Plan for today

• Sensitivity Analysis
  • Addition of a new constraint
  • Addition of a new variable

• Game Theory
  • Mathematical Framework
  • What would we like to understand?
SENSITIVITY ANALYSIS

... where we see how to handle change in data after solving the LP

Here is where we will use two tools that we have learnt before:
1. the matrix form of the simplex method and
2. the dual simplex method
Sensitivity Analysis: Motivation

• Suppose that you have solved the LP via Simplex
• Now, there is some change in data
  • Perhaps, the profit for some item has changed or some resource is available more, etc.
  • Perhaps, your boss gave you some incorrect data to begin with!

How do we solve the changed LP?
- Obvious approach: Re-solve the LP from scratch
- Advantage of Simplex: Can reuse optimal tableau of the starting problem
SENSITIVITY ANALYSIS

1. Change in objective
### Linear Programming – sensitivity analysis

<table>
<thead>
<tr>
<th>Basic Var.</th>
<th>Z</th>
<th>Original Variables</th>
<th>Slack Variables</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>1</td>
<td>$y^T A - c^T$</td>
<td>$y^T$</td>
<td>$y^T b$</td>
</tr>
<tr>
<td>$x_B$</td>
<td>0</td>
<td>$B^{-1} A$</td>
<td>$B^{-1}$</td>
<td>$B^{-1} b$</td>
</tr>
</tbody>
</table>

When there is a change in **coefficients of the objective function**, the current simplex tableau

- Will be feasible
- May not be optimal
  - If optimality conditions are satisfied, then terminate
  - If not optimal, then continue with simplex iterations

Before applying simplex, remember to ensure that the tableau satisfies:
- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
SENSITIVITY ANALYSIS

2. Change in RHS
When there is a change in RHS, the current simplex tableau

- Will satisfy optimality conditions
- May not be feasible
  - If feasible, then terminate
  - If infeasible, then continue with dual simplex iterations

Before applying simplex, remember to ensure that the tableau satisfies:
- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
- When there is a change in RHS, this property will NOT be violated
SENSITIVITY ANALYSIS

3. Addition of a new constraint
sensitivity analysis: adding a new constraint

Suppose that in Example 2, SIMPLEX method is applied, leading to $x_1^* = 2, x_2^* = 6$

\[
\begin{align*}
\text{max } Z &= 3x_1 + 5x_2 \\
x_1 &\leq 4 \\
2x_2 &\leq 12 \\
3x_1 + 2x_2 &\leq 18 \\
x_1 &\geq 0, x_2 \geq 0
\end{align*}
\]

Suppose we have an additional constraint: $x_1 + x_2 \leq 4$

\[
\begin{align*}
\text{max } Z &= 3x_1 + 5x_2 \\
x_1 + x_3 &= 4 \\
2x_2 + x_4 &= 12 \\
3x_1 + 2x_2 + x_5 &= 18 \\
x_1 + x_2 + x_6 &= 4 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{align*}
\]
sensitivity analysis: adding a new constraint

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<td>0</td>
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<td>$B^{-1} b$</td>
</tr>
</tbody>
</table>

Suppose we have an additional constraint: $x_1 + x_2 \leq 4$

Our current optimal solution $x_1^* = 2, x_2^* = 6, x_3^* = 0$ violates this constraint

Qn: What is the new optimal solution?

Q1: which values in the above table change, which ones remain the same?

Q2: is the same solution still feasible?

Q3: are any optimality conditions violated?
sensitivity analysis: adding a new constraint

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<td></td>
</tr>
</tbody>
</table>

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

Converted to augmented form, we have $x_1 + x_2 + x_6 = 4$

To continue, we must add this constraint to the tableau
- Add a new row for the constraint
- Add a new col for the new slack variable
- Make the new slack variable to be a basic variable
sensitivity analysis: adding a new constraint

Example: say we need to add \( x_1 + x_2 \leq 4 \) to the system

<table>
<thead>
<tr>
<th>Basic Var.</th>
<th>( Z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{3}{2} )</td>
<td>1</td>
<td>0</td>
<td>36</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( -\frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \frac{1}{3} )</td>
<td>( -\frac{1}{3} )</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Again tableau is not in proper format!
... since new constraint has more than one basic variable on the LHS

Update the tableau to get it to satisfy:
- Equations corresponding to all rows have: exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one

... substitute for \( x_1 \) and \( x_2 \) into the last constraint and rewrite
sensitivity analysis: adding a new constraint

