IE 511: Integer Programming, Spring 2021 **16 Feb, 2021**

Lecture 7: Minimal faces, Extreme points

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We continue with our discussion on the theory of linear inequalities/polyhedral theory. Last week, we saw how to get a minimal description of a polyhedron: (1) the notion of implicit equalities helped us narrow down to the affine-hull containing the polyhedron and (2) the notion of facets helped us in obtaining an irredundant system to describe the polyhedron. Facets are maximal faces distinct from the polyhedron. We saw a characterization of facets in the previous lecture that helped us in obtaining a minimal inequality description of a polyhedron.

Now that we understand maximal faces, it is natural to study *minimal* faces. How do they look like and do they have any significance? Indeed, they play a significant role in solution techniques for linear programming and also in understanding the lucky case of IPs—the case where an optimal solution to the LP-relaxation is also an IP-optimal solution. We will focus on minimal faces in this lecture.

Recap

Let $P = \{x : Ax \le b\}$.

Definition 1. A subset $F \subseteq P$ is a face of P if either $F = P$ or F is the intersection of P with a supporting hyperplane of P.

Theorem 2 (Characterization of faces)**.** *Let $F \subseteq P$. Then, F is a face of P iff $F = \{x \in P : A'x = b'\}$ for some subsystem $A'x \le b'$ of $Ax \le b$.*

7.1 Minimal Faces

We begin with a formal definition of a minimal face.

Definition 3. A *minimal* face is a face that does not contain any other face.

Example: See Figure 7.1 for two kinds of minimal faces.

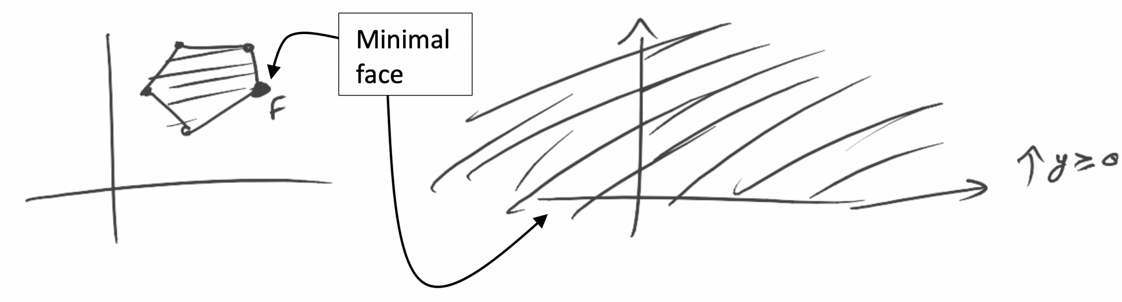


Figure 7.1: Minimal Face: In the right figure, the set $P \cap \{(x, y) : y = 0\}$ is a minimal face.

Observations about minimal faces in the examples in Figure 7.1. Note that the minimal face mentioned in the left figure can be expressed as the set of points satisfying two linear *equations*; similarly, the minimal face mentioned in the right figure can be expressed as the set of points satisfying one linear *equation*, namely $\{(x, y) : y = 0\}$. In particular, in order to describe these minimal faces we need not explicitly specify the fact that we are interested in points of P satisfying these equations; it suffices to say that we are interested in points satisfying the equations. We will now show that this holds in general—minimal faces of a polyhedron should be the set of points satisfying a finite number of linear *equations*.

Definition 4. An *affine subspace* is a set of points satisfying a finite number of linear equations. I.e., an affine subspace is represented by $\{x : Ax = b\}$ for some constraint matrix A and RHS vector b.

Proposition 5. *Let F be a face of P. Then, F a minimal face of P iff it is an affine subspace.*

Proof. Recall that a face of a polyhedron is also a polyhedron. Moreover, by definition, a face is a minimal face iff it has no proper faces. We have the following relationships:

$$\begin{aligned}\text{A polyhedron } Q \text{ has no proper faces} &\iff Q = \{x : Ax = b\} \text{ (by characterization of faces)}\\ &\iff Q \text{ is an affine subspace.}\end{aligned}$$

□

Recall that we characterized maximal faces (i.e., facets) of a polyhedron from the inequality description of the polyhedron in the previous lecture. It is natural to ask if we have a characterization for minimal faces from the inequality description of the polyhedron. Indeed, we do. We will see that Proposition 5 helps us characterize the minimal faces of a polyhedron from its inequality description. The formal characterization is stated below.

