IE 511: Integer Programming, Spring 2021 **4 Feb, 2021**

Lecture 4: Rational IPs, Polyhedron, Decomposition Theorem

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In this lecture, we will continue with our discussion of polyhedral theory. Before that, we will see the significance of rational values in the theory of linear and integer programming.

4.1 Rational Values

Let us see an example to understand the importance of rational values in IPs.

Example:

$$\begin{aligned} \max \quad & -\sqrt{3}x + y \\ \text{s.t.} \quad & -\sqrt{3}x + y \leq 0 \\ & x \geq 1 \\ & y \geq 0 \\ & x, y \in \mathbb{Z} \end{aligned} \tag{4.1}$$

The polyhedron of interest to this problem is $P = \{(x, y) : x \geq 1, y \geq 0, -\sqrt{3}x + y \leq 0\}$ (see Figure 4.1).

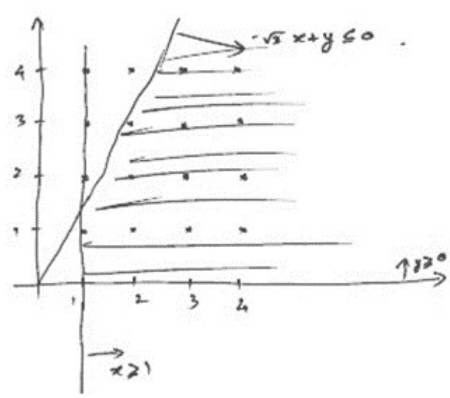

Figure 4.1: Example of an irrational IP

It should be clear that the IP (4.1) is feasible and bounded in objective value. Possible candidates for optimal solutions are integral points which are closest to the line $-\sqrt{3}x + y = 0$. However, note that the IP has no optimal solution. One can get arbitrarily close to the line $-\sqrt{3}x + y = 0$ by picking large non-negative integer x and setting $y = \lfloor \sqrt{3}x \rfloor$ but can never hit the line because no integral solution satisfies $y = \sqrt{3}x$. We summarize this example in the observation below.

Observation. *IPs with irrational inputs can be feasible and bounded and still have no optimal solution.*

Thus, the behaviour of IPs with irrational coefficients are harder to predict (as seen by the above example). To circumvent this issue, we will only consider the case of rational inputs throughout this course. This is justified for two reasons: Firstly, in most real-world problems the input values are rational. Secondly, computers work only with rationals as they are finite-precision machines.

4.1.1 Rational cones

Next, let us formalize rationality in the definitions of cones and state Weyl-Minkowski's theorem (that we saw in the previous lecture) for *rational cones.*

Definition 1.

1. A polyhedral cone $\{x : Ax \leq 0\}$ is a *rational polyhedral cone* if A is rational.
2. A finitely generated cone is rational if its generators are rational. I.e., a cone generated by $\{a_1, \ldots, a_m\}$ is *rational* if $a_1, \ldots, a_m$ are rational.

Here is a simple observation about finitely generated rational cones that will be useful later.

Observation. *A finitely generated rational cone can be assumed to be generated by integral vectors because the cone generated by scaling the generators by positive scalars is the same cone.*

With the above two definitions, we can now restate Weyl-Minkowski's theorem that we proved in the previous lecture as follows. The proof is identical to the proof that we saw in the previous lecture with the additional use of rationality throughout.

Theorem 2 (Farkas, Weyl-Minkowski)**.** *Let $C \subseteq \mathbb{R}^n$. The following are equivalent:*

1. *C is a rational polyhedral cone.*
2. *C is a cone generated by a finite set of rational vectors.*
3. *C is a cone generated by a finite set of integral vectors.*

Recall that this result allows us to move between linear-inequality and linear-combination descriptions of a cone.

4.2 Polyhedral theory: Structure of polyhedra

Techniques for IP depend heavily on the theory of linear inequalities/polyhedral theory. To understand polyhedra, we started with cones. Cones are fundamental geometric tools which are helpful to understand polyhedra. A polyhedron is a slice of a cone as illustrated by the examples below.