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

<table>
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<tr>
<th>Basic Var.</th>
<th>$Z$</th>
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<th>$x_6$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>1</td>
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<td>36</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
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<td>0</td>
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<td>2</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>$-\frac{1}{3}$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$x_6$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\frac{1}{6}$</td>
<td>$-\frac{1}{3}$</td>
<td>1</td>
<td>$-4$</td>
</tr>
</tbody>
</table>

Simplex cannot continue from here

But dual simplex can continue from here!
The tableau is in the format needed to apply dual simplex:
- non-negative $Z$-row
- Equations corresponding to all rows have exactly one basic var, rest of the vars being non-basic and the coefficient of the basic var is one
- (Some RHS entries could be negative)

So reoptimize using dual simplex…

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

Solution is $x_1 = 2, x_2 = 6, x_3 = 2, x_6 = -4$

Solution is infeasible!
sensitivity analysis: adding a new constraint

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

<table>
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<tr>
<th>Basic Var.</th>
<th>$Z$</th>
<th>$x_1$</th>
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<tr>
<td>$Z$</td>
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<td>0</td>
<td>0</td>
<td>$3/2$</td>
<td>1</td>
<td>0</td>
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<tr>
<td>$x_1$</td>
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<td>0</td>
<td>0</td>
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<td>$1/3$</td>
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<td>2</td>
</tr>
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</tr>
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<td>0</td>
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<td>1</td>
<td>$1/3$</td>
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<td>0</td>
<td>0</td>
<td>$-1/6$</td>
<td>$-1/3$</td>
<td>1</td>
<td>$-4$</td>
</tr>
<tr>
<td>ratio</td>
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<td></td>
<td></td>
<td>$3/2$</td>
<td>$1/3$</td>
<td></td>
<td></td>
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</tbody>
</table>
sensitivity analysis: adding a new constraint

