IE 511: Integer Programming, Spring 2021 **9 Feb, 2021**

Lecture 5: Dimension of a polyhedron

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Recall that we dived into the theory of linear inequalities/polyhedral theory in order to gain solid footing towards identifying the lucky case of the IP (i.e., solving the LP-relaxation gives an IP-optimal solution). We began with the fundamental theorem of linear inequalities, which in turn, helped us understand cones. Cones are fundamental building blocks of polyhedron. We proved that polyhedral cones are finitely-generated cones and vice-versa—this result allows us to move between linear inequality description and non-negative linear combination description of polyhedral cones. Next, we viewed a polyhedron as a slice of a cone to argue that every polyhedron is the Minkowski sum of a polytope and a polyhedral cone. Consequently, every polytope is simply a bounded polyhedron. This week, we will focus on understanding minimal description of a polyhedron.

Minimal Description of a Polyhedron

A *minimal* inequality description for a polyhedron P is a system $Ax \leq b$ such that $P = \{x : Ax \leq b\}$ and moreover, for any subsystem $A'x \leq b'$ of $Ax \leq b$, the set $\{x : A'x \leq b'\}$ is a strict superset of $\{x : Ax \leq b\}$.

There are two aspects of getting a minimal description:

1. We need minimal constraints to remove subspaces that do not intersect the polyhedron, i.e., minimal constraints that are sufficient to identify the *affine subspace* containing the polyhedron. See Example 1 for the intuition. We will formally define *affine subspaces* later.

   Example 1: Consider the polyhedron

   $$P = \{(x, y, z) : x + y + z \leq 4, x + y \geq 3, 1 \leq y \leq 2, z \geq 1\}.$$

   The description of P can be re-written as

   $$P = \{(x, y, z) : 1 \leq y \leq 2, x + y = 3, z = 1\}. \tag{5.1}$$

   We have thus obtained a description for P as a subset of $\mathbb{R}^3$ for which $z = 1$ and $x + y = 3$. In particular, the polyhedron P does not intersect any point $(x, y, z) \in \mathbb{R}^3$ for which $z \neq 1$ or $x + y \neq 3$. The constraints $z = 1$ and $x + y = 3$ removes all such points and tells us that P is contained among those points for which $z = 1$ and $x + y = 3$. As a consequence, we observe that the inequality $x + y + z \leq 4$ is unnecessary/redundant and can be removed.

2. We need an irredundant system to describe the polyhedron. See Example 2 for a geometric intuition for redundant constraints. We will formally define *redundant constraints* and an *irredundant system* later.

   Example 2: Consider the polyhedron below for which we have redundant constraints. Dropping such redundant constraints does not change the polyhedron.

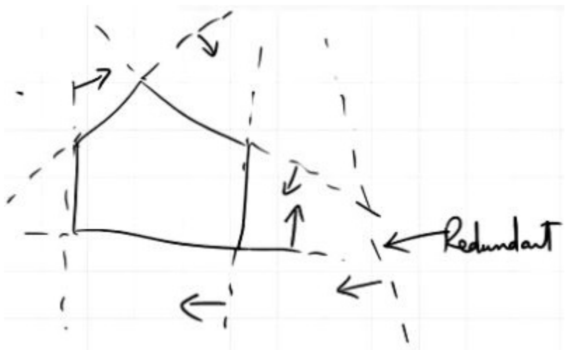


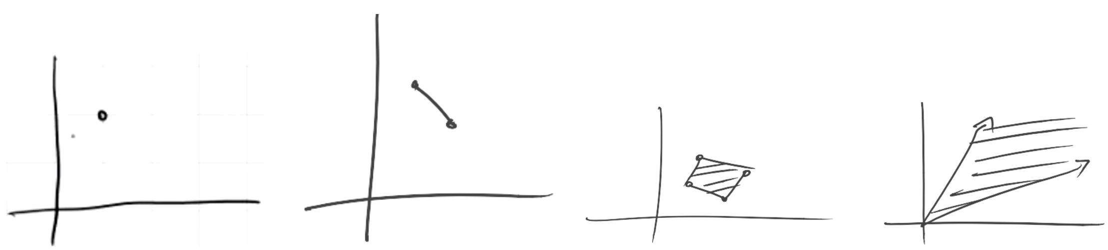
Figure 5.1: A redundant constraint for a polyhedron

To address the first aspect of getting a minimal description of a polyhedron we need to understand the subspace containing the polyhedron. To achieve this, it will be helpful to understand the dimension of the (*affine*) subspace containing the polyhedron.

