 
\[ \lambda^{W_0 \cap Q}_2 = 0. \] Hence, 
\[ \lambda^Q_2 \geq \lambda^{W_0 \cap Q}_2 + \lambda^e_2 - \lambda^e_0. \]

iii) Suppose $e$ intersects exactly one of the sets in \{\(Y_1, \ldots, Y_p\)\}, say $Y_i$ for some 
\(i \in [p]\), and $e$ intersects $W$ but is disjoint from $Y_{p+1}$. Then, 
\[ \lambda^e_0 \geq 1 \] since $e$ crosses the partition \(X_1, \ldots, X_p, W_0, Z_0\). Since 
\[ 2 = \lambda^e_0 \geq \lambda^e_0 + 1, \] it follows that 
\[ \lambda^e_0 = 1. \] This implies that none of cost($W_0, Z_0$), \(\alpha(X_1, \ldots, X_p, W_0, Z_0)\), and 
\(\beta(X_1, \ldots, X_p, Z_0)\) count $e$. Therefore, $e$ is contained in 
\(X_i \cup Z\) and $e$ intersects 
\(X_i \cap Q\) since $e$ has to intersect $W$. Moreover, $e$ intersects $Y_i = X_i \setminus Q$. Thus, $e$ is counted by 
\(d(Q)\) but not $d(W_0 \cap Q)$. Consequently, 
\[ \lambda^Q_2 = 1 \] and 
\[ \lambda^{W_0 \cap Q}_2 = 0. \] Hence, 
\[ \lambda^Q_2 \geq \lambda^{W_0 \cap Q}_2 + \lambda^e_2 - \lambda^e_0. \]

iv) Suppose $e$ is disjoint from $Y_1, \ldots, Y_p$ and intersects both $Y_{p+1}$ and $W$. Then, $e$ intersects at least two of the sets in 
\{\(X_1 \cap Q, \ldots, X_p \cap Q\)\}.

A) Suppose $e$ intersects at least two of the sets in \{\(X_1 \cap Q, \ldots, X_p \cap Q\)\}. Then, 
\[ \lambda^e_0 = 2 \] since $e$ is counted by cost($X_1, \ldots, X_p, W_0, Z_0$) as well as 
\(\alpha(X_1, \ldots, X_p, W_0, Z_0)\). Thus, 
\[ 2 = \lambda^e_0 \geq \lambda^e_0 + 1 + 3, \] a contradiction.

B) Suppose $e$ does not intersect 
\(X_1 \cap Q, \ldots, X_p \cap Q\). Then, $e$ intersects $W_0$ since $e$ is counted by both cost($Y_1, \ldots, Y_{p+1}, W, Z$) and 
\(\alpha(Y_1, \ldots, Y_{p+1}, W, Z)\) (recall that we are in case (b)). Moreover, 
\[ e \subseteq W_0 \cup Z_0. \] Therefore, 
\[ \lambda^e_0 = 2 \] since $e$ is counted by cost($X_1, \ldots, X_p, W_0, Z_0$) as well as cost($W_0, Z_0$). Thus, 
\[ 2 = \lambda^e_0 \geq \lambda^e_0 + 1 + 3, \] a contradiction.

C) Suppose $e$ intersects exactly one of the sets in \{\(X_1 \cap Q, \ldots, X_p \cap Q\)\}, say $X_i \cap Q$ for some $i \in [p]$, and $e$ intersects 
\(W_0 \cap Q\). Then, 
\[ \lambda^e_0 = 2 \] since $e$ is counted by both cost($X_1, \ldots, X_p, W_0, Z_0$) and 
\(\alpha(X_1, \ldots, X_p, W_0, Z_0)\). Thus, 
\[ 2 = \lambda^e_0 \geq \lambda^e_0 + 1 + 3, \] a contradiction.

