Plan for today

• Simplex Method to solve LPs
  • Complete Algorithm
  • Initialization
  • Iterations
  • Termination
  • Other Issues

Announcements:
- HW 3 posted in Gradescope
- Instructor office hours: Zoom
- TA office hours: In-person
SIMPLEX METHOD
How to make selection easier: proper format

Initialization: basic variables \((x_3, x_4, x_5)\), non-basic variables \(x_1 = 0, x_2 = 0\)

Write objective function in terms of only non-basic variables,

\[ Z = 3x_1 + 5x_2 \]

… helps us select the entering variable by a simple comparison of coefficients

Rewrite the equality constraints: Each equality constraint should contain

- one basic variable on the LHS with coefficient 1,
- constants and non-basic variables on the RHS and
- the RHS constant should be non-negative

\[
\begin{align*}
x_3 &= 4 - x_1 \\
x_4 &= 6 - x_2 \\
x_5 &= 18 - 3x_1 - 2x_2 \\
\end{align*}
\]

… helps us select the leaving variable by a simple comparison of the ratios between the RHS constant and the entering variable’s coefficient in each equation

This proper format of objective and equality constraints should be maintained in each step to facilitate the selection of entering and leaving variables
After determining which variables to swap...

- $x_2$ is the entering variable, $x_4$ is the leaving variable

- Step 1: express the entering basic variable as a function of non-basic variables and the leaving variable (use the equality constraint for the leaving basic variable to do this)
  \[ x_2 = 6 - x_4 \]

- Step 2: use the expression to substitute the entering basic variable in the objective function
  \[ Z = 3x_1 + 5x_2 = 30 + 3x_1 - 5x_4 \]

- Step 3: substitute the entering basic variable in the remaining constraints
  \[ x_3 = 4 - x_1 \]
  \[ x_5 = 18 - 3x_1 - 2x_2 = 6 - 3x_1 + 2x_4 \]

- Now $(x_2, x_3, x_5)$ are basic variables and $(x_1, x_4)$ are non-basic variables
  - The objective function contains only non-basic variables
  - Each equality constraint contains:
    - one basic variable on the LHS with coefficient 1
    - constants and non-basic variables on the RHS and
    - the RHS constant is non-negative

Non-basic variables: $x_1, x_2$
Basic variables:
- $x_3 = 4, x_4 = 6, x_5 = 18$

Example
Corresponds to the solution:
- $x_1 = 0, x_2 = 6, x_3 = 4, x_4 = 0, x_5 = 6, Z = 30$
- which is the corner point $x_1 = 0, x_2 = 6$
Next iteration…

- which variable should be the entering variable, which one should be the leaving variable?
  \( x_1 \) enters
  \( x_5 \) leaves

- expression for the entering basic variable
  \[
x_1 = 2 + \frac{2}{3} x_4 - \frac{1}{3} x_5
\]

- Substitute in the objective function
  \[
  Z = 30 + 3x_1 - 5x_4 = 36 - 3x_4 - x_5
\]

- Substitute in all equality constraints
  \[
x_1 = 2 + \frac{2}{3} x_4 - \frac{1}{3} x_5
  
x_3 = 4 - x_1 = 2 - \frac{2}{3} x_4 + \frac{1}{3} x_5
  
x_2 = 6 - x_4
\]

- Next iteration? STOP since no entering variable to improve obj value

- So the optimal solution is
  \[
x_1^* = 2, x_2^* = 6, x_3^* = 2
  
  Z = 36
\]
**Simplex Method: the complete algorithm**

- **Initialization**
  - transform the original LP into the augmented LP, determine basic and non-basic variables
  - Rewrite constraints in proper format:
    - one basic variable on the LHS with coefficient 1,
    - constants and non-basic variables on the RHS and
    - RHS constant should be non-negative
  - Rewrite objective function in proper format: contains only non-basic variables