5.1 Dimension of a Polyhedron

Intuitively, the dimension of a set $K \subseteq \mathbb{R}^n$ (not necessarily a polyhedron) tells us the number of degrees of freedom. See the example below for intuition.

Example: Consider the number of degrees of freedom in the following figures as the intuitive notion of dimension of a polyhedron.



(a) Number of degrees of freedom = 0 (b) Number of degrees of freedom = 1 (c) Number of degrees of freedom = 2 (d) Number of degrees of freedom = 2

Figure 5.2: Dimension of a polyhedron

Now we will formally define the dimension of a set $K \subseteq \mathbb{R}^n$. An important property that we need from the definition of *dimension*(K) is that it should be invariant under translation—i.e., if Dimension(K) is d, then Dimension$(K + \{a\})$ (where the addition is Minkowski sum) should still be d for any vector a. To ensure this property, we will use the notion of affine independence (as opposed to linear independence). We will later see that affine independence is invariant under translation.

Definition 1. Vectors $x^1, \dots, x^m \in \mathbb{R}^n$ are

1. *affinely independent* if the only solution to the system

$$\begin{aligned} \lambda_1 x^1 + \cdots + \lambda_m x^m &= 0 \\ \lambda_1 + \cdots + \lambda_m &= 0 \end{aligned}$$

is $\lambda_1 = \cdots = \lambda_m = 0$.

2. *linearly independent* if the only solution to the system

$$\lambda_1 x^1 + \cdots + \lambda_m x^m = 0$$

is $\lambda_1 = \cdots = \lambda_m = 0$.

Linear independence implies affine independence by definition.

Proposition 2. *$x^1, \ldots, x^m$ are linearly independent $\implies$ $x^1, \ldots, x^m$ are affinely independent.*

In contrast, affine independence does not necessarily imply linear independence. Here is an example.

Example: In Figure 5.4, a_1 and a_2 are affinely independent, i.e.,

$$\lambda_1 + 2\lambda_2 = 0, \quad \lambda_1 + \lambda_2 = 0 \quad \implies \quad \lambda_1 = \lambda_2 = 0.$$

However, a_1 and a_2 are not linearly independent since $a_2 = 2a_1$.

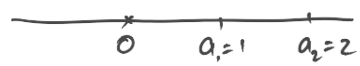


Figure 5.3: Affine independence does not imply linear independence.

Linear independence and affine independence are closely related as shown by the following proposition.

Proposition 3. *The following are equivalent:*

1. *$x^1, \ldots, x^m$ are affinely independent.*
2. *$x^2 - x^1, \ldots, x^m - x^1$ are linearly independent.*
3. *$\binom{x^1}{1}, \ldots, \binom{x^m}{1}$ are linearly independent.*

Proof. **Exercise**. □

Affine independence is invariant under translation as shown below:

Proposition 4. *Let $x^1, \ldots, x^m, w \in \mathbb{R}^n$. If $x^1, \ldots, x^m$ are affinely independent, then $x^1 - \omega, \ldots, x^m - \omega$ are affinely independent.*

Proof. Suppose $\lambda_1(x^1 - \omega) + \cdots + \lambda_m(x^m - \omega) = 0$ and $\lambda_1 + \cdots + \lambda_m = 0$. This implies that

$$\begin{cases} \lambda_1 x^1 + \cdots + \lambda_m x^m = (\sum_{i=1}^m \lambda_i) \cdot \omega \\ \lambda_1 + \cdots + \lambda_m = 0 \end{cases}$$
$$\implies \begin{cases} \lambda_1 x^1 + \cdots + \lambda_m x^m = 0 \\ \lambda_1 + \cdots + \lambda_m = 0. \end{cases}$$

By affine indepenence of $x^1, \ldots, x^m$, this implies that

$$\lambda_1 = \cdots = \lambda_m = 0$$

Thus, we have shown that $x^1 - \omega, \ldots, x^m - \omega$ are affinely independent. □

With this understanding of affine independence, we define the dimension of a set.

Definition 5. The dimension of a set $K \subseteq \mathbb{R}^n$ is one less than the maximum cardinality of an affinely independent subset of K.

