Lecture 9: Birkhoff's theorem, Integral polyhedra

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9.1 Integral polyhedra (IPs with efficient algorithms): min-cost perfect matching in bipartite graphs

Recap

Definition 1. A polyhedron P is *integral* if $P = P_I$ where $P_I := \text{conv}(P \cap \mathbb{Z}^n)$.

Example: See Figure 9.1.

P is *not* integral $\quad$ P is integral

Figure 9.1: Examples of integral polyhedron

Recall that we can solve integral-optimization over the polyhedron P if it is integral by simply solving the LP-relaxation. As an application, we started looking at perfect matching in bipartite graphs.

Application: Min-cost perfect matching in bipartite graphs.
Given: A bipartite graph $G = (A \cup B, E)$ and edge costs $c : E \to \mathbb{R}$.
Goal. $\min\{\sum_{e \in M} c_e : M \text{ is a perfect matching in } G\}$.

Lemma 1.1. *A graph G is bipartite iff G does not contain a cycle with odd number of edges.*

Definition 2. A *perfect matching* in G is a set M of edges such that each vertex is adjacent to exactly one edge in M.

Definition 3. $\text{PM}(G) := \{\chi^M : M \text{ is a perfect matching in } G\}$ where $\chi^M \in \{0,1\}^E$ is defined as

$$\chi^M(e) := \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise.} \end{cases}$$

We are interested in obtaining an inequality description of conv(PM(G)) in order to be able to solve the minimum cost perfect matching problem in bipartite graphs. Let us call conv(PM(G)) as the *perfect matching polytope* of G.

Theorem 4 (Birkhoff)**.** *Let $G = (V, E)$ be a bipartite graph. Let*

$$P(G) := \left\{ x \in \mathbb{R}^E : \sum_{e \in E:\ e \text{ is incident to } v} x(e) = 1\ \forall v \in V, x(e) \geq 0\ \forall e \in E \right\}.$$

Then,

1. *Conv($PM(G)$) $= P(G)$, and*
2. *$P(G)$ is integral.*

Proof. We note that conclusion 1 implies conclusion 2:

$$\begin{aligned}
(P(G))_I &= \text{conv}\left(P(G) \cap \mathbb{Z}^E\right) && \text{(by definition of integral hull)}\\
&= \text{conv}\left(P(G) \cap \text{PM}(G)\right) && \text{(integral points in } P(G) \text{ are indicator vectors of perfect matchings in } G)\\
&= \text{conv}\left(\text{PM}(G)\right) && \text{(Every point in PM}(G) \text{ is present in } P(G))\\
&= P(G) && \text{(from conclusion 1).}
\end{aligned}$$

Now we will prove conclusion 1. For this, we will show containment in both directions.

1. conv(PM(G)) $\subseteq P(G)$: Holds since $\chi^M \in P(G)$ for all perfect matchings M of G. Since $P(G)$ is convex, it follows that conv(PM(G)) is also contained in $P(G)$.
2. $P(G) \subseteq$ conv (PM(G)): Note that $P(G)$ is a polytope (since it is bounded and it is a polyhedron). We will show this containment by showing that all extreme points of $P(G)$ are integral. This is sufficient: note that each integral point in $P(G)$ has to be the indicator vector of a perfect matching in G; So, $P(G) =$ conv(extreme points of $(P(G)) \subseteq$ conv(PM(G)).

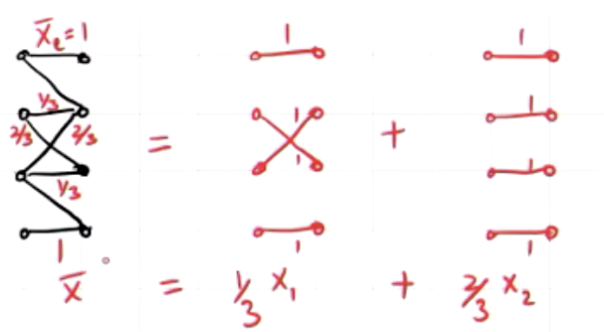

Figure 9.2: Proof idea for $P(G) \subseteq$ conv($PM(G)$): Consider $\bar{x}$ which is not integral (the edges with no value shown next to it have a value of zero) . It can be expressed as a convex combination of points in $P(G)$ and hence, $\bar{x}$ is not an extreme point of $P(G)$.

