IE 511: Integer Programming, Spring 2021	**18 Feb, 2021**

Lecture 8: Integer-hull of rational polyhedra

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In the previous lecture, we saw that in order to solve $\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$ over a polyhedron P, it is sufficient to solve $\max\{c^T x : x \in \text{convex-hull}(P \cap \mathbb{Z}^n)\}$. In today's lecture, we will focus on the set convex-hull$(P \cap \mathbb{Z}^n)\}$. If this set has a nice structure, then we will be able to optimize over this set. What sort of nice structure would help us optimize? Well, since we know how to solve LPs, it would be nice if the set convex-hull$(P \cap \mathbb{Z}^n)\}$ is a polyhedron. Sometimes, wishes do come true—this set is indeed a polyhedron! We will prove it during the lecture today.

Notation: For convenience, we will henceforth write conv(S) to denote convex-hull(S).

8.1 Integer-hull of a polyhedron

Let $P \subseteq R^n$ be a polyhedron. We begin by giving a name to the set $\text{conv}(P \cap \mathbb{Z}^n)$.

Definition 1. The *integer-hull* of P is $P_I := \text{conv}(P \cap \mathbb{Z}^n)$.

Example: Consider the polyhedron $P = \{(x, y) : 2x + 3y \leq 7, x \geq 0, y \geq 0\}$. The polyhedron and its integer-hull are shown in Figure 8.1.

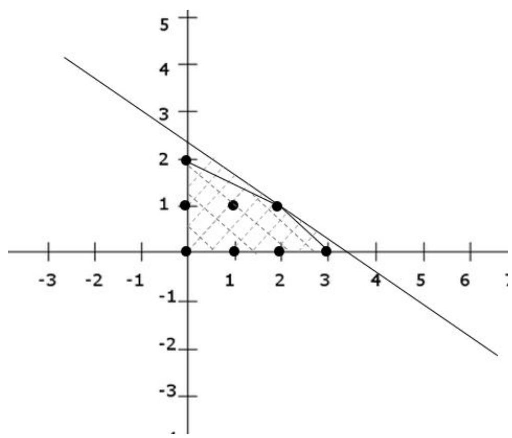

Figure 8.1: Polyhedron P and its integral hull

Note that if P_I is also a polyhedron, then $\max\{c^T x : x \in P_I\}$ would be an LP which would be easier to solve. This motivates the following questions:

Question 1: If P is a polyhedron, then is P_I a polyhedron?

Question 2: If P_I is a polyhedron, then does P_I have a rational inequality description?

Recall the definition of rational polyhedron: A polyhedron $P = \{x : Ax \leq b\}$ is *rational* if A, b are rational. We first note that Question 1 has a negative answer if the polyhedron P is unbounded and irrational.

Observation. If a polyhedron P is unbounded irrational then P_I might not be a polyhedron.

Integer hull of irrational unbounded polyhedron may not be a polyhedron

Example: Consider the example polyhedron from Lecture 4 which showed us that irrational IPs may be feasible and bounded but still have no optimum solution. This is the polyhedron $P = \{(x, y) : x \geq 1, y \geq 0, -\sqrt{3}x + y \leq 0\}$. The number of inequalities needed for the integer hull is infinite which means that the integer hull is not a polyhedron (see Figure 8.2).

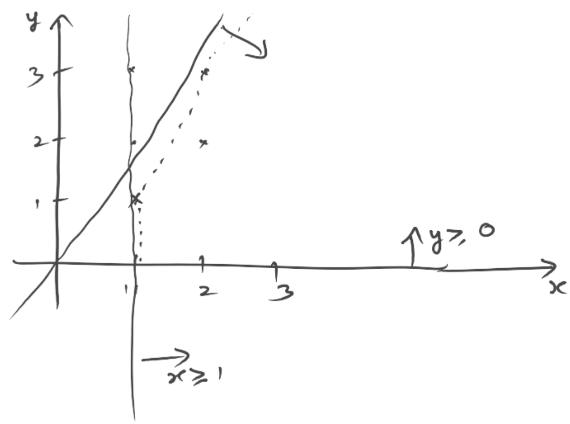

Figure 8.2: An unbounded irrational polyhedron whose integer hull is not a polyhedron.