D) Suppose $e$ intersects exactly one of the sets in \{\(X_1 \cap Q, \ldots, X_p \cap Q\)\}, say $X_i \cap Q$ for some $i \in [p]$, and $e$ is disjoint from 
\(W_0 \cap Q\). Then, 
\[ \lambda^e_0 \geq 1 \] since $e$ crosses the partition \(X_1, \ldots, X_p, W_0, Z_0\). Since 
\[ 2 = \lambda^e_0 \geq \lambda^e_0 + 1, \] it follows that 
\[ \lambda^e_0 = 1. \] This implies that none of cost($W_0, Z_0$), \(\alpha(X_1, \ldots, X_p, W_0, Z_0)\), and 
\(\beta(X_1, \ldots, X_p, Z_0)\) count $e$. Therefore, $e$ is contained in 
\(X_i \cap Q \cup Z_0\) and $e$ intersects 
\(Y_{p+1} = Z_0 \cap Q\) and 
\(Z = Z_0 \setminus Q\). Thus, $e$ is counted by 
\(d(Q)\) but not $d(W_0 \cap Q)$. Consequently, 
\[ \lambda^Q_2 = 1 \] and 
\[ \lambda^{W_0 \cap Q}_2 = 0. \] Hence, 
\[ \lambda^Q_2 \geq \lambda^{W_0 \cap Q}_2 + \lambda^e_2 - \lambda^e_0. \]

The next lemma will help in uncrossing a collection of sets to obtain a partition with small $\sigma$-value. See Figure 2 for an illustration of the sets that appear in the statement of
the lemma.

**Lemma 2.** Let $G = (V, E)$ be a hypergraph and $\emptyset \neq R \subseteq U \subseteq V$. Let $S = \{u_1, \ldots, u_p\} \subseteq U \setminus R$ for $p \geq 2$. Let $(\overline{A}, A)$ be a minimum $((S \cup R) \setminus \{u_i\}, \overline{U})$-terminal cut. Suppose that $u_i \in A_i \setminus \bigcup_{j \in [p] \setminus \{i\}} A_j$ for every $i \in [p]$. Let

$$Z := \cap_{i=1}^p \overline{A}_i, \quad W := \bigcup_{1 \leq i < j \leq p} (A_i \cap A_j), \quad \text{and} \quad Y_i := A_i - W \forall i \in [p].$$

Then, $(Y_1, \ldots, Y_p, W, Z)$ is a $(p+2)$-partition of $V$ with

$$\sigma(Y_1, \ldots, Y_p, W, Z) \leq \min\{d(A_i) + d(A_j) : i, j \in [p], i \neq j\}.$$

**Proof:** For every $i \in [p]$, the set $Y_i$ is non-empty since $u_i \in Y_i$. The set $W$ is non-empty since $\overline{U} \subseteq W$. The set $Z$ is non-empty since $R \subseteq Z$. By definition, the sets $Y_1, \ldots, Y_p, W, Z$ are all disjoint and their union contains all vertices. Hence, $(Y_1, \ldots, Y_p, W, Z)$ is a partition of $V$. Without loss of generality, let $d(A_1) \leq d(A_2) \leq \ldots \leq d(A_p)$. We bound the $\sigma$-value of the partition by induction on $p$.

The base case of $p = 2$ follows from Proposition 1. We show the induction step. Suppose that the statement holds for $p = q$. We prove that it holds for $p = q + 1$. Consider $R_0 := R \cup \{u_{q+1}\}$ and $S_0 := S \setminus \{u_{q+1}\}$. Then, $(\overline{A}, A)$ is still a minimum $((S_0 \cup R_0) \setminus \{u_i\}, \overline{U})$-terminal cut for every $i \in [q]$ and moreover, $u_i \in A_i \setminus \bigcup_{j \in [q] \setminus \{i\}} A_j$ for every $i \in [q]$. By induction hypothesis, we get that for the sets

$$Z_0 := \cap_{i=1}^q \overline{A}_i, \quad W_0 := \bigcup_{1 \leq i < j \leq q} (A_i \cap A_j), \quad \text{and} \quad X_i := A_i - W \forall i \in [q],$$

we have

$$\sigma(X_1, \ldots, X_q, W_0, Z_0) \leq d(A_1) + d(A_2).$$

The partition $(X_1, \ldots, X_q, W_0, Z_0)$ and the set $Q := A_{q+1}$ satisfy the conditions of Lemma 1. By Lemma 1, we obtain that

$$\sigma(Y_1, \ldots, Y_q, Y_{q+1}, W, Z) \leq \sigma(X_1, \ldots, X_q, W_0, Z_0) + d(A_{q+1}) - d(W_0 \cap A_{q+1}).$$