We now show that all extreme points of $P(G)$ are integral (see Figure 9.2 for the approach.) Let $\bar{x}$ be an extreme point of $P(G)$. Suppose $\bar{x}$ has fractional coordinates. Let $F := \{e \in E : 0 < x_e < 1\}$ be the set of fractional edges. Consider $H := G[F]$, the subgraph induced by the fractional edges. Since $\bar{x}$ satisfies all degree constraints, it follows that the each node in H has at least two edges incident to it. Thus, the degree of every node in H is at least two. Therefore, H contains a cycle, say C. Also, H is a subgraph of a bipartite graph and hence, all cycles in H should have even number of edges. So, C has even number of edges. Order the edges of C as they appear in the cycle, say $C = e_1 e_2 \ \ldots \ e_{2r-1} \ e_{2r}$ (see Figure 9.3).

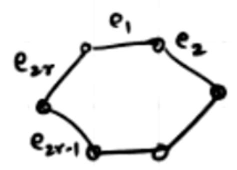

Figure 9.3: An even cycle

For $\epsilon \geq 0$, let

$$y_1(e) := \begin{cases} \bar{x}(e) & \text{if } e \notin C \\ \bar{x}(e) + \epsilon & \text{if } e \in C \text{ and } e = e_{2i} \text{ for some } i \in [r] \ (e \text{ is an even edge in the cycle}) \\ \bar{x}(e) - \epsilon & \text{if } e \in C \text{ and } e = e_{2i+1} \text{ for some } i \in [r-1] \ (e \text{ is an odd edge in the cycle}) \end{cases}$$

and let

$$y_2(e) := \begin{cases} \bar{x}(e) & \text{if } e \notin C \\ \bar{x}(e) - \epsilon & \text{if } e \in C \text{ and } e = e_{2i} \text{ for some } i \in [r] \ (e \text{ is an even edge in the cycle}) \\ \bar{x}(e) + \epsilon & \text{if } e \in C \text{ and } e = e_{2i+1} \text{ for some } i \in [r-1] \ (e \text{ is an odd edge in the cycle}). \end{cases}$$

In particular, for $\epsilon := \min_{e \in C}\{\bar{x}(e), 1 - \bar{x}(e)\}$, the points y_1 and y_2 are in $P(G)$ as they satisfy all the constraints. Moreover, $y_1 \neq y_2$ (because $0 < \bar{x}(e) < 1 \ \forall e \in E$ and hence $\epsilon > 0$). Therefore, $\bar{x} = \frac{y_1 + y_2}{2}$ where $y_1 \neq y_2$ and $y_1, y_2 \in P(G)$. So, $\bar{x}$ is not an extreme point of $P(G)$, which is a contradiction.

□

The following is the main corollary of Birkhoff's theorem (Theorem 4).

Corollary 4.1. *We can find a minimum cost perfect machining in bipartite graphs by solving*

$$\min \left\{ \sum_{e \in E} c_e x(e) : \sum_{e \in E:\ e \text{ is incident to } v} x(e) = 1 \ \forall v \in V,\ x(e) \geq 0 \ \forall e \in E \right\}.$$

Theorem 4 gives us an inequality description of the perfect matching polytope in bipartite graphs. We proved the theorem by showing that $P(G)$ is integral. More generally, it would be helpful if we could recognize integral polyhedra from the inequality description. For this purpose, we will characterize integral polyhedra. Once we have such a characterization, Birkhoff's theorem and more generally, *integrality of several polyhedra*, become much easier to show.

9.2 Integral Polyhedra

Lemma 4.1 (Characterization of integral polyhedra)**.** *Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. The following are equivalent:*

1. *P is integral i.e., $P = P_I$.*
2. *Each face of P contains an integral vector.*
3. *Each minimal face of P contains an integral vector.*
4. *$\max\{c^T x : x \in P\}$ has an integral optimum solution for every objective direction $c \in \mathbb{R}^n$ for which the optimum objective value is finite.*

Note: Suppose P is a polytope. Then, minimal faces of P are extreme points of P; therefore, $1 \iff 3$ is equivalent to saying that P is an integral polytope iff all extreme points of P are integral. Note that we proved the latter in order to show Birkhoff's theorem.

Proof of lemma. We will prove the lemma by showing that $1 \implies 2 \implies 3 \implies 4 \implies 1$.