In today's lecture, we will answer both questions above affirmatively if P is a rational polyhedron. Recall that we briefly touched upon Question 1 towards the end of Lecture 4: we mentioned that if P is a polytope, then P_I is also a polytope. But, what if P is an unbounded polyhedron? We have the following result for any polyhedron:

Theorem 2 (Meyer (1974))**.** *If P is a rational polyhedron, then P_I is a polyhedron.*

How would we prove this statement? Recall that a polyhedron is the sum of a polytope and a cone. So, in order to the prove the theorem, it is natural to begin by studying the integer-hull of polytopes and cones. This will be our approach. Let us begin with polytopes. We mentioned the following result in Lecture 4.

Lemma 2.1. *If P is a bounded polyhedron, then $P_I = conv(P \cap \mathbb{Z}^n)$ is also a bounded (rational) polyhedron.*

Proof. If P is bounded, then $P \cap \mathbb{Z}^n$ is finite. It implies that $\text{conv}(P \cap \mathbb{Z}^n)$ is a (rational) polytope. Therefore $P_I = \text{conv}(P \cap \mathbb{Z}^n)$ is a (rational) polytope. □

Note that in the above lemma, P need not be a rational polytope to conclude that P_I is a polytope—the proof does not rely on rationality.

Now that we understand the integer-hull of polytopes, let us now understand the integer-hull of cones.

Lemma 2.2. *Let $C = \{x : Ax \le 0\}$ be a rational polyhedral cone. Then $C_I = C$.*

Proof. To show that $C_I = C$ we show $C \subseteq C_I$ and $C_I \subseteq C$.

1. $C_I \subseteq C$: This follows by definition of integer-hull.
2. $C \subseteq C_I$: By Weyl-Minkowski, the cone C is generated by finitely many integral vectors, say $a_1, \ldots, a_t$. Recall that $C_I = \text{conv}(C \cap \mathbb{Z}^n)$. Let $x \in C$. Then $x = \sum_{i=1}^t \lambda_i a_i$ for some $\lambda_1, \ldots, \lambda_t \ge 0$. We need to show that $x \in \text{conv}(C \cap \mathbb{Z}^n)$ i.e., x is a convex combination of integral points in C.

   Proof intuition. Look at the example in Figure 8.3. If $x = x_1$, then it is clear that x is a convex combination of integral points in C—namely, it is a convex combination of $(0,0), a_1$, and a_2. How about $x = x_2$? How can we argue that x_2 is a convex combination of integral points in the cone generated by a_1 and a_2? Well, note that x_2 is a convex combination of $(0,0), 2a_1$, and a_2. We will extend this approach to prove that any point x in C is a convex combination of integral points in C.

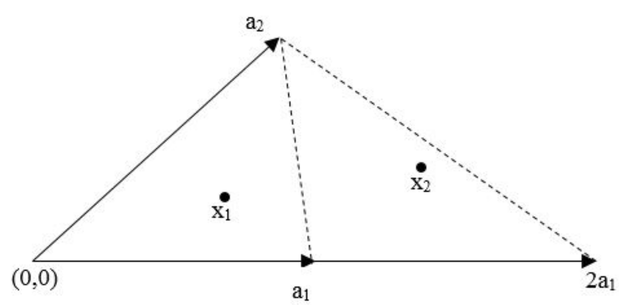

Figure 8.3: Proof idea: x_1 is a convex combination of (0,0), a_1, and a_2 and x_2 is a convex combination of (0,0), $2a_1$, and a_2.

   Recall that x is a point in C and hence $x = \sum_{i=1}^t \lambda_i a_i$ for some $\lambda_1, \ldots, \lambda_t \ge 0$. Let $\mu := \sum_{i=1}^t \lambda_i$ and let $b_i := \lceil \mu \rceil a_i \ \forall i \in [t]$. Then $b_1, \ldots, b_t \in C \cap \mathbb{Z}^n$ and $x = \sum_{i=1}^t \left(\frac{\lambda_i}{\lceil \mu \rceil} \right) b_i$. Set $\alpha_i := \frac{\lambda_i}{\lceil \mu \rceil} \ \forall i \in [t]$ and $\alpha_0 := 1 - \sum_{i=1}^t \alpha_i$. Then $x = \sum_{i=1}^t \alpha_i b_i + (1 - \alpha_0)0$ which implies that $x \in \text{conv}(C \cap \mathbb{Z}^n)$.