Example: See Figure 5.2 and note that the dimension (under this definition) and the intuitive notion of degrees of freedom coincide.

Note: Dimension is invariant under translation since affine independence is invariant under translation.

Dimension of polyhedron? The above definition of dimension is not helpful to understand the dimension of a polyhedron as a polyhedron is given by its inequality description. Recall that a polyhedron is of the form $P = \{x : Ax \leq b\}$. It would be nice to be able to compute the dimension of a polyhedron from its inequality description:

Question 0. Given the constraint matrix A and the RHS vector b, can we compute the dimension of $\{x : Ax \leq b\}$?

Question 1. Given the constraint matrix A and the RHS vector b, can we compute the dimension of $\{x : Ax = b\}$?

We can answer question 1 using the definition of dimension via the following exercise:

Exercise. $\text{Dimension}(\{x : Ax = b\}) = \text{Dimension}(\{x : Ax = 0\}) = n - \text{rank}(A)$.

Now, we return to the question of computing the dimension of a polyhedron from its inequality description. For this, we need to understand the *affine hull* containing the polyhedron. Let us see how to do this.

We recall the following definitions: (we will be interested in the definition of affine-hull; note the similarity in the definition of affine-hull to that of linear-hull and convex-hull)

Definition 6. Let $S \subseteq \mathbb{R}^n$.

1. The *linear-hull* of a set $S \subseteq \mathbb{R}^n$ is the set of all linear combinations of finitely many points in S, i.e.,

$$\text{linear-hull}(S) := \left\{ \sum_{i=1}^{t} \lambda_i a_i : t \geq 1 \text{ and finite, } a_1, \ldots, a_t \in S,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R} \right\}.$$

2. The *affine-hull* of a set $S \subseteq \mathbb{R}^n$ is the set of all affine combinations of finitely many points in S, i.e.,

$$\text{affine-hull}(S) := \left\{ \sum_{i=1}^{t} \lambda_i a_i : t \geq 1 \text{ and finite, } a_1, \ldots, a_t \in S,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R},\ \sum_{i=1}^{t} \lambda_i = 1 \right\}.$$

3. The *convex-hull* of a set $S \subseteq \mathbb{R}^n$ is the set of all convex combinations of finitely many points in X, i.e.,

$$\text{convex-hull}(S) := \left\{ \sum_{i=1}^{t} \lambda_i a_i : t \geq 1 \text{ and finite, } a_1, \ldots, a_t \in S,\ \lambda_1, \ldots, \lambda_t \geq 0,\ \sum_{i=1}^{t} \lambda_i = 1 \right\}.$$

Example: Suppose $S = \{a_1, a_2\}$ is as shown in Figure 5.4. Then, linear-hull(S) is $\mathbb{R}^2$, affine-hull(S) is the line joining a_1 and a_2, and convex-hull(S) is the line segment joining a_1 and a_2.

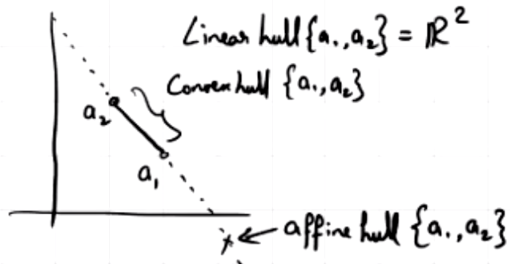


Figure 5.4: Affine-hull(S) for $S = \{a_1, a_2\}$

Next, we need the notion of an implicit equality in a system $Ax \leq b$.

Definition 7. Consider the system of inequalities $Ax \leq b$. An inequality $a_i^T x \leq b_i$ in $Ax \leq b$ is an *implicit equality in the system* $Ax \leq b$ if

$$a_i^T \bar{x} = b_i \ \forall \bar{x} \in \{x : Ax \leq b\}.$$

Example: Suppose $P := \{(x, y, z) : x+y+z \leq 4,\ x+y \geq 3,\ z \geq 1,\ 1 \leq y \leq 2\}$. Then $x+y+z \leq 4$, $x + y \geq 3$, and $z \geq 1$ are the implicit equalities in the system of inequalities defining P.

The following notation will be crucial in understanding the dimension of a polyhedron from its constraint matrix and RHS vector.

Notation. For a system $Ax \leq b$, let

1. $A^= x \leq b^=$ be the system of implicit equalities in the system $Ax \leq b$ and
2. $A^+ x \leq b^+$ be the system of remaining inequalities in the system $Ax \leq b$.