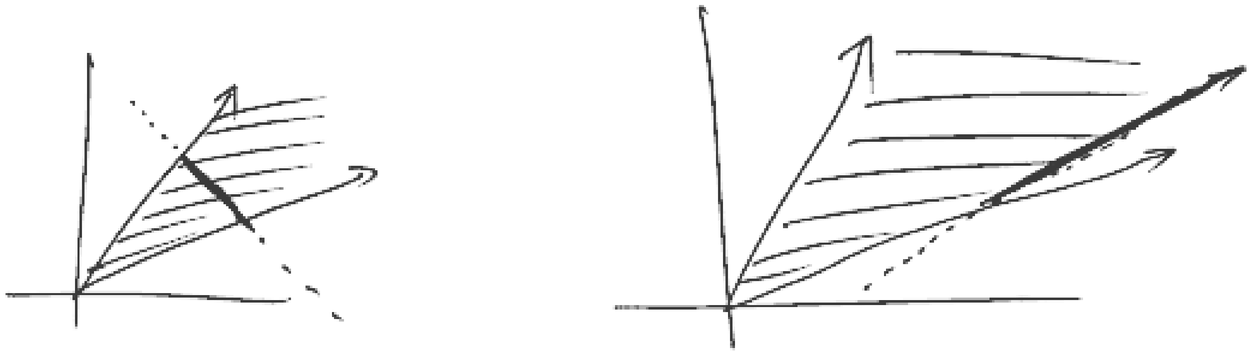

Figure 4.2: Two polyhedra shown as a slice of a cone

We will next formalize this geometric intuition that a polyhedron is a slice of some cone. This "structural" viewpoint that *"a polyhedron is a slice of some cone"* will help us understand polyhedron much better (since we have understood cones already!). We begin by recalling the following definitions.

Definition 3. 1. A *polyhedron* $P \subseteq \mathbb{R}^n$ is a set of points that satisfy a finite number of linear inequalities, i.e.,

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}.$$

2. A set $K \subseteq \mathbb{R}^n$ is *bounded* if there exists $w \in \mathbb{R}$ such that

$$K \subseteq \{x \in \mathbb{R}^n : |x_j| \leq w, \forall j \in [n]\}.$$

See Figure 4.3 for an example of a polyhedron: note that the left-side polyhedron is bounded while the right-side polyhedron is unbounded. We will see an alternative viewpoint of bounded polyhedron shortly.

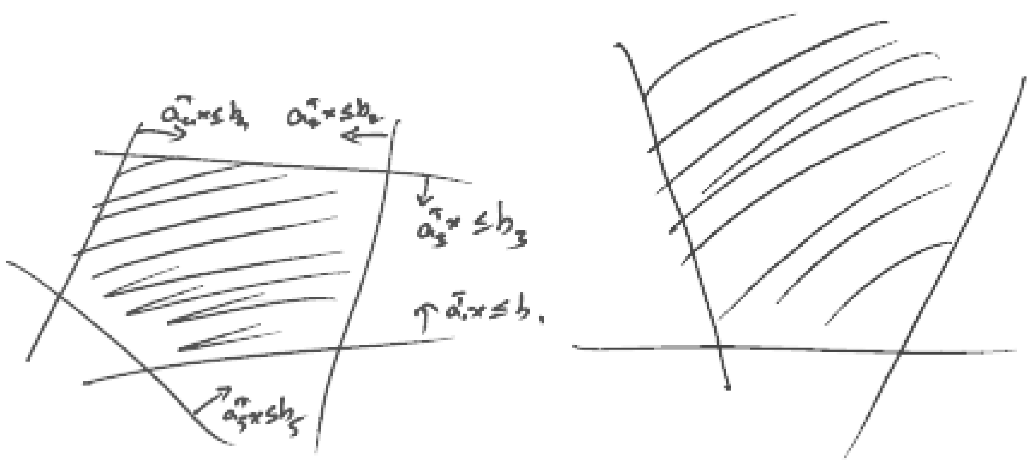

Figure 4.3: Two examples of polyhedra:
A bounded polyhedron (left) and an unbounded polyhedron (right)

A polyhedron is convex. Recall the definition of convexity:

Definition 4. A set $Q \subseteq \mathbb{R}^n$ is *convex* if $\forall x_1, x_2 \in Q, \lambda \in [0,1]$, we have $\lambda x_1 + (1-\lambda)x_2 \in Q$.

