Plan for today

- LP Solving in Excel
  - Interpreting reports

- Transportation Problem
  - LP Formulation
  - Transportation Simplex Method
    - Initialization: Northwest corner rule

**Announcements:**
- Exam 1 Topics: Until the end of Game Theory
- Permitted:
  - Lecture notes/slides/videos, HWs, Quizzes
- **NOT permitted:**
  - collaboration, solution sources, softwares
Recap

LP SOLVING IN EXCEL
Outline

• Step 0: Install Solver

• Step 1: Input Instance and Solve
  • Step 1.1: Input Data, Declare Variables, Objective, and Constraints
  • Step 1.2: Set up solver
  • Step 1.3: Solve (and get reports)

• Step 2: Interpret reports
STEP 0: INSTALL SOLVER

... To be done only if your Excel Software does not have it already installed
Step 0: Install Solver

- Windows: https://www.youtube.com/watch?v=g7C3XXyMV4A
- MacOS: https://www.youtube.com/watch?v=g7C3XXyMV4A
STEP 1: INPUT INSTANCE AND SOLVE
Example 2: Formulation

- Tesla makes two models of cars
  - Model I: makes a profit of $3 million per batch
  - Model II: sells for $5 million per batch
- Tesla has three plants with limited working hours
  - Plant 1: Frame I
    - at most 4 working hours per week
    - 1 hour to prepare a batch of Frame I
  - Plant 2: Frame II
    - at most 12 working hours per week
    - 2 hours to prepare a batch of Frame II
  - Plant 3: Assembly
    - at most 18 working hours per week
    - 3 hours to assemble a batch of Model I (using Frame I) and
      2 hours to assemble a batch of Model II (using Frame II)

Question: What is the best product mix?

\[
\text{max } Z = 3x_1 + 5x_2 \quad \text{(profit)}
\]
\[
\begin{align*}
   x_1 & \leq 4 & \text{(hour constraint for plant 1)} \\
   2x_2 & \leq 12 & \text{(hour constraint for plant 2)} \\
   3x_1 + 2x_2 & \leq 18 & \text{(hour constraint for plant 3)} \\
   x_1 & \geq 0 & \text{(non-negative amount of commodity 1)} \\
   x_2 & \geq 0 & \text{(non-negative amount of commodity 2)}
\end{align*}
\]
### Step 1.1.1: Input data

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tesla Production Problem</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of working hours at Plant 1</td>
<td>Model I</td>
<td>Model II</td>
<td>Availability</td>
</tr>
<tr>
<td>Number of working hours at Plant 2</td>
<td>0</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>Number of working hours at Plant 3</td>
<td>3</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>Profit (in millions)</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>
### Step 1.1.2: Declare variables and objective

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tesla Production Problem</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Decision Variables</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of working hours at Plant 1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Number of working hours at Plant 2</td>
<td>0</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>Number of working hours at Plant 3</td>
<td>3</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>Profit (in millions)</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td><strong>Objective</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total profit</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Profit cell B14 is a changing cell
- It is defined as an excel function “=B7*B10+C7*B11”
- It represents $3x_1 + 5x_2$
Step 1.1.3: Declare constraints

- LHS are changing cells: they are the LHS of constraints

- Examples:
  - B17 cell is defined as an excel function “=B4*B10+C4*B11”
    - It represents $1x_1 + 0x_2$
  - B18 cell is defined as an excel function “=B5*B10+C5*B11”
    - It represents $0x_1 + 2x_2$
Step 1.2: Setup Solver

- **Step 2.1: Open Solver Dialog**
  - Data -> Solver

- **Step 2.2: Setup Solver**
  - Select the cell that represents the “Objective”
    - Choose object cell B14
  - Check “Max” to indicate maximization
  - Select the cells that represent the “Variables”
    - Choose B10:B11 (press “Shift” key to select many cells)
  - Select the cells that represent the “Constraints”
    - Constraints can only involve two adjacent columns
    - Choose B17-B19 and C17-C19
  - Check “Make Unconstrained Variables Non-Negative” (if needed)
  - Select “Simplex LP” to tell the Solver that this is an LP
  - Click ”Solve”
Step 1.2: Setup Solver

Solver Parameters

Set Objective: SBS14

To: Max

Value Of: 0

By Changing Variable Cells:
SBS10:SBS11

Subject to the Constraints:
SBS17:SBS19 <= SCS17:SCS19

Make Unconstrained Variables Non-Negative

Select a Solving Method:
Simplex LP

Solving Method
Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.
Step 1.2: Setup Solver