□

Note that Lemma 2.2 is Theorem 2 for the special case when the polyhedron is a cone. We now have the tools to prove Theorem 2.

Proof of Theorem 2. Let P be a rational polyhedron. Then by the decomposition theorem we have that $P = Q + C$ where Q is a polytope and C is a rational polyhedral cone. A rational polyhedral cone is also a finitely generated cone, with the generators being integral. I.e., $C = \text{cone}\{a_1, \ldots, a_t\}$ where $a_1, \ldots, a_t \in \mathbb{Z}^n$ are integral.

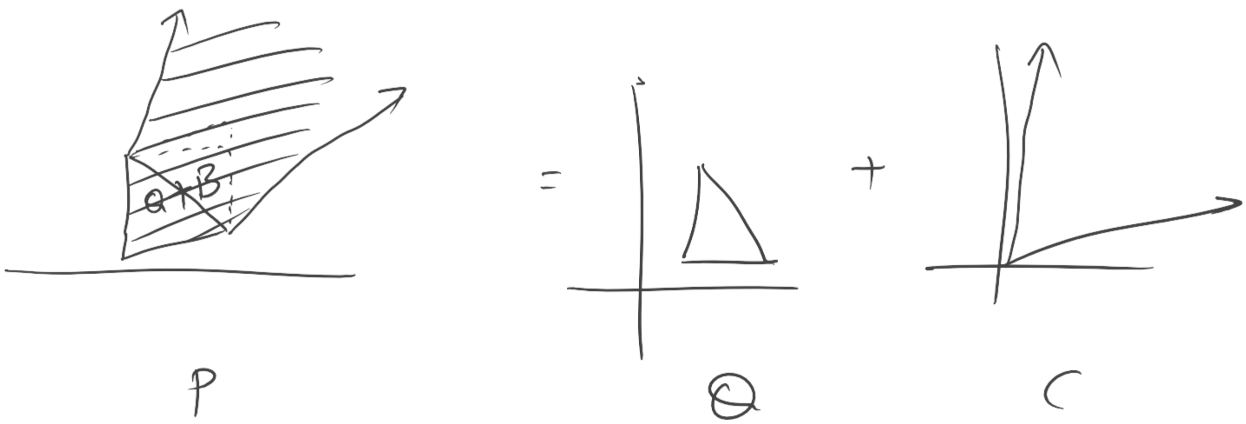

Figure 8.4: A polyhedron P is sum of a polytope Q and a cone C.

Let $B := \{\sum_{i=1}^{t} \lambda_i a_i : 0 \leq \lambda_i \leq 1 \; \forall i \in [t]\}$, i.e., B is the parallelepiped of the integral generators of C (see Figure 8.5 for an example).

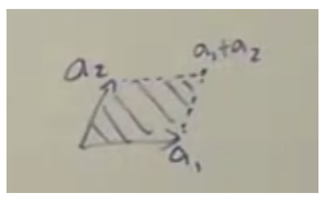

Figure 8.5: Parallelepiped of a_1 and a_2

The following claim implies the theorem, since $(Q + B)_I$ is a polytope and C is a cone. So, by the polyhedral decomposition theorem, P_I is a polyhedron. This completes the proof of the theorem. □

Claim 2.1. $P_I = (Q + B)_I + C$.

Proof.

1. $(Q + B)_I + C \subseteq P_I$:

   We know that $Q + B \subseteq Q + C = P$. So, $(Q + B)_I \subseteq P_I$. Therefore, $(Q + B)_I + C \subseteq P_I + C = P_I + C_I \subseteq (P + C)_I = P_I$. The containment $P_I + C_I \subseteq (P + C)_I$ is left as an **exercise**.