- $1 \implies 2$:

  Let F be a face of P. Then $F = P \cap H$ for some supporting hyperplane H of P. Let $x \in F$ (see Figure 9.4). From 1, we have that $P = P_I = \text{conv}(P \cap \mathbb{Z}^n)$ which implies that x is a convex combination of integral points in P. These integral points should also be in H since H is a supporting hyperplane of P. So, we have integral points in $P \cap H = F$.

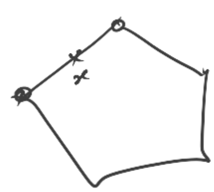

Figure 9.4: Integrality of the polyhedron implies each face contains an integral vector

- $2 \implies 3$: Holds since minimal faces are also faces.
- $3 \implies 4$: Let $\delta := \max\{c^T x : x \in P\}$ be finite. Then $F := \{x \in P : c^T x = \delta\}$ is a face of P. From 3, we have that there exists an integral point in F.
- $4 \implies 1$: In order to show that $P = P_I$, we will show that $P_I \subseteq P$ and $P \subseteq P_I$.
  1. $P_I \subseteq P$: By definition of P_I.
  2. $P \subseteq P_I$: Suppose $P \not\subseteq P_I$. Then there exists $y \in P \setminus P_I$ and an inequality $w^T x \leq \delta$ that is valid for P_I but violated by y (see Figure 9.5). Therefore,

$$\max\{w^T x : x \in P_I\} \leq \delta < \max\{w^T x : x \in P\}.$$

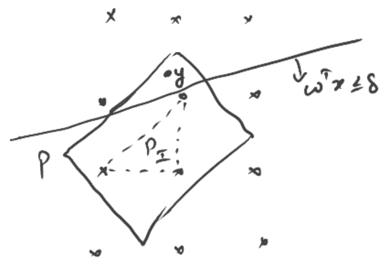

Figure 9.5: $P \not\subseteq P_I$

Now we will show that $\max\{w^T x : x \in P\}$ is finite: Say not. Then there exists a rational $z \in P$ such that $w^T z > 0$ and $\alpha z \in P \; \forall \alpha > 0$ (i.e., there exists an extreme ray in P along the direction of w). Therefore, there exists an infinite sequence $\alpha_1, \alpha_2, \ldots$ such that $\alpha_i z \in P \cap \mathbb{Z}^n$ and $w^T(\alpha_i z) > w^T(\alpha_{i-1} z)$ for every $i = 2, 3, \ldots$. Therefore, $\max\{w^T x : x \in P_I\}$ is unbounded. This contradicts $\max\{w^T x : x \in P_I\} < \delta$.
Moreover, $\max\{w^T x : x \in P_I\} \leq \delta < \max\{w^T x : x \in P\}$. Hence, $\max\{w^T x : x \in P\}$ is finite but without an integral optimum solution, thus contradicting 4.

□

More characterization of integral polyhedra are also known:

Lemma 4.2 (More characterizations of integral polyhedra)**.** *Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. The following are equivalent.*

1. *P is integral.*
2. $\max\{c^T x : x \in P\}$ *has an integral optimum solution for every objective direction $c \in \mathbb{Z}^n$ for which the optimum objective value is finite.*
3. *The value $z = \max\{c^T x : x \in P\}$ is an integer for every objective direction $c \in \mathbb{Z}^n$ for which the optimum objective value is finite.*
4. *Every rational supporting hyperplane of P contains an integral vector.*

These characterizations are helpful to argue integrality of a polyhedron by carefully inspecting the inequality description of the polyhedron. Based on the structure of the polyhedron, one characterization might be easier to show than another—but they all imply integrality of the polyhedron.

Remark 1. Note the subtle difference between the fourth item of Lemma 4.1 and the second item of Lemma 4.2—the former is about real-valued objective directions while the latter is about integer-valued objective directions.

Remark 2. In Lemma 4.2, it is easy to see that $2 \implies 3$ easily. What is interesting is the fact that $3 \implies 2$ and $3 \implies 1$. For several polyhedra, it is easier to show that condition 3 holds. We will indeed use this later.

The proof of Lemma 4.2 is left as an **exercise**.

In the next lecture, we will see a family of integral polyhedra defined by special kinds of constraint matrices. Integrality of those polyhedra will be easy to show using the above characterizations.