Since $(W_0 \cap A_{q+1}, W_0 \cap A_{q+1})$ is a feasible $((S \cup R) \setminus \{u_{q+1}\}, \overline{U})$-terminal cut, we have that $d(A_{q+1}) \leq d(W_0 \cap A_{q+1})$. Hence,

$$\sigma(Y_1, \ldots, Y_q, Y_{q+1}, W, Z) \leq \sigma(X_1, \ldots, X_q, W_0, Z_0) \leq d(A_1) + d(A_2).$$

The next lemma will help in aggregating the parts of a $2k$-partition $P$ to a $k$-partition $K$ so that the cost of $K$ is at most half the $\sigma$-value of $P$.

**Lemma 3.** Let $G = (V, E)$ be a hypergraph, $k \geq 2$ be an integer, and $(Y_1, \ldots, Y_p, W, Z)$ be a partition of $V$ for some integer $p \geq 2k - 2$. Then, there exist distinct $i_1, \ldots, i_{k-1} \in [p]$ such that $2 \cdot \min\{d(A_i) + d(A_j) : i, j \in [p], i \neq j\}$.
Proof: By applying Lemma 2, we obtain a \((p + 2)\)-partition \((Y_1, \ldots, Y_p, W, Z)\) such that
\[
\sigma(Y_1, \ldots, Y_p, W, Z) \leq \min\{d(A_i) + d(A_j) : i, j \in [p], i \neq j\}
\]
and moreover, \(\mathcal{U} \subseteq W\). We recall that \(p \geq 2k - 2\). Hence, by applying Lemma 3 to the \((p + 2)\)-partition \((Y_1, \ldots, Y_p, W, Z)\), we obtain a \(k\)-partition \((P_1, \ldots, P_k)\) of \(V\) such that \(W \cup Z \subseteq P_k\) and
\[
cost(P_1, \ldots, P_k) \leq \frac{1}{2}\sigma(Y_1, \ldots, Y_p, W, Z)
\]
\[
\leq \frac{1}{2}\min\{d(A_i) + d(A_j) : i, j \in [p], i \neq j\}.
\]
We note that \(\mathcal{U}\) is strictly contained in \(P_k\) since \(\mathcal{U} \cup Z \subseteq W \cup Z \subseteq P_k\) and \(Z\) is non-empty.

IV. PROOF OF THEOREM 2

In this section, we prove Theorem 2. We start with a useful containment property captured by the next lemma.

Lemma 4. Let \(G = (V, E)\) be a hypergraph, \((V_1, \ldots, V_k)\) be a maximal minimum \(k\)-partition in \(G\) for an integer \(k \geq 2\), and \(S \subseteq V_1\), \(T \subseteq \overline{V_1}\) such that \(T \cap V_j \neq \emptyset\) for every \(j \in \{2, \ldots, k\}\). Suppose \((U, \mathcal{U})\) is a minimum \((S, T)\)-terminal cut. Then, \(U \subseteq V_1\).

Proof: For the sake of contradiction, suppose \(U \setminus V_1 \neq 0\). We will obtain another minimum \(k\)-partition that will contradict the maximality of \(V_1\) in the minimum \(k\)-partition \((V_1, \ldots, V_k)\). We observe that
\[
d(U) \leq d(U \cap V_1)
\]
(2)
since \((U \cap V_1, U \cap V_1)\) is a \((S, T)\)-terminal cut. We need the following claim:

Claim 3.
\[
d(V_1) \leq d(U \cup V_1).
\]

Proof: For the sake of contradiction, suppose \(d(U \cup V_1) < d(V_1)\). Then, consider \(W_1 := U \cup V_1\) and \(W_j := V_j \setminus U\) for every \(j \in \{2, \ldots, k\}\) (see Figure 5). We have \(d(W_1) < d(V_1)\). Since \(S \subseteq W_1\) and \(T \cap W_j \neq \emptyset\) for every \(j \in \{2, \ldots, k\}\), we have that \((W_1, \ldots, W_k)\) is a \(k\)-partition. We will show that cost\((W_1, \ldots, W_k)\) is strictly smaller than cost\((V_1, \ldots, V_k)\), thus contradicting the optimality of the \(k\)-partition \((V_1, \ldots, V_k)\).