• Step 2.1: Open Solver Dialog
  • Data -> Solver

• Step 2.2: Setup Solver
  • Select the cell that represents the “Objective”
    • Choose objective cell B14
  • Check “Max” to indicate maximization
  • Select the cells that represent the “Variables”
    • Choose B10:B11 (press “Shift” key to select many cells)
  • Select the cells that represent the “Constraints”
    • Constraints can only involve two adjacent columns
    • Choose B17-B19 and C17-C19
  • Check “Make Unconstrained Variables Non-Negative” (if needed)
  • Select “Simplex LP” to tell the Solver that this is an LP
  • Click ”Solve”
The constraint

\[ \text{B17:B19} \leq \text{C17:C19} \]

is the same as the system of constraints:

\[ \text{B17} \leq \text{C17} \]
\[ \text{B18} \leq \text{C18} \]
\[ \text{B19} \leq \text{C19} \]

But much more convenient to use!

**Tips for declaring constraints**
## Step 1.3: Solve (and get reports)

After clicking solve, see two changes:

- **Change 1**: values in “Decision Variables”, “Objective”, and LHS cells will change
- Optimum decision variable values: (2,6)
- Optimum objective value: 36

### Tesla Production Problem

<table>
<thead>
<tr>
<th></th>
<th>Model I</th>
<th>Model II</th>
<th>Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of working hours at Plant 1</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Number of working hours at Plant 2</td>
<td>0</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>Number of working hours at Plant 3</td>
<td>3</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>Profit (in millions)</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

### Decision Variables

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of batches of model I, x_1</td>
<td>2</td>
</tr>
<tr>
<td>Number of batches of model II, x_2</td>
<td>6</td>
</tr>
</tbody>
</table>

### Objective

- Total profit: 36

### Constraints

<table>
<thead>
<tr>
<th></th>
<th>LHS</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of working hours at Plant 1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Number of working hours at Plant 2</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>Number of working hours at Plant 3</td>
<td>18</td>
<td>18</td>
</tr>
</tbody>
</table>
Step 1.3: Solve (and get reports)

- **Change 2:** solver window becomes

  ![Solver Results Window]

  - There are 3 possible reports. Each one will be a separate tab in the excel file.
STEP 2: INTERPRET REPORTS

1. Answer Report
2. Sensitivity Report
3. Limits Report: No useful information
1. ANSWER REPORT
Answer Report

- Gives optimum (and initial) values of objective function
- Gives optimum (and initial) values of variables
- For each constraint:
  - Gives the amount of 'slack' between LHS and RHS at optimum
  - Whether the constraint is satisfied as an equation (i.e., binding) at optimum
2. SENSITIVITY REPORT
<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Reduced Cost</th>
<th>Objective Coefficient</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$10</td>
<td>Number of batches of model I, x_1 Model I</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>4.5</td>
<td>3</td>
</tr>
<tr>
<td>$B$11</td>
<td>Number of batches of model II, x_2 Model I</td>
<td>6</td>
<td>0</td>
<td>5</td>
<td>1E+30</td>
<td>3</td>
</tr>
</tbody>
</table>

**Constraints**

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Shadow Price</th>
<th>Constraint R.H. Side</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$17</td>
<td>Number of working hours at Plant 1 LHS</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>1E+30</td>
<td>2</td>
</tr>
<tr>
<td>$B$18</td>
<td>Number of working hours at Plant 2 LHS</td>
<td>12</td>
<td>1.5</td>
<td>12</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$B$19</td>
<td>Number of working hours at Plant 3 LHS</td>
<td>18</td>
<td>1</td>
<td>18</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>
Part 1: Constraints

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Shadow Price</th>
<th>Constraint R.H. Side</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$17</td>
<td>Number of working hours at Plant 1 LHS</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>1E+30</td>
<td>2</td>
</tr>
<tr>
<td>$B$18</td>
<td>Number of working hours at Plant 2 LHS</td>
<td>12</td>
<td>1.5</td>
<td>12</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$B$19</td>
<td>Number of working hours at Plant 3 LHS</td>
<td>18</td>
<td>1</td>
<td>18</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

- **Constraint R.H. Side**: refers to the constant in the RHS of each constraint
- **Final Value**: refers to the total value of LHS at optimum
- **Shadow price**: rate of increase of optimal objective when RHS changes

The shadow price is valid only for its **allowable increase** and **allowable decrease**

- The **allowable increase (decrease)** on a constraint RHS is the maximum amount the RHS can increase (decrease) without the shadow price changing
- If RHS changes to a value in [RHS - allowable decrease, RHS + allowable decrease], then the change in optimum objective value can be directly computed based on shadow price

To see the impact of changing more than one constraint or going beyond the allowable range, we need to re-run the problem
Exercise

Q1: The range of the RHS $b_1$ for which the shadow prices are optimal is

Q2: Change $b_1$ from 4 to 2; what are the new shadow prices?