- **Iteration**
  1. select the entering variable: the variable with the largest positive coefficient in the obj. function
  2. select the leaving variable *(min-ratio test)*: the first basic variable that reaches 0 as the value of the entering variable increases
  3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
  4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

- **Termination: Optimality Test**
  - if every variable in the objective function is with a negative coefficient, then no entering variable can be found ⇒ **the optimal solution has been found**
  - if the entering variable can be increased to infinity without driving any other basic variable to below zero, then no leaving variable can be found ⇒ **the problem is unbounded**
SIMPLEX METHOD

initialization

To begin simplex, which variables should be set to basic variables?
Simplex Method: initialization (easy case, example)

original LP

\[
\text{max } Z = 7x_1 + 6x_2 \\
9x_1 - 2x_2 \leq 12 \\
3x_1 + x_2 \leq 30 \\
x_1, x_2 \geq 0
\]

augmented LP

\[
\text{max } Z = 7x_1 + 6x_2 \\
9x_1 - 2x_2 + x_3 = 12 \\
3x_1 + x_2 + x_4 = 30 \\
x_1, x_2, x_3, x_4 \geq 0
\]

If every equality has a non-negative RHS

- all slack variables are initially basic variables
- all other (original) variables are non-basic variables and are moved to the right

Initialization: \((x_3, x_4 \text{ basic variables, } x_1, x_2 \text{ non-basic variables})\)

\[
Z = 7x_1 + 6x_2 \\
x_3 = 12 - 9x_1 + 2x_2 \\
x_4 = 30 - 3x_1 - x_2
\]

initial solution:
Non-basic variables: \(x_1 = 0, x_2 = 0\),
Basic variables: \(x_3 = 12, x_4 = 30\)
**Simplex Method: initialization**

**complicated case example: step 1**

**original LP**

\[
\begin{align*}
\text{max } Z &= 7x_1 + 6x_2 \\
9x_1 - 2x_2 &\leq -12 \\
3x_1 + x_2 &\leq 30 \\
x_1, x_2 &\geq 0
\end{align*}
\]

**augmented LP**

\[
\begin{align*}
\text{max } Z &= 7x_1 + 6x_2 \\
9x_1 - 2x_2 + x_3 &= -12 \\
3x_1 + x_2 + x_4 &= 30 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

If the RHS of an equation is **negative**

**Constraints in proper format:**
- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

**Obj function in proper format:**
- contains only non-basic variables

**Not proper format!**

Cannot initialize slack variables as basic variables and original variables as non-basic variables!

**step (i): multiply both sides of such an equation in the augmented LP by -1 to make the RHS positive**

**augmented LP**

\[
\begin{align*}
\text{max } Z &= 7x_1 + 6x_2 \\
-9x_1 + 2x_2 - x_3 &= 12 \\
3x_1 + x_2 + x_4 &= 30 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

(P1)
Simplex Method: initialization

complicated case example: step 2

step (ii): add another slack variable (say $x_5$) to the same equation

\[
\begin{align*}
\text{max } Z &= 7x_1 + 6x_2 \\
-9x_1 + 2x_2 - x_3 &= 12 \\
3x_1 + x_2 + x_4 &= 30 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

\[\text{(P1)}\]

\[
\begin{align*}
\text{max } Z &= 7x_1 + 6x_2 \\
-9x_1 + 2x_2 - x_3 + x_5 &= 12 \\
3x_1 + x_2 + x_4 &= 30 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{align*}
\]

\[\text{(P2)}\]

• but (P2) is a different problem from (P1), since it has a larger feasibility region
  • e.g., $x_1 = 1, x_2 = 6$ is feasible for (P2), in which case $x_3 = 0, x_4 = 21, x_5 = 9$, but not feasible for (P1)

• however, any solution of (P2) given by $(x_1, x_2, x_3, x_4, x_5)$ is feasible for (P1) if $x_5 = 0$
Simplex Method: initialization
complicated case example: step 3