We recall that for a subset \(A\) of vertices, the graph \(G[A]\) is obtained from \(G\) by discarding the vertices in \(A\) and by discarding the hyperedges that intersect \(A\). With this notation, we can write
\[
cost_G(W_1, \ldots, W_k) = d(W_1) + cost_G(W_2, \ldots, W_k)\) and
\[
cost_G(V_1, \ldots, V_k) = d(V_1) + cost_G(V_2, \ldots, V_k).
\]
Moreover, every hyperedge that is disjoint from \(W_1 = U \cup V_1\) but crosses the \((k - 1)\)-partition \((W_2 = V_2 \setminus U, \ldots, W_k = V_k \setminus U)\) is also disjoint from \(V_1\) but crosses the \((k - 1)\)-partition \((V_2, \ldots, V_k)\). Hence, \(cost_{G[W_1]}(W_2, \ldots, W_k) \leq cost_{G[V_1]}(V_2, \ldots, V_k)\). We also have \(d(W_1) < d(V_1)\). Therefore,
\[
cost(W_1, \ldots, W_k) < cost(V_1, \ldots, V_k),
\]
a contradiction to optimality of the \(k\)-partition \((V_1, \ldots, V_k)\).

By inequality (2), Claim 3, and submodularity of the hypergraph cut function, we have that
\[
d(U) + d(V_1) \leq d(U \cup V_1) + d(U \cup V_1) \leq d(U) + d(V_1).
\]

Therefore, the inequality in Claim 3 should in fact be an equation, i.e.,
\[
d(V_1) = d(U \cup V_1).
\]

Going through the proof of Claim 3 with this additional fact, we obtain that the \(k\)-partition \((U \cup V_1, V_2 \cup U, \ldots, V_k \cup U)\) has cost at most that of \((V_1, \ldots, V_k)\). Hence, the \(k\)-partition \((U \cup V_1, V_2 \cup U, \ldots, V_k \cup U)\) is also a minimum \(k\)-partition and it contradicts the maximality of \(V_1\).

We now restate and prove Theorem 2.

Theorem 2. Let \(G = (V, E)\) be a hypergraph and let \((V_1, \ldots, V_k)\) be a maximal minimum \(k\)-partition in \(G\) for an integer \(k \geq 2\). Suppose \(|V_1| \geq 2k - 2\). Then, for every subset \(T \subseteq \overline{V_1}\) such that \(T\) intersects \(V_j\) for every \(j \in \{2, \ldots, k\}\), there exists a subset \(S \subseteq V_1\) of size \(2k - 2\) such that \((V_1, \overline{V_1})\) is the source maximal minimum \((S, T)\)-terminal cut.

Proof: For the sake of contradiction, suppose that the theorem is false for some subset \(T \subseteq \overline{V_1}\) such that \(T \cap V_j \neq \emptyset\) for all \(j \in \{2, \ldots, k\}\). Our proof strategy is to obtain a cheaper \(k\)-partition than \((V_1, \ldots, V_k)\), thereby contradicting the optimality of \((V_1, \ldots, V_k)\). For a subset \(X \subseteq V_1\), let \((V_X, \overline{V_X})\) be the source maximal minimum \((X, T)\)-terminal cut.

Among all possible subsets of \(V_1\) of size \(2k - 2\), pick a subset \(S\) such that \(d(V_S)\) is maximum. By Lemma 4 and assumption, we have that \(V_S \subseteq V_1\). By source maximality of the minimum \((S, T)\)-terminal cut \((V_S, \overline{V_S})\), we have that
\[d(V_S) < d(V_1).\] Let \(u_1, \ldots, u_{2k-2}\) be the vertices in \(S\).
Since \(V_S \subseteq V_1\), there exists a vertex \(u_{2k-1} \in V_1 \setminus V_S\). Let
\(C := \{u_1, \ldots, u_{2k-1}\} = S \cup \{u_{2k-1}\}.\) For \(i \in [2k-1]\), let
\((B_i, B_i^c)\) be the source maximal minimum \((C - \{u_i\}, T)\)-terminal cut. We note that
\((B_{2k-1}, B_{2k-1}^c) = (V_S, V_1)\) and the size of \(C - \{u_i\}\) is \(2k-2\) for some \(i \in [2k-1]\).
By Lemma 4 and assumption, we have that \(B_i \subseteq V_1\) for every \(i \in [2k-1]\). Hence, we have
\[d(B_1) \leq d(V_S) < d(V_1)\] and \(B_i \subseteq V_1\) for every \(i \in [2k-1]\).