Example: say we need to add \( x_1 + x_2 \leq 4 \) to the system

```
<table>
<thead>
<tr>
<th>Basic Var.</th>
<th>Z</th>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>0</td>
<td>-(\frac{1}{2})</td>
<td>0</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>x_2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>x_3</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>(\frac{1}{2})</td>
<td>0</td>
<td>-1</td>
<td>6</td>
</tr>
<tr>
<td>x_5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>1</td>
<td>-3</td>
<td>12</td>
</tr>
<tr>
<td>ratio</td>
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<td></td>
<td></td>
<td></td>
<td>(\frac{1}{1/2})</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

After dual simplex iteration
sensitivity analysis: adding a new constraint

Example: say we need to add $x_1 + x_2 \leq 4$ to the system

<table>
<thead>
<tr>
<th>Basic Var.</th>
<th>$Z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
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<th>$x_6$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>1</td>
<td>2</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>20</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>10</td>
</tr>
<tr>
<td>ratio</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After another dual simplex iteration

No leaving variable, so feasibility is attained

Tableau is optimal

Optimal profit reduces to $Z^* = 20$
Optimal solution becomes $x_2^* = 4, x_3^* = 4, x_4^* = 4, x_5^* = 10$
SENSITIVITY ANALYSIS

4. Addition of a new variable
sensitivity analysis: adding a new variable

Suppose that in Example 2, SIMPLEX method is applied, leading to $x_1^* = 2, x_2^* = 6$

\[
\begin{align*}
\text{max } Z &= 3x_1 + 5x_2 \\
x_1 &\leq 4 \\
2x_2 &\leq 12 \\
3x_1 + 2x_2 &\leq 18 \\
x_1, x_2 &\geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\text{max } Z &= 3x_1 + 5x_2 + 8x_{\text{new}} \\
x_1 + 2x_{\text{new}} &\leq 4 \\
2x_2 + x_{\text{new}} &\leq 12 \\
3x_1 + 2x_2 + x_{\text{new}} &\leq 18 \\
x_1, x_2, x_{\text{new}} &\geq 0 \\
\end{align*}
\]

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<td>$Z$</td>
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<td>36</td>
<td></td>
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<tr>
<td>$x_1$</td>
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<td></td>
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</tbody>
</table>

Suppose we add a new variable $x_{\text{new}}$ to the system with coefficients as above

Qn: What is the new optimal solution?
sensitivity analysis: adding a new variable

Suppose we add a new variable $x_{new}$ to the system with coefficients as above

Qn: What is the new optimal solution?

Q1: which values in the above table change, which ones remain the same?

Q2: is the same solution still feasible?

Q3: are any optimality conditions violated?

max $Z = 3x_1 + 5x_2 + 8x_{new}$  
$x_1 + 2x_{new} \leq 4$  
$2x_2 + x_{new} \leq 12$  
$3x_1 + 2x_2 + x_{new} \leq 18$  
$x_1, x_2, x_{new} \geq 0$

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<td>0</td>
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Example
sensitivity analysis: adding a new variable

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New constraint matrix and objective vector

$A_{\text{new}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$
$c_{\text{new}}^T = (3 \ 5 \ 8)$

From the final tableau of the original problem, we have

$$B^{-1} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$
$$y^T = \begin{pmatrix} 0 & 3 & 2 \end{pmatrix}$$

So we can compute the entries of the tableau for the new parameters

$$y^T A_{\text{new}} - c_{\text{new}}^T = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} - (3 \ 5 \ 8) = \begin{pmatrix} 0 & 0 & -\frac{11}{2} \end{pmatrix}$$

$$B^{-1} A_{\text{new}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 2 \end{pmatrix}$$
sensitivity analysis: adding a new variable

Resulting tableau is

<table>
<thead>
<tr>
<th>Basic Var.</th>
<th>Z</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_{\text{new}}$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>RHS</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
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<td>$-\frac{11}{2}$</td>
<td>0</td>
<td>3/2</td>
<td>1</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>2</td>
<td></td>
</tr>
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<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
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<td></td>
</tr>
<tr>
<td>$x_3$</td>
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<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
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<td></td>
</tr>
</tbody>
</table>

Non-optimal

So reoptimize using simplex…
sensitivity analysis: adding a new variable

<table>
<thead>
<tr>
<th>Basic Var.</th>
<th>Z</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_{\text{new}}$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>RHS</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$-\frac{11}{2}$</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\frac{1}{3}$</td>
<td>1</td>
<td>$\frac{3}{2}$</td>
<td>2</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>$x_3$</td>
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<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Example 1:

**Iteration**

\[
\begin{align*}
\text{Row}_1 - \left( -\frac{11}{2} \right) \times \text{Row}_{4}^{\text{new}} &= Z \\
\text{Row}_2 - 0 \times \text{Row}_{4}^{\text{new}} &= x_1 \\
\text{Row}_3 - \left( \frac{1}{2} \right) \times \text{Row}_{4}^{\text{new}} &= x_2 \\
\text{Row}_{4}^{\text{new}} \leftarrow \text{Row}_4 / 2 &= x_{\text{new}}
\end{align*}
\]
sensitivity analysis: adding a new variable

<table>
<thead>
<tr>
<th>Basic Var.</th>
<th>Z</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_{new}$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>RHS</th>
<th>ratio</th>
</tr>
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<tbody>
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<td>Z</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\frac{-11}{2}$</td>
<td>0</td>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{-1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{-1}{3}$</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Simplex Iteration