**Constraints in proper format:**
- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

**Obj function in proper format:**
- contains only non-basic variables

---

**Step (iii): change the objective function to force \( x_5 = 0 \) in the optimal solution**

\[
\begin{align*}
\text{(P1)} & : & \max Z &= 7x_1 + 6x_2 \\
& & -9x_1 + 2x_2 - x_3 &= 12 \\
& & 3x_1 + x_2 + x_4 &= 30 \\
& & x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{align*}
\]

\[
\begin{align*}
\text{(P2)} & : & \max Z &= 7x_1 + 6x_2 \\
& & -9x_1 + 2x_2 - x_3 + x_5 &= 12 \\
& & 3x_1 + x_2 + x_4 &= 30 \\
& & x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{align*}
\]

\[
\begin{align*}
\text{(P3)} & : & \max Z &= 7x_1 + 6x_2 - Mx_5 \\
& & -9x_1 + 2x_2 - x_3 + x_5 &= 12 \\
& & 3x_1 + x_2 + x_4 &= 30 \\
& & x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{align*}
\]

- Choose a very large \( M \), so that an optimal solution for (P3) can never have \( x_5 > 0 \)

- Finding an optimal solution for (P1) is the same as finding an optimal solution for (P3) if we use a very large \( M \)

… The Big-\( M \) method
Initialization of (P3) is easy: we can start with $x_4, x_5$ as basic variables, and $x_1, x_2, x_3$ as non-basic variables.

$$x_5 = 12 + 9x_1 - 2x_2 + x_3$$
$$x_4 = 30 - 3x_1 - x_2$$

Are we done with the initialization? Not yet, need to write obj in proper format.

**step (iv): substitute out the basic variable in the objective function**

$$Z = 7x_1 + 6x_2 - Mx_5$$
$$= 7x_1 + 6x_2 - M(12 + 9x_1 - 2x_2 + x_3)$$
$$= (7 - 9M)x_1 + (6 + 2M)x_2 - Mx_3 - 12M$$

initial solution:
Non-basic variables: $x_1 = 0, x_2 = 0, x_3 = 0$
Basic variables: $x_4 = 30, x_5 = 12$

Now, we are done with the initialization!
SIMPLEX METHOD

iterations

\[ Z = (7 - 9M)x_1 + (6 + 2M)x_2 - Mx_3 - 12M \]

\[ x_5 = 12 + 9x_1 - 2x_2 + x_3 \]
\[ x_4 = 30 - 3x_1 - x_2 \]

initial solution:
Non-basic variables: \( x_1 = 0, x_2 = 0, x_3 = 0 \)
Basic variables: \( x_4 = 30, x_5 = 12 \)
**Simplex Method: Iteration steps 1 & 2**

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable (min-ratio test): the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

- **select the entering variable:** $x_2$
- **select leaving variable by min ratio test:**

\[
x_5 = 12 + 9x_1 - 2x_2 + x_3
\]
\[
x_4 = 30 - 3x_1 - x_2
\]

⇒ $x_5$ is the leaving variable

\[Z = (7 - 9M)x_1 + (6 + 2M)x_2 - Mx_3 - 12M\]
\[x_5 = 12 + 9x_1 - 2x_2 + x_3
\]
\[x_4 = 30 - 3x_1 - x_2
\]

Non-basic vars: $x_1 = 0, x_2 = 0, x_3 = 0$
Basic vars: $x_4 = 30, x_5 = 12$

Min-ratio test (for entering variable $x_2$):
for all $i = 1, \ldots, m$ such that $a_{i2} < 0$, calculate $\frac{b_i}{|a_{i2}|}$
choose the equation $i$ that gives the smallest $\frac{b_i}{|a_{i2}|}$

$x_{n+i}$ is the leaving variable
Simplex Method: Iteration steps 3 & 4

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable (min-ratio test): the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