Note that the set $\mathbb{Z}^n$ in not convex.

Exercise. Every polyhedron is convex.

Recall the definition of convex-hull of a set.

Definition 5. Convex-hull(S) = Set of convex combinations of finitely many points of S.

Definition 6. A *polytope* is the convex hull of *finitely* many vectors.

The left polyhedron in Figure 4.2 is a polytope as it is the convex-hull of finitely many vectors. In contrast, the right polyhedron in that figure is not a polytope. Note that the convex-hull of finitely many points is always bounded and hence we have the following proposition:

Proposition 7. *Every polytope is bounded.*

Polyhedron and polytope are closely related. To understand this connection, we need the notion of sum of sets (also known as Minkowski sum).

Definition 8 (Minkowski sum)**.** For sets $S, T \subseteq \mathbb{R}^n$, define $S + T := \{s + t : s \in S, t \in T\}$.

Example:

(i) Let $a, v \in \mathbb{R}^n$ and $C = \text{Cone}\{v\}$. Then $C + \{a\}$ is shown in Figure 4.4.

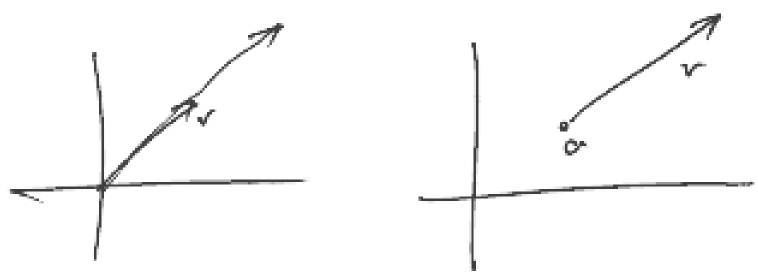

Figure 4.4: Left: $C = \text{Cone}\{v\}$, Right: $C + \{a\}$

(ii) Let $C = \text{Cone}\{v_1, v_2\}$, $a \in \mathbb{R}^n$. Then $C + \{a\}$ is shown in Figure 4.5.

