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Lecture 6: Faces, Facets

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

Recall that we are attempting to get a minimal description of a polyhedron. We saw how to obtain a minimal set of constraints to eliminate *affine* subspaces that do not intersect the polyhedron. Today, we will aim to get an *irredundant system* to describe a polyhedron.

6.1 Redundant Inequalities

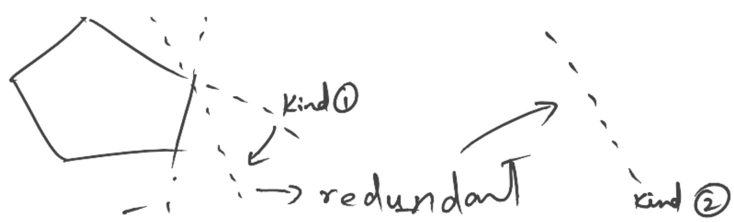
We would like to understand which inequalities are *necessary* to describe a polyhedron.

An inequality $a_i^T x \leq b_i$ in the system $Ax \leq b$ is said to be *redundant* if it is implied by the rest of the inequalities in the system. What does 'implied by' mean mathematically? Here is the formal definition.

Definition 1. Consider the system $Ax \leq b$.

1. An inequality $a_i^T x \leq b_i$ in the system $Ax \leq b$ is redundant if it is a non-negative linear combination of the other inequalities in $Ax \leq b$.
2. An *irredundant system* has no redundant inequality.

We seek an irredundant system to describe a polyhedron. There are two kinds of redundant inequalities as seen in the figure below.



It is helpful to distinguish between the two kinds as follows:

Definition 2. Let $P = \{x : Ax \leq b\}$.

1. An inequality $c^T x \leq \delta$ is *valid* for P if $c^T \bar{x} \leq \delta \ \forall \bar{x} \in P$.
2. A hyperplane $\{x : c^T x = \delta\}$ is a *supporting hyperplane* of P if $\delta = \max\{c^T x : Ax \leq b\}$ and $c \neq 0$. Equivalently, the hyperplane $\{x : c^T x = \delta\}$ is a *supporting hyperplane* of P if $c^T x \leq \delta$ is valid for P and the hyperplane $\{x : c^T x = \delta\}$ intersects P.

Note that both kind 1 and kind 2 in the figure are valid, but kind 1 gives a supporting hyperplane while kind 2 does not.

Using these tools, we will define *faces* of a polyhedron. We will later see that *maximal faces* give a minimal inequality description for the polyhedron and *minimal faces* correspond to the *vertices* of the polyhedron.

6.2 Faces

Let P be a polyhedron.

Definition 3. A set $F \subseteq P$ is a *face* of P if either $F = P$ or if F is the intersection of P and a supporting hyperplane of P.

Example: See Figure 6.1.

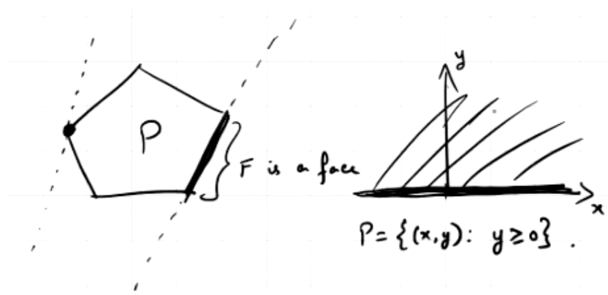


Figure 6.1: Example of faces of a polyhedron: Left polyhedron has 11 faces two of which are marked. Right polyhedron has only two faces: $\{(x, y) : y = 0\}$ and the polyhedron itself.

Note that by definition, a face of a polyhedron is also a polyhedron (i.e., a face of a polyhedron is a set of points satisfying a finite number of linear inequalities). Next we will see a convenient way to understand the faces of a polyhedron P from the inequality description of P.

Theorem 4 (Characterization of faces)**.** *Let $P = \{x : Ax \leq b\}$ and let $F \subseteq P$. Then, F is a face of P iff $F \neq \emptyset$ and $F = \{x : x \in P, A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$.*

Proof. If $F = P$, then the statement is immediate. We consider the case where $F \neq P$.

$\Longrightarrow$:

Let F be a face of P. Then $F = P \cap \{x : c^T x = \delta\}$ for some $c \neq 0$ and $\delta = \max\{c^T x : x \in P\}$ where δ is finite. By duality theorem we have that $\delta = \min\{y^T b : y^T A = c^T, y \geq 0\}$. Let y^* be an optimal solution for $\min\{y^T b : y^T A = c^T, y \geq 0\}$. Then $y^* \neq 0$ because $c \neq 0$.