Theorem 6 (Characterization of minimal faces)**.** *Let $P = \{x : Ax \le b\}, F \subseteq P, F \neq \emptyset$. Then F is a minimal face of P iff $F = \{x : A'x = b'\}$ for some subsystem $A'x \le b'$ of $Ax \le b$.*

Proof.
$\Longleftarrow$: We note that $F \subseteq P = \{x : Ax \le b\}$ and $F = \{x : A'x = b'\}$. That is, $F = \{x : A'x = b', Ax \le b\}$. Hence, F is obtained by setting some inequalities in $Ax \le b$ to equations. Therefore, by the characterization of faces, we have that F is a face. Also, $F = \{x : A'x = b'\}$ is an affine subspace. By Proposition 5, it means that F is a minimal face.

$\Longrightarrow$: Let F be a minimal face of P. Then F is a face. So $F = \{x : A'x = b', A''x \leq b''\}$ where $A'x \leq b', A''x \leq b''$ is a partitioning of the system $Ax \leq b$. Choose the partitioning that defines F so that the system $A''x \leq b''$ has as few inequalities as possible. With this choice, we will show that there are no inequalities in the system $A''x \leq b''$. For the sake of contradiction, suppose that there is at least one inequality in the system $A''x \leq b''$. Then all inequalities in $A''x \leq b''$ should be irredundant in the system

$$\begin{cases} A''x \leq b'' \\ A'x = b'. \end{cases}$$

If $A''x \leq b''$ is not empty then we get a face $F' \subsetneq F$ by setting some of the inequalities in $A''x \leq b''$ to equations. It implies that F' is a face strictly contained in F which contradicts the minimality of F. Therefore, the system $A''x \leq b''$ is empty and consequently, $F = \{x : A'x = b'\}$. □

We saw that all maximal faces of a polyhedron have the same dimension, namely one less than the dimension of the polyhedron. How about minimal faces? We will now see that all minimal faces of a polyhedron also have the same dimension.

Recall the characterization of faces of a polyhedron: a set $F \subseteq P = \{Ax \leq b\}$ is a face of P iff $F = \{x \in P : A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.

Corollary 6.1. *Suppose $F = \{x : A'x = b'\}$ is a minimal face of $P = \{x : Ax \leq b\}$ with $A'x \leq b'$ being a subsystem of $Ax \leq b$. Then rank(A') = rank(A).*

Proof. We know that $\text{rank}(A') \leq \text{rank}(A)$ because A' is sub-matrix of A.

We now show that $\text{rank}(A') = \text{rank}(A)$: Suppose that the rank of the two matrices are not equal. Then there exists $a_i^T x \leq b_i$ in $Ax \leq b$ with $a_i \notin$ row-space(A'). Now, observe that

$$F \subseteq \{x : A'x = b', a_i^T x \leq b_i\}$$

since $F \subseteq P$. Moreover,

$$\{x : A'x = b', a_i^T x \leq b_i\} \subsetneq \{x : A'x = b'\}$$

since a_i is not in row-space(A'). Thus, we have

$$F \subseteq \{x : A'x = b', a_i^T x \leq b_i\} \subsetneq \{x : A'x = b'\} = F$$

which is a contradiction. □

Corollary 6.1 implies that all minimal faces of a polyhedron have the same dimension. This is due to the following exercise (which was posed in Lecture 5).

Exercise. Consider the affine subspace $K = \{x : A'x = b'\}$. Prove that dimension$(K) = n-$ rank(A').

Minimal faces that contain only one point have a special significance. We will focus on such minimal faces now.

Definition 7. Let P be a polyhedron.

1. A face F of P is a *vertex* if it consists of exactly one point.

2. The polyhedron P is *pointed* if it contains a face that is a vertex.

Example: See Figure 7.2. The polyhedra on the left and in the middle are pointed while the one on the right is not pointed.

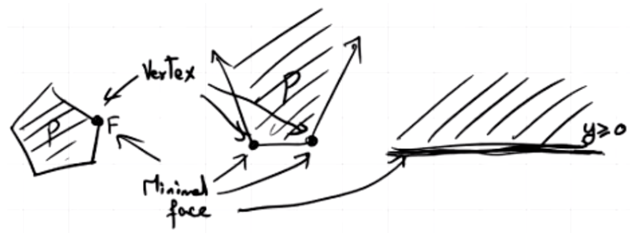


Figure 7.2: Vertex of a polyhedron

Observation 1. If P is a pointed polyhedron, then every minimal face of P is a vertex (because the dimension of all minimal faces are the same).