Note that the system $Ax \leq b$ is partitioned in two systems $A^= x \leq b^=$ and $A^+ x \leq b^+$.

Here is a nice reason to do this partitioning of $Ax \leq b$. Geometrically, we have the simple fact that a non-empty polyhedron P always contains an interior point. How do we express this fact algebraically? The following proposition about implicit inequalities and the remaining inequalities allows us to express this fact algebraically:

Proposition 8. *Let $P = \{x : Ax \leq b\}$. Then, there exists a point $\bar{x} \in P$ such that*

$$A^= \bar{x} = b^=,$$
$$A^+ \bar{x} < b^+.$$

I.e., $\bar{x}$ satisfies all inequalities in $A^+ x \leq b^+$ in a strict fashion and all inequalities in $A^= x \leq b^=$ as equations.

Proof idea. We can assume that the polyhedron is non-empty (otherwise, there is nothing to prove). We know that there exists $\bar{x}$ such that

$$A^= \bar{x} = b^=, A^+ \bar{x} \leq b^+, a_i^T \bar{x} = b_i \text{ for some inequality } a_i^T x \leq b_i \text{ in } A^+ x \leq b^+.$$

Since the inequality $a_i^T x \le b_i$ is in $A^+ x \le b^+$ but not in $A^= x \le b^=$, it follows that there exists a point $x_0 \in P$ for which $a_i^T x_0 < b_i$. That is, we have a point x_0 such that

$$A^= x_0 = b^=, A^+ x_0 \le b^+, a_i^T x_0 < b_i.$$

Now move from $\bar{x}$ towards x_0, i.e., reset $\bar{x}$ to be a point in the interior of the line segment joining $\bar{x}$ and x_0, and repeat the same argument. □

With these notations, we can obtain the affine-hull of a polyhedron from its inequality description as described in the lemma below.

Lemma 9. *Let $P = \{x : Ax \le b\}$. Then*

$$\textit{affine-hull}(P) = \{x \in \mathbb{R}^n : A^= x = b^=\} = \{x \in \mathbb{R}^n : A^= x \le b^=\}.$$

Proof. We prove this lemma by proving three containments:

1. affine-hull$(P) \subseteq \{x : A^= x = b^=\}$:

   By definition we have that $P \subseteq \{x : A^= x = b^=\}$. Let $\bar{x} \in$ affine-hull(P). Then

   $$\bar{x} = \lambda_1 x^1 + \cdots + \lambda_t x^t \text{ for some } x^1, \ldots, x^t \in P,\ \lambda_1, \ldots, \lambda_t \in \mathbb{R},\ \sum_{i=1}^{t} \lambda_i = 1$$

   $$\implies A^= \bar{x} = \lambda_1 A^= x^1 + \cdots + \lambda_t A^= x^t = \left(\sum_{i=1}^{t} \lambda_i\right) b^= = b^=$$

   $$\implies \bar{x} \in \{x : A^= x = b^=\}.$$

2. $\{x : A^= x = b^=\} \subseteq \{x : A^= x \le b^=\}$: This is immediate.

3. $\{x : A^= x \le b^=\} \subseteq$ affine-hull(P):

   Let $\bar{x}$ satisfy $A^= x \le b^=$ and let $x' \in P$ such that $A^= x' = b^=, A^+ x' < b^+$ (i.e., x' is an interior point of P that is guaranteed to exist by Proposition 8). If $\bar{x} = x'$ then $\bar{x}$ is in P and hence $\bar{x}$ is in affine-hull(P). So, we may assume that $\bar{x} \neq x'$.

   The line segment joining $\bar{x}$ and x' contains more than one point in P since $A^+ x' \le b^+$ and $A^= y \le b^=$ for all y in the line segment (see Figure 5.5 for an example).

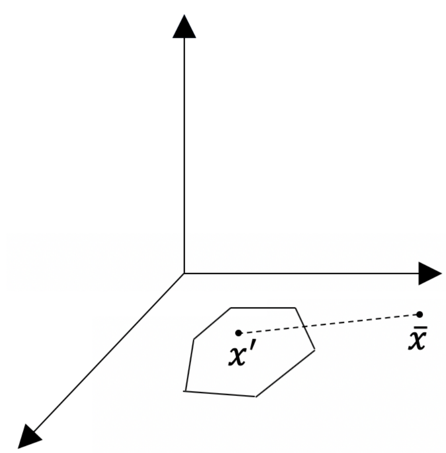


Figure 5.5: Line segment joining $\bar{x}$ and x'

Let x_0 be another point in this line segment in P besides x' (see Figure 5.6 for an example). Now affine-hull(P) $\supseteq$ affine-hull$\{x_0, x'\} \ni \bar{x}$. Therefore, $\bar{x} \in$ affine-hull(P).