Let $A'x \leq b'$ be a subsystem of $Ax \leq b$ corresponding to the positive components of y^*. Since $y^* \neq 0$, the subsystem is non-empty. By complementary slackness conditions we have that a point x^* is optimal for $\max\{c^T x : x \in P\}$ iff $x^* \in P$ and $A'x^* = b'$. Also, $x \in F$ iff x is optimal for $\max\{c^T x : x \in P\}$. Therefore, $F = \{x : x \in P, A'x = b'\}$.

$\Longleftarrow$:

Suppose $F = \{x \in P : A'x = b'\}$ for some subsystem $A'x \le b'$ of $Ax \le b$ and $F \ne \emptyset$. Let c be the sum of the linearly independent rows of A' and δ be the sum of the corresponding rows of b'. Then $c \ne 0$. We will show the following two claims which together imply that F is a face of P. □

Claim 4.1.

$$F = \{x \in P : c^T x = \delta\}.$$

Proof. Let $\bar{x} \in P$. We have that

- If $\bar{x} \in F$, then $c^T \bar{x} = \delta$ by the choice of c and δ. Hence, $\bar{x} \in \{x \in P : c^T x = \delta\}$
- Suppose $\bar{x} \notin F$. So, $\bar{x} \in P \setminus F$. Then, $A\bar{x} \le b$ since $\bar{x} \in P$. Since $\bar{x} \notin F$, there exists an inequality $a_i^T x \le b_i$ in the system $A'x \le b'$ for which $a_i^T \bar{x} < b_i$. Consequently, $c^T \bar{x} < \delta$. Hence $\bar{x} \notin \{x \in P : c^T x = \delta\}$.

□

Claim 4.2. *$\{x : c^T x = \delta\}$ is a supporting hyperplane of P.*

Proof. Since $c^T x \le \delta$ is a non-negative combination of the inequalities in $A'x \le b'$, we have that $c^T x \le \delta$ is valid for P. Also, since $F = \{x \in P : c^T x = \delta\}$ by the previous claim and $F \ne \emptyset$, we have that $P \cap \{x : c^T x = \delta\} \ne \emptyset$. So, the hyperplane $\{x : c^T x = \delta\}$ is a supporting hyperplane of P. □

Interpreting Theorem 4. The characterization of faces in Theorem 4 tells us that all faces of a polyhedron $P = \{x : Ax \le b\}$ are obtained by *converting* some of the inequalities in the system $Ax \le b$ into equations. In particular, this implies that the number of faces in a polyhedron is finite.

Corollary 4.1. *A polyhedron has finite number of faces.*

The characterization of faces also implies that the face of a face of a polyhedron is also a face of the original polyhedron and vice-versa.

Corollary 4.2. *Let F be a face of P and $F' \subseteq F$. Then, F' is a face of P iff F' is a face of F.*

6.3 Facets

We note that faces do not give a minimal description of a polyhedron as an inequality $c^T x \le \delta$ corresponding to a supporting hyperplane $\{x : c^T x = \delta\}$ of P may not be needed to describe P (e.g., see Figure 6.2).

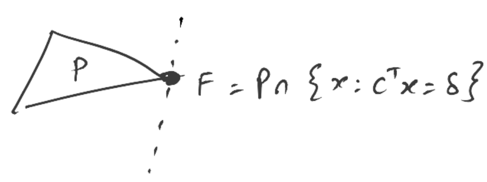


Figure 6.2: A polyhedron and an inequality that determines a face, but the inequality will not be in the minimal description of P.

However, we will next see that *maximal faces* give a minimal description of a polyhedron.

Definition 5. Let P be a polyhedron.

1. A *proper face* is a face distinct from P.
2. A *facet* is a maximal face distinct from P. I.e., a facet is a face that is not contained in any other *proper* face of P.

Example: See Figure 6.3.

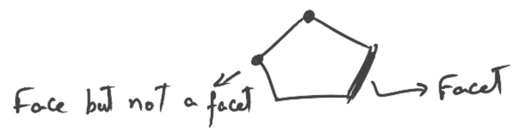


Figure 6.3: Facet

Let us see how to obtain the facets of a polyhedron from the inequality description of the polyhedron. Recall that $A^{=}x \leq b^{=}$ are the implicit equalities of the system $Ax \leq b$ and $A^{+}x \leq b^{+}$ are the remaining inequalities of $Ax \leq b$.

Theorem 6 (Characterization of facets)**.** *Let $P = \{x : Ax \leq b\}$. Suppose no inequality in $A^{+}x \leq b^{+}$ is redundant in $Ax \leq b$. Then there is a one to one correspondence between facets of P and the inequalities in $A^{+}x \leq b^{+}$ given by $F = \{x \in P : a_i^T x = b_i\}$ for facets F and inequalities $a_i^T x \leq b_i$ in $A^{+}x \leq b^{+}$. In words: if the non-implicit inequalities form an irredundant system, then the facets are obtained by turning exactly one of the non-implicit inequalities into an equation.*

Proof. We prove the theorem by showing Lemmas 6.1 and 6.2. □

Lemma 6.1. *Each facet of P can be represented as $\{x \in P : a_i^T x = b_i\}$ for some $a_i^T x \leq b_i$ in $A^{+}x \leq b^{+}$.*

Proof. Let F be a facet of P. Then, F is a face and $F \neq P$. By the characterization of faces, we have that $F = \{x \in P : A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$. We may assume that $A'x \leq b'$ is a subsystem of $A^{+}x \leq b^{+}$ (since discarding those inequalities in $A'x \leq b'$ that are from $A^{=}x \leq b^{=}$ will not change F).