Observation 2. Every bounded polyhedron is pointed.

Faces of dimension 1 also have a special significance.

Definition 8. 1. A face of dimension 1 is an edge.

2. If an edge is a half-line then it is an *extreme ray.*

See example below.

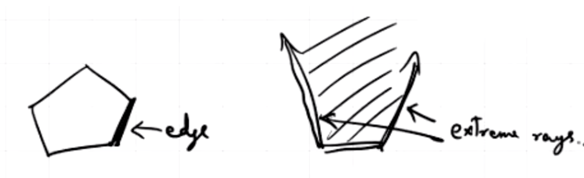


Figure 7.3: Edge

7.2 Extreme Points

We now define the notion of extreme points. The characterization of extreme points is the fundamental result that drives the Simplex method for solving linear programs.

Definition 9. A point $\bar{x}$ in a polyhedron P is an *extreme point* if it cannot be expressed as a convex combination of two distinct points in P. I.e., $\bar{x}$ is an extreme point of P if

$$\bar{x} = \lambda x + (1-\lambda) y \ \text{ for } \ x, y \in P, \ 0 \le \lambda \le 1 \implies \bar{x} = x = y.$$

Example: See Figure 7.4.

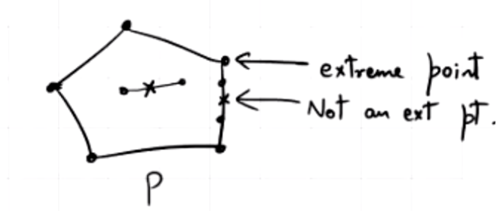


Figure 7.4: Extreme point

From the examples, it appears like extreme points are vertices and vice-versa. We now state this formally. The proof of this fact relies on the characterization of faces that we have seen already.

Theorem 10 (Characterization of extreme points)**.** *Let* $P = \{x \in \mathbb{R}^n : Ax \leq b\}, \bar{x} \in P$*. Then the following are equivalent:*

(i) $\bar{x}$ *is an extreme point of* P*.*

(ii) If $A'x \leq b'$ *is the subsystem of* $Ax \leq b$ *satisfied as equality by* $\bar{x}$*, then rank* $(A') = n$

(iii) $F = \{\bar{x}\}$ *is a face of* P *(with dim*$(F) = 0$*).*

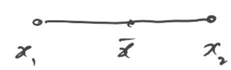
(iv) There exist $c \in \mathbb{R}^n$ *such that* $\bar{x}$ *is the unique optimum solution of the LP* $\max\{c^T x : x \in P\}$*.*

Proof of characterization of extreme points

Proof. 1. $(i) \implies (ii)$:

Let $\bar{x}$ be an extreme point of P and let $A'x \leq b'$ be the subsystem of $Ax \leq b$ with $A'\bar{x} = b'$. Also, let $A''x \leq b''$ be the remaining inequalities in $Ax \leq b$. If $\text{rank}(A') < n$ (rank of A' cannot be more than n because A has only n columns), then the null-space of A' has dimension at least one, and consequently, there exists a point $y \neq 0$ such that $A'y = 0$.

Let $x_1 := \bar{x} + \epsilon y$ and $x_2 := \bar{x} - \epsilon y$. Then there exists $\epsilon > 0$ such that $x_1, x_2 \in P$ (because $A''\bar{x} < b''$, $A'x_1 = b'$, $A'x_2 = b'$). Moreover, if $\epsilon > 0$, then $x_1 \neq x_2$.



Now $\bar{x} = \left(\dfrac{x_1 + x_2}{2}\right)$ which contradicts $\bar{x}$ being an extreme point of P.

2. $(ii) \implies (iii)$:

Let $A'x \leq b'$ be a subsystem of $Ax \leq b$ satisfied as equality by $\bar{x}$. It implies that there exist n linearly independent rows of A'—say $a_1^T, \ldots, a_n^T$. Then $\text{rank}(A') = n$ and the system

$$\begin{cases} a_1^T x = b_1 \\ \vdots \\ a_n^T x = b_n \end{cases} \tag{7.1}$$

has a unique solution which is $\bar{x}$.

Therefore $F = \{x \in P : a_1^T x = b_1, \ldots, a_n^T x = b_n\}$ is a face of P as it is obtained by setting some inequalities to equations. By uniqueness of the solution $\bar{x}$ to system (7.1) we have that $\{x \in P : a_1^T x = b_1, \ldots, a_n^T x = b_n\} = \{\bar{x}\}$.