• now that $x_2$ is the entering variable and $x_5$ is the leaving variable
• express the entering variable as a function of non-basic variables
$$x_2 = 6 - \frac{1}{2} x_5 + \frac{9}{2} x_1 + \frac{1}{2} x_3$$
• substitute the entering basic variable in the objective function
$$Z = (7 - 9M)x_1 + (6 + 2M)x_2 - Mx_3 - 12M$$

Non-basic vars: $x_1 = 0, x_2 = 0, x_3 = 0$
Basic vars: $x_4 = 30, x_5 = 12$

$$x_5 = 12 + 9x_1 - 2x_2 + x_3$$
$$x_4 = 30 - 3x_1 - x_2$$

• substitute the entering basic variable in other equations
$$x_4 = 30 - 3x_1 - x_2$$
$$= 30 - 3x_1 - \left(6 - \frac{1}{2} x_5 + \frac{9}{2} x_1 + \frac{1}{2} x_3\right)$$
$$= 24 + \frac{1}{2} x_5 - \frac{15}{2} x_1 - \frac{1}{2} x_3$$
Simplex Method: after the first iteration

\[ Z = 36 - (3 + M)x_5 + 34x_1 + 3x_3 \]
\[ x_2 = 6 - \frac{1}{2}x_5 + \frac{9}{2}x_1 + \frac{1}{2}x_3 \]
\[ x_4 = 24 + \frac{1}{2}x_5 - \frac{15}{2}x_1 - \frac{1}{2}x_3 \]

- the problem is still in the same format as the one required for initialization

**Constraints in proper format:**
- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

**Obj function in proper format:**
- contains only non-basic variables

- so we can continue to the second iteration in the same manner
**Simplex Method: Iteration, steps 1 & 2**

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable *(min-ratio test)*: the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

- select the entering variable: $x_1$
- select leaving variable:

\[ x_2 \text{ vs } x_4 \]
\[ - \text{ vs } \frac{24}{15/2} \]

\[ x_4 \text{ is the leaving variable} \]

\[ Z = 36 + 34x_1 + 3x_3 - (3 + M)x_5 \]
\[ x_2 = 6 + \frac{9}{2}x_1 + \frac{1}{2}x_3 - \frac{1}{2}x_5 \]
\[ x_4 = 24 - \frac{15}{2}x_1 - \frac{1}{2}x_3 + \frac{1}{2}x_5 \]

Basic variables: $x_2 = 6, x_4 = 24$
Non-basic variables: $x_1 = x_3 = x_5 = 0$

\[ Z = 36 \]

**Min-ratio test** (for entering variable $x_1$):
for all $i = 1, \ldots, m$ such that $a_{i1} < 0$, calculate \[ \frac{b_i}{|a_{i1}|} \]
choose the equation $i$ that gives the smallest \[ \frac{b_i}{|a_{i1}|} \]

$x_{n+i}$ is the leaving variable
Simplex Method: Iteration, steps 3 & 4

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable *(min-ratio test)*: the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

- now that $x_1$ is the entering variable and $x_4$ is the leaving variable
- express the entering variable as a function of non-basic variables
  \[ x_1 = \frac{16}{5} - \frac{1}{15}x_3 - \frac{2}{15}x_4 + \frac{1}{15}x_5 \]
- substitute the entering basic variable in the objective function
  \[
  Z = 36 + 34x_1 + 3x_3 - (3 + M)x_5
  \]
  \[
  = 36 + 34\left(\frac{16}{5} + \frac{1}{15}x_5 - \frac{1}{15}x_3 - \frac{2}{15}x_4\right) + 3x_3 - (3 + M)x_5
  \]
  \[
  = \frac{724}{5} + \frac{11}{15}x_3 - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5
  \]
- substitute the entering basic variable in other equations
  \[
  x_2 = 6 + \frac{9}{2}x_1 + \frac{1}{2}x_3 - \frac{1}{2}x_5 = 6 + \frac{9}{2}\left(\frac{16}{5} + \frac{1}{15}x_5 - \frac{1}{15}x_3 - \frac{2}{15}x_4\right) + \frac{1}{2}x_3 - \frac{1}{2}x_5
  \]
  \[
  = \frac{102}{5} + \frac{1}{5}x_3 - \frac{3}{5}x_4 - \frac{1}{5}x_5
  \]