Q3: Change $b_1$ from 4 to 1000; what are the new shadow prices?

Q4: Change $b_1$ from 4 to 1.5; what are the new shadow prices?
Part 2: Variables

• **Final value**: the value of each variable at optimum
• **Reduced cost**: will discuss later
• **Objective coefficient** (from problem): coefficient of the objective function of each variable

• **Allowable increase**:  
  How much can the objective coefficient increase before the optimal solution changes  
  (in this example, $c_1$ can go up to $3 + 4.5 = 7.5$)

• **Allowable decrease**:  
  How much can the objective coefficient decrease before the optimal solution changes  
  (in this example: $c_1$ can go down to $3 - 3 = 0$)

Remark: Allowable increase/decrease data are correct only for changes made to coefficient of one variable at a time. If you change more than one variable, need to re-solve the LP.
Exercise

Q1. If the profit per batch of Model I changes from $3 to $6 (i.e., $c_1$ changes from 3 to 6), will the optimal solution change?

Q2. If the profit per batch of Model II changes from $3 to $8 (i.e., $c_1$ changes from 3 to 8), will the optimal solution change?

Q3. If the profit per batch of Model II changes to $1000 (i.e., $c_2$ changes from 5 to 1000), will the optimal solution change?

Exercise: Change the profits for (1) and (3) above, and run Excel to verify.
“Reduced cost”: slack at dual constraints

\[ \text{max } Z = c^T x \]
\[ Ax \leq b \]
\[ x \geq 0 \]

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Reduced Cost</th>
<th>Objective Coefficient</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B10$</td>
<td>Number of batches of model I, x_1 Model I</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>4.5</td>
<td>3</td>
</tr>
<tr>
<td>$B11$</td>
<td>Number of batches of model II, x_2 Model I</td>
<td>6</td>
<td>0</td>
<td>5</td>
<td>1E+30</td>
<td>3</td>
</tr>
</tbody>
</table>

Reduced cost: \( c^T - y^T A \) (negation of slack variable values of the dual LP at optimum)
--appear in Z-row of simplex tableau
--coefficients of original variables in the objective equation

The column in red are two 0’s because of
A. Strong duality
B. Weak duality
C. Complementary slackness Conditions
D. Symmetry of LP
TRANSPORTATION PROBLEM

... where we see an algorithm to solve the transportation problem
**Example**

<table>
<thead>
<tr>
<th>Supply</th>
<th>Supply points</th>
<th>Demand points</th>
<th>Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>75</td>
<td>$S_1$</td>
<td>$D_1$</td>
<td>80</td>
</tr>
<tr>
<td>125</td>
<td>$S_2$</td>
<td>$D_2$</td>
<td>65</td>
</tr>
<tr>
<td>100</td>
<td>$S_3$</td>
<td>$D_3$</td>
<td>70</td>
</tr>
<tr>
<td></td>
<td>$S_3$</td>
<td>$D_4$</td>
<td>85</td>
</tr>
</tbody>
</table>

- Single product, say oil, needs to be transported from supply to demand
- Supply constraints: Entire supply at every $S_i$ must be distributed to the demand points
- Demand constraints: Entire demand at every $D_j$ must be received from the supply points
- Transportation Cost per unit from $S_i$ to $D_j$ is $c_{ij} \geq 0$ for all $i \in \{1, 2, 3\}, j \in \{1, 2, 3, 4\}$

**Question:** How much to transport from each supply point to each demand point to minimize cost?
Formulation of the Example

Step 1: identify decision variables
\[ x_{ij} \]: quantity to be transported from \( S_i \) to \( D_j \), \( i \in \{1,2,3\}, j \in \{1,2,3,4\} \)