3. $(iii) \implies (iv)$:

Let $F = \{\bar{x}\}$ be a face of P with $\dim(F) = 0$. Then there exists a valid inequality $c^T x \leq \delta$ for P such that $\{\bar{x}\} = \{x \in P : c^T x = \delta\}$. It implies that $\bar{x}$ is the unique optimum of the LP $\max\{c^T x : x \in P\}$.

4. $(iv) \implies (i)$:

Let $\bar{x}$ be the unique optimum of $\max\{w^T x : x \in P\}$. If $\bar{x} = \lambda x + (1 - \lambda)y$ for some $x, y \in P, x, y \neq \bar{x}$, $0 < \lambda < 1$, then $c^T\bar{x} = \lambda c^T x + (1-\lambda)c^T y \leq \lambda c^T \bar{x} + (1-\lambda)c^T\bar{x} = c^T\bar{x}$. Therefore, $c^T\bar{x} = c^T x = c^T y$ which means that x and y are also optimum solutions. This contradicts the uniqueness of $\bar{x}$.

□

We emphasize on a few significant consequences of Theorem 10:

- $(i) \equiv (iii)$ tells us that vertices correspond to extreme points and vice-versa. Hence, vertices cannot be expressed as a convex combination of two distinct points in the polyhedron.
- $(i) \equiv (iv)$ is a formal statement of the intuition that every extreme point can be singled out by optimizing along a suitable direction over the polyhedron.

An important consequence of Theorem 10 is that it narrows the search space to extreme points in order to solve a linear program. We formalize this next.

Corollary 10.1. *If P is pointed and the LP $\max\{c^T x : x \in P\}$ has an optimal solution then it has an optimal solution which is also an extreme point.*

Proof. Optimal solutions correspond to a face F of P. Since P is pointed, the face F has a vertex. Therefore, there exists an optimal solution which is a vertex of P. By Theorem 10 (equivalence between (i) and (iii)), this vertex is also an extreme point of P. □

Corollary 10.1 is exploited by the simplex algorithm to solve LPs. The simplex algorithm systematically searches the extreme points for an optimal solution. Why is this sufficient? Well, Corollary 10.1 shows that this is indeed sufficient. But, what if the search space is still infinite? The next corollary (Corollary 10.2) tells us that this search space is in fact finite. Thus, if the simplex algorithm does not cycle during its search over extreme points, then it will terminate within a finite number of iterations!

Corollary 10.2. *Every polyhedron has a finite number of extreme points.*

Proof. Every extreme point is a face (by Theorem 10). Recall that the number of faces is finite. □

Let us turn our focus back to IPs: $\max\{c^T x : x \in \mathcal{S}\}$, where $\mathcal{S}$ encodes the set of feasible solutions of the IP—e.g., $\mathcal{S}$ could be $P \cap \mathbb{Z}^n$ for some polyhedron P. Now that we understand extreme points, we will now argue that it suffices to solve

$$\max\{c^T x : x \in \text{ convex-hull}(\mathcal{S})\}.$$

Lemma 10.1. *Let $\mathcal{S} \subseteq \mathbb{R}^n$. Then all extreme points of convex-hull$(\mathcal{S})$ are in $\mathcal{S}$.*

Proof. Let $\bar{x}$ be an extreme point of convex-hull$(\mathcal{S})$. Then $\bar{x} \in$ convex-hull$(\mathcal{S})$ but $\bar{x}$ is not a convex combination of two points in convex-hull$(\mathcal{S})$. By definition, it means that $\bar{x} = x$ for some $x \in \mathcal{S}$.

□

Lemma 10.2. *Let $\mathcal{S} \subseteq \mathbb{R}^n, c \in \mathbb{R}^n$. Then*

$$\max\{c^T x : x \in \mathcal{S}\} = \max\{c^T x : x \in \textit{convex-hull}(\mathcal{S})\}.$$

Proof. We will show equality by arguing the inequality in both directions. $LHS \leq RHS$: Follows since RHS is a relaxation of LHS.