The next claim will set us up to apply Theorem 4.

**Claim 4.** For every \(i \in [2k-1]\), we have that \(u_i \in T_i^c\).

**Proof:** The claim holds for \(i = 2k-1\) by choice of \(u_{2k-1}\). For the sake of contradiction, suppose \(u_i \in B_i\) for some \(i \in [2k-2]\). Then, the 2-partition \((V_S \cap B_i, V_S \cap B_i^c)\) is a \((S, T)\)-terminal cut and hence
\[d(V_S \cap B_i) \geq d(V_S).\]
We also have that
\[d(V_S \cup B_i) \geq d(V_S)\]
since \((V_S \cup B_i, V_S \cup B_i^c)\) is a \((S, T)\)-terminal cut. Thus,
\[2d(V_S) \geq d(V_S) + d(B_i) \quad \text{(By choice of } S)\]
\[\geq d(V_S \cup B_i) + d(V_S \cap B_i) \quad \text{(By submodularity)}\]
\[\geq 2d(V_S).\]
Therefore, \(d(V_S) = d(V_S \cup B_i)\). Moreover, \(B_i \setminus V_S\) is non-empty since the vertex \(u_{2k-1} \in B_i \setminus V_S\). Hence, the 2-partition \((V_S \cup B_i, V_S \cup B_i^c)\) is a minimum \((S, T)\)-terminal cut. However, this contradicts source maximality of the minimum \((S, T)\)-terminal cut \((V_S, V_S)\) since \(u_{2k-1} \in B_i\) and \(u_{2k-1} \notin V_S\).

We note that for every \(i \in [2k-1]\), the 2-partition \((B_i, B_i^c)\) is a minimum \((C - \{u_i\}, \overline{V_i})\)-terminal cut since \(V_i \subseteq B_i^c\).

We will now apply Theorem 4. We consider \(U := V_1\), \(R := \{u_{2k-1}\} \subseteq U\), \(S = \{u_1, \ldots, u_{2k-2}\} \subseteq U \setminus R\). Let \(p := 2k - 2\) and let \((\overline{A_i}, A_i) := (B_i, B_i^c)\) for every \(i \in [p]\).
The 2-partition \((\overline{A_i}, A_i)\) is a minimum \(((S \cup R) \setminus \{u_i\}, U)\)-terminal cut for every \(i \in [p]\). By Claim 4, we have that \(u_i \in A_i\) for every \(i \in [p]\). Since \((B_j, B_j^c)\) is a \((C - \{u_j\}, T)\)-terminal cut, we have that \(u_i \notin B_j\) for every distinct \(i, j \in [p]\). Thus, \(u_i \in A_i \setminus (\cup_{j \in [p]} \{a_j\})\) for every \(i \in [p]\).
Therefore, the sets \(U, R, S\) and the 2-partitions \((\overline{A_i}, A_i)\) for \(i \in [p]\) satisfy the conditions of Theorem 4. By Theorem 4, symmetry of the cut function, and statement (3), we obtain a \(k\)-partition \((P_1, \ldots, P_k)\) of \(V\) such that
\[\text{cost}(P_1, \ldots, P_k) \leq \frac{1}{2} \min\{d(A_i) + d(A_j) : i, j \in [p], i \neq j\}\]
\[= \frac{1}{2} \min\{d(B_i) + d(B_j) : i, j \in [p], i \neq j\}\]
\[< d(V_1) \leq OPT_k.\]

Thus, we have obtained a \(k\)-partition whose cost is smaller than \(OPT_k\), a contradiction.

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**REFERENCES**