Step 2: determine the objective function
\[
\min Z = 464 x_{11} + 513 x_{12} + 654 x_{13} + 867 x_{14} + 352 x_{21} + 416 x_{22} + 690 x_{23} + 791 x_{24} + 995 x_{31} + 682 x_{32} + 388 x_{33} + 685 x_{34}
\]

Step 3: identify constraints
\[
\begin{align*}
    x_{11} + x_{12} + x_{13} + x_{14} &= 75 \\
    x_{21} + x_{22} + x_{23} + x_{24} &= 125 \\
    x_{31} + x_{32} + x_{33} + x_{34} &= 100 \\
    x_{11} + x_{21} + x_{31} &= 80 \\
    x_{12} + x_{22} + x_{32} &= 65 \\
    x_{13} + x_{23} + x_{33} &= 70 \\
    x_{14} + x_{24} + x_{34} &= 85 \\
    x_{ij} &\geq 0, i = 1,2,3, j = 1,2,3,4
\end{align*}
\]

Supply constraints: Total outgoing from a supply point to all demand points should be equal to the supply at that point.

Demand constraints: Total incoming into a demand point from all supply points should be equal to the demand requirement at that point.
Formulation of the Example

\[
\begin{align*}
\min Z &= 464 x_{11} + 513 x_{12} + 654 x_{13} + 867 x_{14} \\
&+ 352 x_{21} + 416 x_{22} + 690 x_{23} + 791 x_{24} \\
&+ 995 x_{31} + 682 x_{32} + 388 x_{33} + 685 x_{34} \\
\end{align*}
\]

\[
\begin{align*}
x_{11} + x_{12} + x_{13} + x_{14} &= 75 \\
x_{21} + x_{22} + x_{23} + x_{24} &= 125 \\
x_{31} + x_{32} + x_{33} + x_{34} &= 100 \\
x_{11} + x_{21} + x_{31} &= 80 \\
x_{12} + x_{22} + x_{32} &= 65 \\
x_{13} + x_{23} + x_{33} &= 70 \\
x_{14} + x_{24} + x_{34} &= 85 \\
x_{ij} &\geq 0, i = 1,2,3, j = 1,2,3,4
\end{align*}
\]
Transportation Problem: General Case

Supply points

Demand points

- Single product to be transported from supply to demand
- Transportation Cost per unit from \( S_i \) to \( D_j \) is \( c_{ij} (\geq 0) \) for all \( i \in \{1, ..., m\}, j \in \{1, ..., n\} \)
- Supply constraints: Entire supply at every \( S_i \) must be distributed to the demand points
- Demand constraints: Entire demand at every \( D_j \) must be received from the supply points

\[
\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j
\]

Question: How much to transport from each supply point to each demand point to minimize cost?
Formulation of the Example

Step 1: identify decision variables
\( x_{ij} \): quantity to be transported from \( S_i \) to \( D_j \), \( i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\} \)

Step 2: determine the objective function
\[
\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}
\]

Step 3: identify constraints
\[
\sum_{j=1}^{n} x_{ij} = s_i \text{ for every } i \in \{1, \ldots, m\}
\]
\[
\sum_{i=1}^{m} x_{ij} = d_j \text{ for every } j \in \{1, \ldots, n\}
\]
\( x_{ij} \geq 0 \text{ for every } i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\} \)

Supply constraints: Total outgoing from a supply point to all demand points should be equal to the supply at that point

Demand constraints: Total incoming into a demand point from all supply points should be equal to the demand requirement at that point
Formulation of the general case transportation problem

\[
\begin{align*}
\text{min } Z &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} \\
\sum_{j=1}^{n} x_{ij} &= s_i \text{ for every } i \in \{1, \ldots, m\} \\
\sum_{i=1}^{m} x_{ij} &= d_j \text{ for every } j \in \{1, \ldots, n\} \\
x_{ij} \geq 0 \text{ for every } i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}
\end{align*}
\]
ALGORITHM FOR THE TRANSPORTATION PROBLEM

- Can use Simplex method, but it is typically slow
- So we have a special-purpose algorithm
  … known as the “Transportation Simplex Method”
A Convenient Representation of the Transportation Problem