$LHS \geq RHS$: Let $\bar{x}$ be an optimum solution to $\max\{c^T x : x \in \text{convex-hull}(\mathcal{S})\}$. Since $\bar{x}$ is in convex-hull($\mathcal{S}$), it implies that $\bar{x} = \sum_{i=1}^{t} \lambda_i x^i$ for some $x^1, \dots, x^t \in \mathcal{S}$ and $\lambda_i \geq 0$ $\forall i \in [t]$ with $\sum_{i=1}^{t} \lambda_i = 1$. Therefore,

$$c^T \bar{x} = \sum_{i=1}^{t} \lambda_i c^T x^i \leq \max\{c^T x^i : i \in [t]\} \leq \max\{c^T x^i : i \in \mathcal{S}\}.$$

$\square$

7.3 Representation of Polyhedra

A polyhedron can be represented by different means. Each representation has its advantages and disadvantages. So far, we have seen the following two representations of a polyhedron:

1. Facet description (or inequality description)
2. Sum of a convex-hull and a cone description (see figure 7.5)

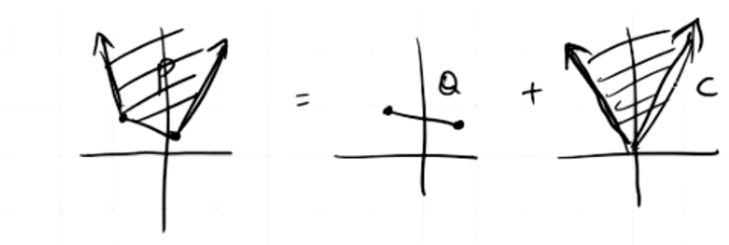


Figure 7.5: Polyhedron Description

We will see one more representation of a polyhedron via minimal faces:

- Extreme point and extreme ray description.

What is this description? Recall that a polyhedron P can be decomposed as $P = Q + C$ where Q is a polytope and C is a polyhedral cone. We can be more specific about the polytope Q and the cone C.

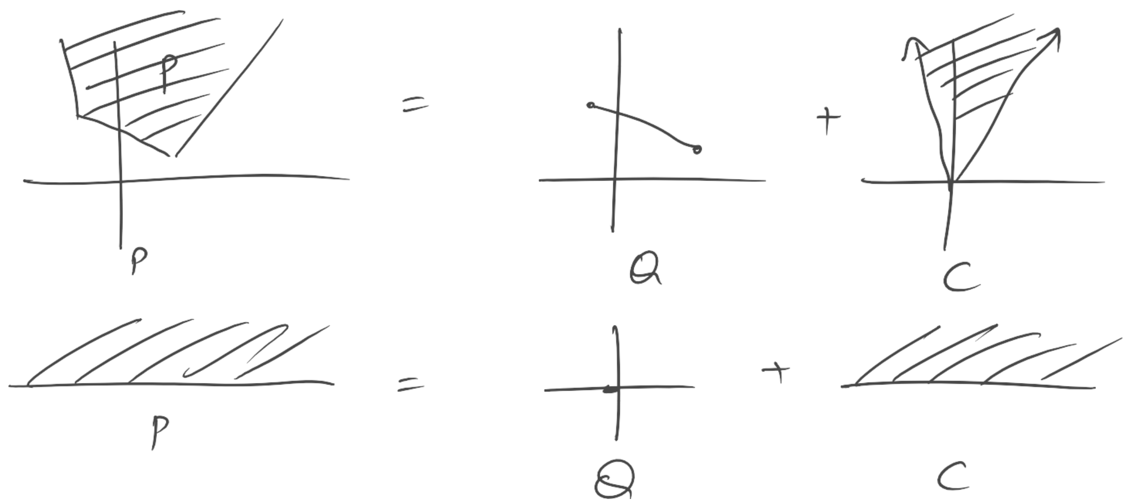


Figure 7.6: Example of the description polyhedra

In particular, if P is a polyhedron, then the top example of Figure 7.6 seems to suggest taking Q to be the convex-hull of extreme points of P and taking C to be the cone generated by the extreme rays of P. Even though this seems to work in the top example, it fails in the bottom example of Figure 7.6 since P has no extreme points (i.e., since P is not pointed). To account for examples like the bottom one in Figure 7.6, we have the following general result.

Theorem 11. *Let $F_1, \dots, F_r$ be minimal faces of a polyhedron $P = \{x : Ax \leq b\}$ with points $x_i \in F_i$ $\forall i \in [r]$. Then $P =$ convex-hull$\{x_1, \dots, x_r\} + C$ where $C = \{x : Ax \leq 0\}$.*

Note that the inequality description of P immediately tells us the cone that we need to consider. Specializing the theorem for pointed polyhedra gives the following corollary.

Corollary 11.1. *If P is a pointed polyhedron, then P can be written as the sum of the convex-hull of extreme points of P and the cone generated by the extreme rays of P.*