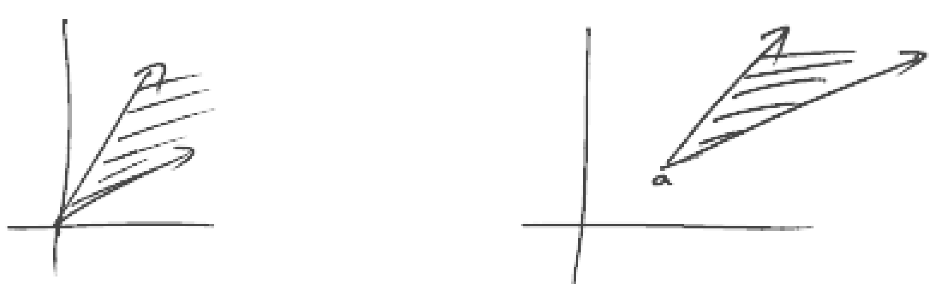

Figure 4.5: Left: $C = \text{Cone}\{v_1, v_2\}$, Right: $C + \{a\}$

(iii) Generally,

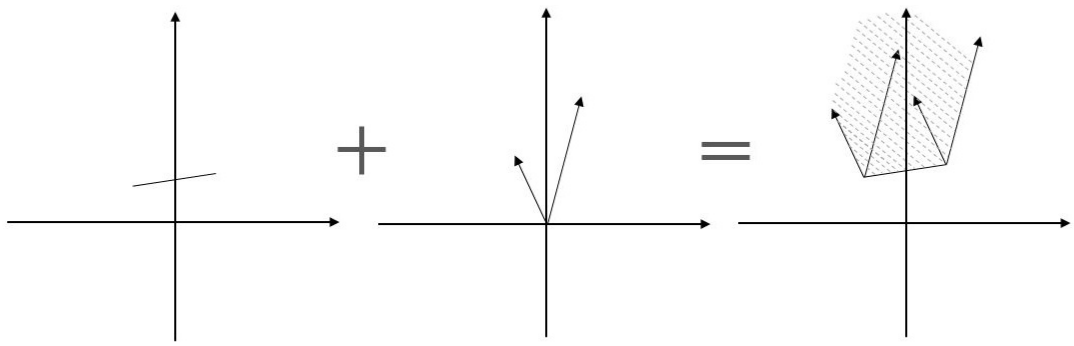

Figure 4.6: The sum of a cone and a polytope is a polyhedron

Note that the figures on the right side of examples (i) and (ii) are polyhedra because they can be written as a set of points satisfying a finite number of linear inequalities. Thus, it appears like the sum of a polytope and a cone is a polyhedron. We now formally prove this.

Theorem 9 (Decomposition theorem for polyhedron)**.** *Let $P \subseteq \mathbb{R}^n$. Then, P is a polyhedron if and only if $P = Q + C$ for some polytope Q and some polyhedral cone C.*

Proof. We will show that it follows from Weyl-Minkowski. For this, we will rely on the geometric intuition that a polyhedron is a slice of a cone. We prove the two directions now.

$\Longrightarrow$:

Let $P = \{x : Ax \le b\}$ be a polyhedron. We need to find a polytope Q and a cone C such that $Q + C = P$. Let

$$T := \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : x \in \mathbb{R}^n, \lambda \in \mathbb{R}, Ax - \lambda b \le 0, \lambda \ge 0, \right\}.$$

Note that T is a polyhedral cone. By Weyl-Mikowski's theorem, the polyhedral cone T is finitely generated. Let the generators be $\binom{x_1}{\lambda_1}, \ldots, \binom{x_m}{\lambda_m}$. Then, we have that $T = \text{Cone}\left\{\binom{x_1}{\lambda_1}, \ldots, \binom{x_m}{\lambda_m}\right\}$. We may assume that $\lambda_i \in \{0, 1\}\ \forall i \in [m]$ by scaling because all λs are non-negative rationals. We have that

$$\begin{aligned}
\bar{x} \in P &\iff \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \in T \\
&\iff \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} = \gamma_1 \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix} + \cdots + \gamma_m \begin{pmatrix} x_m \\ \lambda_m \end{pmatrix} \quad \text{for some } \gamma_1, \ldots, \gamma_m \ge 0 \\
&\iff \bar{x} = \sum_{i=1}^{m} \gamma_i x_i \quad \text{for some } \gamma_1, \ldots, \gamma_m \ge 0 \text{ with } \sum_{i:\lambda_i=1} \gamma_i = 1 \\
&\iff \bar{x} = \sum_{i\in[m]:\lambda_i=0} \gamma_i x_i + \sum_{i\in[m]:\lambda_i=1} \gamma_i x_i \quad \text{for some } \gamma_1, \ldots, \gamma_m \ge 0 \text{ with } \sum_{i\in[m]:\lambda_i=1} \gamma_i = 1
\end{aligned} \tag{4.2}$$

Consider

$$C := \text{Cone}\{x_i : \lambda_i = 0, i \in [m]\} \text{ and } Q := \text{Convex-hull}\{x_i : \lambda_i = 1, i \in [m]\}.$$

Then, C is a cone and Q is a polytope. Statement (4.2) implies that $P = Q + C$.