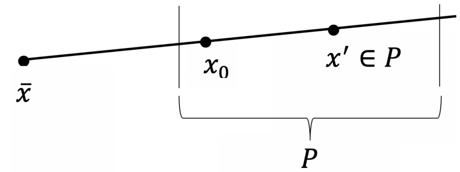


Figure 5.6: Line segment joining $\bar{x}$ and x' contains more than one point in P

□

Now that we know how to identify the affine-hull of a polyhedron from its inequality description, we can also find the dimension of a polyhedron from its inequality description as shown in the following corollary.

Corollary 10. *Let $P = \{x : Ax \leq b\}$. Then*

$$dim(P) = dim\,(\textit{affine-hull}(P)) = n - rank(A^=).$$

Proof. The first equality holds since P and affine-hull(P) contain the same number of affinely independent vectors. For the second equality, by Lemma 9, we have that

$$\text{affine-hull}(P) = \{x \in \mathbb{R}^n : A^= x = b^=\}.$$

Furthermore,

$$\begin{aligned} \dim(\{x \in \mathbb{R}^n : A^= x = b^=\}) &= \dim(\{x \in \mathbb{R}^n : A^= x = 0\}) \qquad \text{(see exercise after Question 1)} \\ &= \dim(\text{null space of columns of } A^=) \\ &= n - \text{rank}(A^=). \end{aligned}$$

□

Corollary 10 allows us to use the inequality description of the polyhedron to understand its dimension.

How is the above discussion on dimension related to minimal inequality description of a polyhedron? Let $P = \{x : Ax \leq b\}$. Consider the system $Ax \leq b$ defining the polyhedron. Note that we can replace $A^= x \leq b^=$ in this system by a *minimal* description $A'x \leq b'$ with the property

$$\text{affine-hull}\{x : A'x \leq b'\} = \text{affine-hull}\{x : A^= x \leq b^=\}.$$

(Recall that a system $Mx \leq d$ is a *minimal* description for $Ax \leq b$ if $\{x : Mx \leq d\} = \{x : Ax \leq b\}$ and moreover, for any subsystem $M'x \leq d'$ of $Mx \leq d$, the set $\{x : M'x \leq d\}$ is a strict superset of $\{x : Ax \leq b\}$.) Doing this replacement would still give a description of P but we would be using minimal constraints to remove subspaces that do not intersect the polyhedron, thus addressing the first aspect of getting a minimal inequality description of a polyhedron. See below for an example.

Example: Consider

$$P = \{(x, y, z) : x + y + z \leq 4, x + y \geq 3, 1 \leq y \leq 2, z \geq 1\}.$$

Note that $x + y = 3, z = 1$ is a minimal description for $x + y + z \leq 4, x + y \geq 3, z \geq 1$ with the property that

$$\text{affine-hull}\{(x, y, z) : x + y = 3, z = 1\} = \text{affine-hull}\{(x, y, z) : x + y + z \leq 4, x + y \geq 3, z \geq 1\}.$$

Consequently, we have that

$$P = \{(x, y, z) : x + y = 3, 1 \leq y \leq 2, z = 1\}.$$

More consequences of Lemma 9. The next corollary follows immediately from Lemma 9. Its proof is left as an exercise.

Corollary 11. *If $a_i^T x \leq b_i$ is an implicit equality in the system $Ax \leq b$, then $a_i^T x \leq b_i$ is also an implicit equality in the system $A^= x \leq b^=$.*

We can also identify if a polyhedron is full-dimensional from its inequality description.

Definition 12. A polyhedron $P \subseteq \mathbb{R}^n$ has *full dimension* if $\dim(P) = n$.

The next corollary follows immediately from Corollary 10. Its proof is left as an exercise.

Corollary 13. *A polyhedron $\{x : Ax \leq b\}$ has full dimension iff there are no implicit equalities in $Ax \leq b$.*