2. $P_I \subseteq (Q + B)_I + C$:

   Let p be an arbitrary point in $P \cap \mathbb{Z}^n$. We will show that $p \in (Q + B)_I + C$. This is sufficient to prove the claim since the RHS set is a polyhedron and is convex, so if all integral points of P are in the RHS set, then their convex-hull is also in the RHS set.

   We now show that $p \in (Q + B)_I + C$. Since $p \in P$, we have that $p = q + c$ for some $q \in Q, c \in C$. If $q \in Q \cap \mathbb{Z}^n$ then the proof is done but q may not be in $Q \cap \mathbb{Z}^n$. Consider the set of integral points which are non-negative integral combinations of $a_1, \ldots, a_t$—recall that $a_1, \ldots, a_t$ are the integral vectors that generate the cone C (see Figure 8.6 for an example).

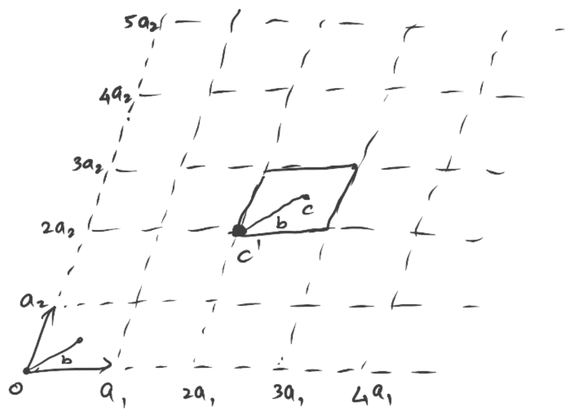

Figure 8.6: Set of non-negative integral combinations of a_1, a_2

Note that we can write $c = b + c'$ for $b \in B$ and $c' = \sum_{i=1}^{t} \alpha_i a_i$ for some $\alpha_1, \ldots, \alpha_t \in \mathbb{Z}_{\geq 0}$. Therefore, $p = q + b + c'$. We need to show that $q + b \in (Q + B)_I$. To show this, it is sufficient to show that $q + b \in (Q + B) \cap \mathbb{Z}^n$. It is clear that $q + b \in Q + B$ by definition. It suffices to show that $q + b \in \mathbb{Z}^n$. For this, we observe that

$$q + b = p - c',$$

where $c' \in C \cap \mathbb{Z}^n$ because α_is and a_is are integral and $p \in P \cap \mathbb{Z}^n$. Thus, both p and c' are integral and hence $q + b$ is also integral.

□

Note: Rationality of P_I (in the statement of Theorem 2 also follows from the same proof by using the rational variant of the polyhedral decomposition theorem.

Using Theorem 2, we have the following corollary. This is the main significance of Theorem 2.

Corollary 2.1. *Let P be a polyhedron. Then, the RHS in*

$$\max\{c^T x : x \in P \cap \mathbb{Z}^n\} = \max\{c^T x : x \in P_I\}$$

is an LP.

Due to Corollary 2.1, we can use linear programming techniques to solve IPs. In particular, if we can obtain the inequality description of P_I, then we can solve integer optimization over the polyhedron P.

We also note a geometric consequence of our proof technique of Theorem 2.

Corollary 2.2. *Let P be a rational polyhedron. If $P_I \neq \emptyset$, then the extreme rays of P and P_I coincide.*

Proof. This is because, the cone in the decomposition of P and P_I are the same. □

8.2 Efficiently Solvable IPs (IP: $\max\{c^T x : P \cap \mathbb{Z}^n\}$)

As a thumb rule, if a family of IP instances has an efficient algorithm, then it is usually because of one of the following reasons:

1. An explicit and concise inequality description of P_I is known.
2. An efficient separation oracle for P_I is known.

   It means that given $x^* \in \mathbb{R}^n$, there exists an efficient algorithm to verify if $x^* \in P_I$ and if not, then the algorithm finds an inequality $w^T x \le \delta_0$ with $w^T x^* > \delta_0$ such that $\{x : w^T x = \delta_0\}$ is a supporting hyperplane for P_I.
3. A strong dual problem is known.