<table>
<thead>
<tr>
<th>Supply</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>16</td>
<td>16</td>
<td>13</td>
<td>22</td>
<td>17</td>
<td>50</td>
</tr>
<tr>
<td>$S_2$</td>
<td>14</td>
<td>14</td>
<td>13</td>
<td>19</td>
<td>15</td>
<td>60</td>
</tr>
<tr>
<td>$S_3$</td>
<td>19</td>
<td>19</td>
<td>20</td>
<td>23</td>
<td>99</td>
<td>50</td>
</tr>
<tr>
<td>$S_4$</td>
<td>99</td>
<td>0</td>
<td>99</td>
<td>0</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>Demand</td>
<td>30</td>
<td>20</td>
<td>70</td>
<td>30</td>
<td>60</td>
<td></td>
</tr>
</tbody>
</table>

Per unit shipment cost $c_{34}$
A Convenient Representation of the Transportation Problem

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>16</td>
<td>16</td>
<td>13</td>
<td>22</td>
<td>17</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>$x_{11} = 20$</td>
<td>$x_{12} = 20$</td>
<td>$x_{13} = 10$</td>
<td>$x_{14} = 0$</td>
<td>$x_{15} = 0$</td>
<td></td>
</tr>
<tr>
<td>$S_2$</td>
<td>14</td>
<td>14</td>
<td>13</td>
<td>19</td>
<td>15</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>$x_{21} = 10$</td>
<td>$x_{22} = 0$</td>
<td>$x_{23} = 50$</td>
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<td>60</td>
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</tbody>
</table>
A Convenient Representation of the Transportation Problem

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<tr>
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<td>60</td>
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</tr>
</tbody>
</table>

We will avoid writing the variables and simply write the allocations in the respective cells.

Allocation of values for $x_{ij}$
## A Convenient Representation of the Transportation Problem

We will avoid writing the variables and - simply write the allocations in the respective cells - simply write the relevant allocations

<table>
<thead>
<tr>
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<th>$D_4$</th>
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<td>20</td>
<td>70</td>
<td>30</td>
<td>60</td>
<td>$Z = ??$</td>
</tr>
</tbody>
</table>

Test your understanding: What is the objective value of this solution?
Transportation Simplex Method: Two Stage Algorithm

Initialization: Identify an initial **allocation** (i.e., an initial basic feasible solution)

... Via Northwest corner rule

Iterations: Iterative method to arrive at an optimal solution

\[
\begin{align*}
\min Z &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\sum_{j=1}^{n} x_{ij} &= s_i \text{ for every } i \in \{1, \ldots, m\} \\
\sum_{i=1}^{m} x_{ij} &= d_j \text{ for every } j \in \{1, \ldots, n\} \\
x_{ij} &\geq 0 \text{ for every } i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}
\end{align*}
\]
TRANSPORTATION
SIMPLEX METHOD

Initialization: identifying an initial basic feasible solution
# Northwest Corner Rule

1. Start from northwest corner
2. Allocate as much as possible at \( x_{ij} \) while meeting demand and supply constraints
3. If allocation exhausts all supply from \( S_i \), move one row down
4. Else if \( S_i \) has any supply remaining, move one column to the right

<table>
<thead>
<tr>
<th></th>
<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( D_3 )</th>
<th>( D_4 )</th>
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<td>20</td>
<td>70</td>
<td>30</td>
<td>60</td>
<td>( Z = 3460 )</td>
</tr>
</tbody>
</table>
Initialization Stage: Which rule to use?

• Northwest corner rule is quick and easy
• But it does not take cost into account
  • So, it does not provide an optimum solution
• Other rules that take cost into account exist
  • But none of them provide an optimum solution
  • They only provide an initial basic feasible solution
BASIC FEASIBLE SOLUTION

... to a transportation problem.

- Needed for iterations of the Transportation Simplex Method
A basic feasible solution is an allocation satisfying the following conditions:
1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly $m + n - 1$ allocations
3. The allocations do not form a loop

The corresponding variables are basic variables.
1. Supply, demand and non-negativity constraints are satisfied
2. Number of allocations = 8, which is \( m + n - 1 = 5 + 4 - 1 \)
## Northwest Corner Rule

<table>
<thead>
<tr>
<th></th>
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<th>$D_4$</th>
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<td>$Z = 3460$</td>
</tr>
</tbody>
</table>

A cell with a value written is an allocation and corresponds to a **basic variable**. An empty (with no written value) corresponds to a non-basic variable.
Basic Feasible Solution to a Transportation Problem

A basic feasible solution is an allocation satisfying the following conditions:
1. Supply, demand constraints and non-negativity constraints are satisfied
2. There are exactly \( m + n - 1 \) allocations
3. The allocations do not form a loop

the corresponding variables are basic variables