Basic variables: $x_2 = 6, x_4 = 24$
Non-basic variables: $x_1 = x_3 = x_5 = 0$
\[ Z = 36 \]
Simplex Method: after the second iteration

\[ Z = \frac{724}{5} + \frac{11}{15} x_3 - \frac{68}{15} x_4 - \left( \frac{11}{15} + M \right) x_5 \]

\[ x_2 = \frac{102}{5} + \frac{1}{5} x_3 - \frac{3}{5} x_4 - \frac{1}{5} x_5 \]

\[ x_1 = \frac{16}{5} - \frac{1}{15} x_3 - \frac{2}{15} x_4 + \frac{1}{15} x_5 \]

- the constraints and objective are still in proper format

Basic variables: \( x_2 = \frac{102}{5}, x_1 = \frac{16}{5} \)
Non-basic variables: \( x_3 = x_4 = x_5 = 0 \)
\[ Z = \frac{724}{5} \]

Constraints in proper format:
- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

Obj function in proper format:
- contains only non-basic variables

- so we can continue the third iteration in the same manner
**Simplex Method: Iteration, steps 1 & 2**

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable *(min-ratio test)*: the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

**Min-ratio test** (for entering variable $x_3$): for all $i = 1, \ldots, m$ such that $a_{i3} < 0$, calculate $\frac{b_i}{|a_{i3}|}$ and choose the equation $i$ that gives the smallest value. $x_{n+i}$ is the leaving variable.
Simplex Method: Iteration, steps 3 & 4

1. select the entering variable: the variable with the largest positive coefficient in the obj. function
2. select the leaving variable (min-ratio test): the first basic variable that reaches 0 as the value of the entering variable increases
3. express the entering variable as a function of non-basic variables using the constraint that identified the leaving variable
4. use the expression obtained at step 3 to replace the entering basic variable in the objective function and in all constraints

• now that \( x_3 \) is the entering variable and \( x_1 \) is the leaving variable
• express the entering variable as a function of non-basic variables
  \[
  x_3 = 48 - 15x_1 - 2x_4 + x_5
  \]
• substitute the entering basic variable in the objective function
  \[
  Z = \frac{724}{5} + \frac{11}{15}x_3 - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5
  \]
  \[
  = \frac{724}{5} + \frac{11}{15}(48 - 15x_1 - 2x_4 + x_5) - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5
  \]
  \[
  = 180 - 11x_1 - \frac{46}{15}x_4 - Mx_5
  \]
• substitute the entering basic variable in other equations
  \[
  x_2 = \frac{102}{5} + \frac{1}{5}x_3 - \frac{3}{5}x_4 - \frac{1}{5}x_5 = \frac{102}{5} + \frac{1}{5}(48 - 2x_4 - 15x_1 + x_5) - \frac{3}{5}x_4 - \frac{1}{5}x_5
  \]
  \[
  = 30 - 3x_1 - x_4
  \]

Iteration 3

\[
Z = \frac{724}{5} + \frac{11}{15}x_3 - \frac{68}{15}x_4 - \left(\frac{11}{15} + M\right)x_5
\]

Basic variables: \( x_2 = \frac{102}{5}, x_1 = \frac{16}{5} \)

Non-basic variables: \( x_3 = x_4 = x_5 = 0 \)

\[
Z = \frac{724}{5}
\]
Simplex Method: after the third iteration

\[ Z = 180 - 11x_1 - \frac{46}{15}x_4 - Mx_5 \]
\[ x_2 = 30 - 3x_1 - x_4 \]
\[ x_3 = 48 - 15x_1 - 2x_4 + x_5 \]