   Recall that a strong dual problem (D) $\min\{w(u) : u \in U\}$ to the IP satisfies

   $$x^* \in P \cap \mathbb{Z}^n \text{ is optimal} \iff \exists\, u^* \in U \text{ with } c^T x^* = w(u^*).$$

Note that 1 implies 2 and 3 via LP-duality. In the next few lectures, we will see families of IPs where these three properties hold. We will repeatedly allude to all three properties wherever one of them holds. If you suspect an IP to be solvable efficiently, then you could try to design an efficient algorithm by showing that one of the three properties above hold.

8.3 Integral Polyhedron

An important special case of efficiently solvable IPs are over polyhedra whose integer-hull is the polyhedron itself. We will call such a polyhedron to be an integral polyhedron.

Definition 3. A polyhedron P is *integral* if $P = P_I$.

It is easy to solve IPs over integral polyhedra as we could simply solve the LP to obtain an integral optimum solution. So, IPs defined over integral polyhedra correspond to lucky cases: e.g., consider the IP $\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$ – if P is an integral polyhedron, then solving $\max\{c^T x : x \in P\}$ should give the optimal objective value for the IP. So, it would be helpful if we can verify whether a given polyhedron (given by its constraint matrix A and RHS vector b) is integral. We will now focus on a concrete example as an application of integral polyhedra.

8.3.1 Application: Perfect matching polyhedron of bipartite graphs

Definition 4. A graph $G = (V, E)$ is *bipartite* if there exists a partition A, B of V with $A \cup B = V$ such that every edge $e \in E$ is of the form $e = \{a, b\}$ for some $a \in A, b \in B$.

Example:

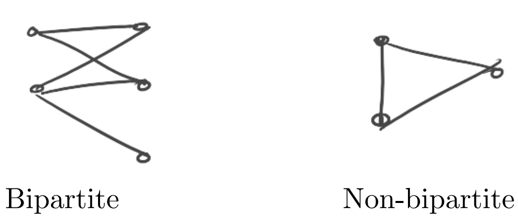

Figure 8.7: A bipartite and a non-bipartite graph

Bipartite graphs are recognized by the following lemma.

Lemma 4.1. *A graph G is bipartite iff G does not contain a cycle with odd number of edges.*

Proof. **Exercise**. □

We will focus on *perfect matchings* in bipartite graphs.

Definition 5. Let G be a graph. A *perfect matching* in G is a set M of edges such that each vertex is adjacent to exactly one edge in M.

In the minimum cost perfect matching problem in bipartite graphs, we are given a bipartite graph with edge costs and our goal is to find a minimum cost perfect matching. This is also known as the assignment problem. The assignment problem has numerous applications, e.g., consider the problem of finding an assignment of four swimmers to the four legs of swimming medley relay to minimize the total swimming time.

Definition 6. Let $\text{PM}(G) := \{\chi_M : M \text{ is a perfect matching in } G\}$ where $\chi_M \in \{0,1\}^E$ is the indicator vector of M, i.e.,

$$\chi_M(e) := \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{otherwise.} \end{cases}$$

In order to solve minimum cost perfect matching problem, it suffices to solve $\min\{c^T x : x \in \text{PM}(G)\}$. Note that $\text{PM}(G)$ contains a collection of discrete vectors and hence $\text{PM}(G)$ is non-convex. However, we do know that

$$\min\{c^T x : x \in \text{PM}(G)\} = \min\{c^T x : x \in \text{convPM}(G)\}.$$

If we can optimize over $\text{conv}(\text{PM}(G))$, i.e., solve $\min\{\sum_{e\in E} c_e X_e : X \in \text{conv}(\text{PM}(G))\}$, then we can get a minimum cost perfect matching (because extreme points of $\text{conv}(\text{PM}(G))$ will correspond to indicator vectors of perfect matching).

So, how do we optimize over $\text{conv}(\text{PM}(G))$? Observe that $\text{conv}(\text{PM}(G))$ is a polytope. So, if we have the inequality description of $\text{conv}(\text{PM}(G))$, then we can optimize over it. In the next lecture, we will see an inequality description of $\text{conv}(\text{PM}(G))$.