Row 1 $- \left( -\frac{11}{2} \right)$ Row 4$\text{new}$

Row 2 $- 0 \times$ Row 4$\text{new}$

Row 3 $- \left( \frac{1}{2} \right) \times$ Row 4$\text{new}$

Row 4$\text{new} \leftarrow$ Row 4/2
sensitivity analysis: adding a new variable

<table>
<thead>
<tr>
<th>Basic Var.</th>
<th>Z</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_{new}$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>RHS</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{11}{4}$</td>
<td>$\frac{7}{12}$</td>
<td>$\frac{23}{12}$</td>
<td>$\frac{83}{2}$</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{-1}{3}$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\frac{-1}{4}$</td>
<td>$\frac{7}{12}$</td>
<td>$\frac{-1}{12}$</td>
<td>$\frac{11}{2}$</td>
<td></td>
</tr>
<tr>
<td>$x_{new}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{-1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

No entering variable, so optimality is attained

Tableau is optimal

Optimal profit increases to $Z^* = 41.5$
Optimal solution becomes $x_1^* = 2, x_2^* = 5.5, x_{new}^* = 1$
sensitivity analysis: adding a new variable

When we added the variable, we computed \( x_1 \) and \( x_2 \) cols from scratch using matrix operations

- These cols did not change from their previous optimal basis!
- We could have taken a shortcut and only computed the \( x_{new} \) col

Shortcut:

- \( a_{new} \) be the column of constraint coefficient for the new var
- \( c_{new} \) be its objective coefficient
- Updated coefficients for \( x_{new} \) are:

  \[
  Z\text{-row: } y^T a_{new} - c_{new}
  \]

  Constraint rows: \( B^{-1} a_{new} \)
sensitivity analysis: adding a new variable

<table>
<thead>
<tr>
<th>Basic Var.</th>
<th>$Z$</th>
<th>Original Variables</th>
<th>Slack Variables</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>1</td>
<td>$y^T A - c^T$</td>
<td>$y^T$</td>
<td>$y^T b$</td>
</tr>
<tr>
<td>$x_B$</td>
<td>0</td>
<td>$B^{-1} A$</td>
<td>$B^{-1}$</td>
<td>$B^{-1} b$</td>
</tr>
</tbody>
</table>

The coefficients in the col of $x_{new}$ are

$$y^T a_{new} - c_{new} = \begin{pmatrix} 0 & \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 8 = -\frac{11}{2}$$

$$B^{-1} a_{new} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{2}{3} \end{pmatrix}$$

$a_{new} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$c_{new} = 8$
Change management: Beware

If lot of entries among $A, c, b$ change, then the changes are not small …
Summary of Linear Programming

1. Formulation

2. Solution methods
   • Graphic Method
   • SIMPLEX Method
     • Algebraic Form
     • Tabular Form

3. Duality Theory
   • Shadow Prices
   • Dual linear program
   • Duality Theorems, Weak and Strong duality, Complementary Slackness

4. Dual SIMPLEX Method

5. Matrix Form of Simplex Method

6. Sensitivity Analysis
GAME THEORY

... where we see how to compute the optimal strategy for 2-player 0-sum games
Significance

- Mathematics to understand competitive situations
- Commonly used in business, economics to identify best strategies and policies

Nobel Prizes in Economic Sciences

- Aumann-Schelling (2005): Why do some countries/individuals promote cooperation while others suffer from conflict?

Our topic: 2-player 0-sum games
WHAT IS A GAME?

... a mathematical framework for games
Matching Pennies Game

- Two players, each has a penny
- Secretly turn the penny to head or tail in their hand
- Reveal their choices simultaneously
- If pennies match, then player A wins both pennies
- If not, then player B wins both pennies

Representation:

Payoff table for Player A:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

Payoff table for Player B:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

⇒ zero-sum game
Two-player, zero-sum game

- A two-player zero-sum game is specified by
- \( S_1, \ldots, S_m \): strategies for Player A,
- \( T_1, \ldots, T_n \): strategies for Player B

- **Payoff table for A:** Shows the gain for Player A for each combination of strategies for the two players

\[
\begin{array}{c|cccc}
 & T_1 & \ldots & T_j & \ldots & T_n \\
\hline
S_1 & p_{11} & \ldots & p_{1j} & \ldots & p_{1n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
S_i & p_{i1} & \ldots & p_{ij} & \ldots & p_{in} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
S_m & p_{m1} & \ldots & p_{mj} & \ldots & p_{mn} \\
\end{array}
\]

Payoff table for Player A
Main assumption in Game Theory

- Both players are intelligent and rational!
What would we like to understand?

- Two players, each has a penny
- Secretly turn the penny to head or tail in their hand
- Reveal their choices simultaneously
- If pennies match, then player A wins both pennies
- If not, then player B wins both pennies

Representation:

Payoff table for Player A:

<table>
<thead>
<tr>
<th></th>
<th>A 1</th>
<th>B -1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

⇒ zero-sum game

Payoff table for Player B:

<table>
<thead>
<tr>
<th></th>
<th>A -1</th>
<th>B 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>B</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

Matching Pennies Game Example
What would we like to understand?

- Player $A$ has two strategies
- Assume that they play the game a large number of times
- What strategy would Player $A$ play?

Payoff table for Player $A$
What would we like to understand?

- Player $A$ has two strategies
- Assume that they play the game a large number of times
- What strategy would Player $A$ play?

Matching Pennies Game Example

\[
\begin{array}{c|cc}
 & y & 1-y \\
\hline
x & 1 & -1 \\
1-x & -1 & 1
\end{array}
\]

Payoff table for Player $A$
What would we like to understand?

- Given: payoff table for $A$
- Assuming that players are intelligent and rational

Question:
- With what probability (i.e., proportion) $x_i$ should player $A$ play each strategy $S_i$ and
- With what probability (i.e., proportion) $y_j$ should player $B$ play each strategy $T_j$ so that $A$ maximizes her profit and $B$ minimizes his loss

Player $A$ knows that Player $B$ is an intelligent player and so will not allow Player $A$ to get more and more profit
- So Player $A$’s objective will be to maximize the minimum profit that she can get
  - Player $A$: **Maximin criterion**
- Similarly, Player $B$’s objective will be to minimize the maximum loss
  - Player $B$: **Minimax criterion**
- **Value of the game** = Payoff to player $A$ when both players play optimally