- the constraints and objective are still in proper format

**Constraints in proper format:**
- one basic variable on the LHS with coefficient 1
- constants and non-basic variables on the RHS
- RHS constant should be non-negative

**Obj function in proper format:**
- contains only non-basic variables

- so we can continue the fourth iteration in the same manner
- But no more entering variable, so terminate

Optimum solution is
\[ x_1^* = 0, x_2^* = 30, x_3^* = 48, x_4^* = 0, x_5^* = 0 \]
\[ Z^* = 180 \]
SIMPLEX METHOD

termination
Termination: Optimality Test

- no entering variable can be found
  all coefficients (except the constant) in the objective function are negative

- there is an entering variable, but no leaving variable
  - In this case, the LP is **unbounded**

original LP
\[
\begin{align*}
\text{max } Z &= x_1 + 2x_2 \\
3x_1 - 2x_2 &\leq 5 \\
x_1 - x_2 &\leq 4 \\
x_1, x_2 &\geq 0
\end{align*}
\]

augmented LP
\[
\begin{align*}
\text{max } Z &= x_1 + 2x_2 \\
3x_1 - 2x_2 + x_3 &= 5 \\
x_1 - x_2 + x_4 &= 4 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

Initialization
\[
\begin{align*}
Z &= x_1 + 2x_2 \\
x_3 &= 5 - 3x_1 + 2x_2 \\
x_4 &= 4 - x_1 + x_2
\end{align*}
\]

Basic variables: \(x_3 = 5, x_4 = 4\)
Non-basic variables: \(x_1 = x_2 = 0\)
\[Z = 0\]

Entering variable: \(x_2\)
Leaving variable: None \(\Rightarrow\) Unbounded
OTHER ISSUES

... that could happen while executing the Simplex method
Initialization

Original LP

\[
\begin{align*}
\text{max } Z &= -11x_1 + 9x_2 \\
2x_1 + 8x_2 &\leq 5 \\
-x_1 + x_2 &\leq 0 \\
x_1, x_2 &\geq 0
\end{align*}
\]

Augmented LP

\[
\begin{align*}
\text{max } Z &= -11x_1 + 9x_2 \\
2x_1 + 8x_2 + x_3 &= 5 \\
-x_1 + x_2 + x_4 &= 0 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

Initialization

\[
\begin{align*}
Z &= -11x_1 + 9x_2 \\
x_3 &= 5 - 2x_1 - 8x_2 \\
x_4 &= 0 + x_1 - x_2 \\
\text{Basic variables: } x_3 = 5, x_4 = 0 \\
\text{Non-basic variables: } x_1 = x_2 = 0 \\
Z &= 0
\end{align*}
\]
Iteration

Iteration 1:

\[
\begin{align*}
Z &= -11x_1 + 9x_2 \\
x_3 &= 5 - 2x_1 - 8x_2 \\
x_4 &= 0 + x_1 - x_2
\end{align*}
\]

Basic variables: \(x_3 = 5, x_4 = 0\)
Non-basic variables: \(x_1 = x_2 = 0\)

\(Z = 0\)

Entering variable: \(x_2\)
Leaving variable: \(x_4\)

\[
\begin{align*}
x_2 &= 0 + x_1 - x_4 \\
x_3 &= 5 - 2x_1 - 8(0 + x_1 - x_4) \\
    &= 5 - 10x_1 + 8x_4
\end{align*}
\]

\[
Z = -11x_1 + 9(0 + x_1 - x_4) = -2x_1 - 9x_4
\]

\[
\begin{align*}
Z &= -2x_1 - 9x_4 \\
x_2 &= 0 + x_1 - x_4 \\
x_3 &= 5 - 10x_1 + 8x_4
\end{align*}
\]

Basic variables: \(x_2 = 0, x_3 = 5\)
Non-basic variables: \(x_1 = x_4 = 0\)

\(Z = 0\)