Let $a_i^T x \leq b_i$ be an inequality in $A'x \leq b'$. Consider $F' = \{x \in P : a_i^T x = b_i\}$. The set F' is a face of P by characterization of faces (Theorem 4). Moreover, $F' \subsetneq P$: containment is by definition and strict containment is because the inequality $a_i^T x \leq b_i$ is in the system $A^{+}x \leq b^{+}$ and is hence, not an implicit equality. We note that $F \subseteq F' \subsetneq P$. We have that F' is a face of P that is distinct from P. But F is a facet, i.e., a maximal face of P that is distinct from P. So, we should have that $F = F' = \{x \in P : a_i^T x = b_i\}$. □

Lemma 6.2. *For each inequality $a_i^T x \leq b_i$ in $A^{+}x \leq b^{+}$, the set $F = \{x \in P : a_i^T x = b_i\}$ is a facet of P.*

Proof. Let $a_i^T x \le b_i$ be an inequality in $A^+x \le b^+$ and let $A'x \le b'$ be the other inequalities in $A^+x \le b^+$. Let $F = \{x \in P : a_i^T x = b_i\}$. By characterization of faces (Theorem 4), the set F is a face of P. Since $a_i^T x \le b_i$ is not in $A^= x \le b^=$, we have that $F \subsetneq P$.

We need to show that F is a facet of P, i.e., we need to show that the only face of P containing the face F is P. We will use the fact that there is an interior point in F to prove this. The formal way of saying that there is an interior point in F is stated below:

Claim 6.1. *There exists a point x^0 such that $A^= x^0 = b^=, A'x^0 < b', a_i^T x^0 = b_i$.*

Proof. Recall that there is an interior point in P. Let x^1 be such a point, i.e., it satisfies

$$A^= x^1 = b^=, \quad A'x^1 < b', \quad a_i^T x^1 < b_i.$$

Since $a_i^T x \le b_i$ is not redundant in $Ax \le b$, there exists a point x^2 such that

$$A^= x^2 = b^=, \quad A'x^2 \le b', \quad a_i^T x^2 > b_i.$$

Consequently, there exists a point x^0 on the line segment between x^1 and x^2 for which

$$A^= x^0 = b^=, \quad A'x^0 < b', a_i^T x^0 = b_i.$$

□

Claim 6.1 implies that the only face containing F is P: suppose for contradiction that there exists a face F' such that $F \subseteq F' \subsetneq P$. Then $F' = \{x \in P : A''x = b''\}$ for some subsystem $A''x \le b''$ of $A^+x \le b^+$ by the characterization of faces. Then, $x^0 \in F$. Since $F \subseteq F'$ we have that $x^0 \in F'$. This implies that the system $A''x \le b''$ is exactly $a_i^T x \le b_i$ and hence $F = F'$. Therefore, F is a facet. □

Theorem 6 shows that if a system has no implicit equations and is irredundant, then it is a unique minimal description of P.

Corollary 6.1 (Informal)**.** *Let $P = \{x : Ax \le b\}$. If P is full dimensional and $Ax \le b$ is irredundant then $Ax \le b$ is the unique minimal "representation" of P (up to multiplication of inequalities by positive scalars), i.e., each inequality is necessary and the system $Ax \le b$ is sufficient.*

Note: Theorem 6 tells that facets are necessary and sufficient to describe a polyhedron.

Let us see some consequences of the characterization of facets given by Theorem 6.

Corollary 6.2. *Each proper face of P is the intersection of facets of P.*

The next corollary tells us that all facets of a polyhedron have the same dimension.

Corollary 6.3. *If F is a facet of P then $dim(F) = dim(P) - 1$.*

Proof. Recall that

$$\dim(P) = n - \operatorname{rank}\left(A^{=}\right).$$

Hence, we have that

$$\dim(F) = n - \operatorname{rank}\left(\begin{bmatrix} A^{=} \\ a_i^T \end{bmatrix}\right) = \dim(P) - 1.$$

The last equality holds because $a_i^T x = b_i$ is not redundant. $\square$

Note 1: To show that an inequality is necessary in the minimal description of a polyhedron, it is sufficient to argue that it is a facet.

Note 2: To show that F is a facet of P, it is sufficient to prove that F is a face of P and $\dim(F) = \dim(P) - 1$.