$\Longleftarrow$:

Let $P = Q+C$ for a polytope Q and a polyhedral cone C. We need to show that P is a polyhedron. Let $C = \text{Cone}\{y_1, \ldots, y_t\}$ and $Q = \text{Convex-hull}\{x_1, \ldots, x_m\}$. We have that

$$\begin{aligned}
\bar{x} \in P \iff \bar{x} &= \sum_{i=1}^{t} \lambda_i y_i + \sum_{i=1}^{m} \gamma_i x_i \qquad \text{for some } \lambda_i \geq 0\ \forall i \in [t], \gamma_i \geq 0\ \forall i \in [m], \sum_{i=1}^{m} \gamma_i = 1 \\
\iff \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} &= \lambda_1 \begin{pmatrix} y_1 \\ 0 \end{pmatrix} + \cdots + \lambda_m \begin{pmatrix} y_m \\ 0 \end{pmatrix} + \gamma_1 \begin{pmatrix} x_1 \\ 1 \end{pmatrix} + \cdots + \gamma_m \begin{pmatrix} x_m \\ 1 \end{pmatrix} \\
&\qquad \text{for some } \lambda_i \geq 0\ \forall i \in [t], \gamma_i \geq 0\ \forall i \in [m] \\
\iff \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} &\in \text{Cone}\left\{ \begin{pmatrix} y_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} y_t \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ 1 \end{pmatrix} \right\} =: S
\end{aligned}$$

By Weyl-Minkowski's Theorem, the finitely generated cone S is also a polyhedral cone. Hence, $S = \left\{ \binom{x}{\lambda} : Ax + \lambda b \leq 0 \right\}$ for some constraint matrix $[A\ b]$. Therefore,

$$\begin{aligned}
\bar{x} \in P &\iff \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix} \in \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : Ax + \lambda b \leq 0 \right\} \\
&\iff A\bar{x} \leq -b
\end{aligned}$$

Therefore, $P = \{x : A\bar{x} \leq -b\}$ and hence, P is a polyhedron. □

Significance of the Decomposition Theorem. The Decomposition Theorem allows us to move between the inequality description of a feasible set and linear combination description (by using Farkas, Weyl-Minkoswki's theorem for polyhedral cones). An important corollary of Theorem 9 is that polytopes are bounded polyhedra.

Corollary 10. *Let $P \subseteq \mathbb{R}^n$. Then, P is a polytope iff P is a bounded polyhedron.*

Proof.
$\Longleftarrow$:
P is a polyhedron implies that $P = Q + C$ for a polytope Q and a polyhedral cone C by the decomposition theorem. If $C \neq \{0\}$, then Q+C is unbounded which implies that P is unbounded, which is a contradiction. Therefore, $C = \{0\}$ and hence $P = Q + \{0\} = Q$. Therefore, P is a polytope.

$\Longrightarrow$:

$$\begin{aligned}
P \text{ is a polytope} &\implies P = \text{convex-hull}\{X^1, \ldots, X^m\} \text{ for some points } X^1, \ldots, X^m \in \mathbb{R}^n \\
&\implies P = \text{convex-hull}\{X^1, \ldots, X^m\} + \{0\} \\
&\implies P = Q + C \text{ for the polytope } Q = \text{convex-hull}\{X^1, \ldots, X^m\} \text{ and Cone } C = \{0\} \\
&\implies P \text{ is a polyhedron (by the decomposition theorem).}
\end{aligned}$$

Moreover, P is bounded by definition of a polytope. □

Significance of Corollary 10. Corollary 10 is a remarkable result telling us that if P is a polytope then convex-hull$(P \cap \mathbb{Z}^n)$ (which is also a polytope) is indeed a polyhedron, i.e., it has an inequality description:

$$\text{convex-hull}(P \cap \mathbb{Z}^n) = \{x \in \mathbb{R}^n : Ax \leq b\} \text{ for some } A, b.$$

Knowing this inequality description reduces the optimization problem:

$$\max\{c^T x : x \in \text{convex-hull}(P \cap \mathbb{Z}^n)\} \equiv \max\{c^T x : Ax \leq b\}.$$

We know how to solve the RHS problem since it is an LP. Note the distinction between the LHS problem and the IP $\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$. Now, wouldn't it be nice if the LHS problem is the same as the IP $\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$? Indeed, we will later see that in order to solve the IP $\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$, it is sufficient to solve the LHS problem.