No more entering variable, so terminate

Optimum solution is
\[
\begin{align*}
x_1^* &= 0, x_2^* = 0, x_3^* = 5, x_4^* = 0 \\
Z^* &= 0
\end{align*}
\]
Degeneracy

- If the RHS of the leaving variable is zero, then
  - The entering variable will also take the same RHS value of zero in the new basis
  - The objective value will not increase in the new basis
- We call this situation degeneracy
- In simple terms, degeneracy happens when one or more basic variables take a value of 0
- Note that objective stays the same even though we have an entering variable and a leaving variable

Degeneracy is bad
It can cause the simplex method to cycle around the same set of basic feasible solutions without improving the objective value and without terminating
Bland’s rule for entering and leaving variables

• Entering variable:
  • Among the choices (non-basic variables with positive coefficient in the objective function $Z$ expressed in proper format)
    • pick the one with the smallest index
      i.e., $x_j$ with smallest index $j$

• Leaving variable:
  • Among the choices (basic variables whose equation has a negative coefficient for the entering variable)
    • Pick the one with with the least ratio between RHS constant and absolute value of the coefficient of the entering variable
    • In case of tie, pick the one with the smallest index, i.e., $x_j$ with smallest index $j$

With this choice of entering and leaving variable, it can be provably shown that the Simplex method does not cycle and hence, will terminate in finite number of iterations.
OTHER ISSUES

... that could happen while executing the Simplex method
Iteration step maintains positive RHS

after each iteration, the constants on the RHS must be positive

Let us consider an example:

Situation:

\[
\begin{align*}
    x_3 &= 1 - 6x_1 - 4x_2 + 8x_5 - 7x_6 \\
    x_4 &= 12 - 9x_1 - 3x_2 + 8x_6 \\
    x_7 &= 100 - 3x_2 - 2x_5 \\
    x_8 &= 19 - 7x_1 - 6x_5 + 8x_6
\end{align*}
\]

\(x_3, x_4, x_7, x_8\) are basic variables
\(x_1, x_2, x_5, x_6\) are non-basic variables
Suppose \(x_5\) is the entering variable

- use the min ratio test to select the leaving variable

<table>
<thead>
<tr>
<th>RHS</th>
<th>Coefficient of (x_5)</th>
<th>(\text{RHS coefficient})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>-</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>100</td>
<td>-2</td>
<td>(\frac{100}{2} = 50)</td>
</tr>
<tr>
<td>19</td>
<td>-6</td>
<td>(\frac{19}{6})</td>
</tr>
</tbody>
</table>
Iteration step maintains positive RHS

\[ x_3 = 1 - 6x_1 - 4x_2 + 8x_5 - 7x_6 \]
\[ x_4 = 12 - 9x_1 - 3x_2 + 8x_6 \]
\[ x_7 = 100 - 3x_2 - 2x_5 \]
\[ x_8 = 19 - 7x_1 - 6x_5 + 8x_6 \]

Situation: \( x_3, x_4, x_7, x_8 \) are basic variables
\( x_1, x_2, x_5, x_6 \) are non-basic variables
Suppose \( x_5 \) is the entering variable

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<td>–6</td>
<td>( \frac{19}{6} )</td>
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</table>

- Entering variable: \( x_5 \), Leaving variable: \( x_8 \)
- Substitute \( x_5 \) in other equalities by using
  \[ x_5 = \frac{19}{6} - \frac{7}{6}x_1 + \frac{8}{6}x_6 - \frac{1}{6}x_8 \]
- New constant in the equalities
  \[ 1 + 8 \times \frac{19}{6} \]
  \[ 12 + 0 \]
  \[ 100 - 2 \times \frac{19}{6} \]
- \( \text{min ratio test ensures the constant in the third equality stays positive:} \)
  \[ \frac{100}{2} > \frac{19}{6}, \text{which ensures that} \ 100 - 2 \times \frac{19}{6} > 0 \]