IE 511: Integer Programming, Spring 2021	**22 Apr, 2021**

Lecture 25: Mixed Integer Cuts

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

Today, we will learn how to obtain cuts for Mixed IPs.

Recall that mixed IPs (MIPs) have two kinds of variables: some variables are required to be integers while the rest can take real values. The standard form of MIP is as follows:

MIP: $\max\{c^T x : Ax = b, x \geq 0, x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}\}$.

The feasible set of a MIP is known as a *mixed-integer region (MIR).*

Example 1. Consider the mixed-integer region $S = \{(x, y) : x, y \geq 0, x + y \geq 5/2, y \in \mathbb{Z}\}$. This set is plotted below. Note that the convex-hull of feasible points (i.e., convex-hull(S)) is different from the convex-hull of integral points in S (i.e., convex-hull($S \cap \mathbb{Z}^2$)).

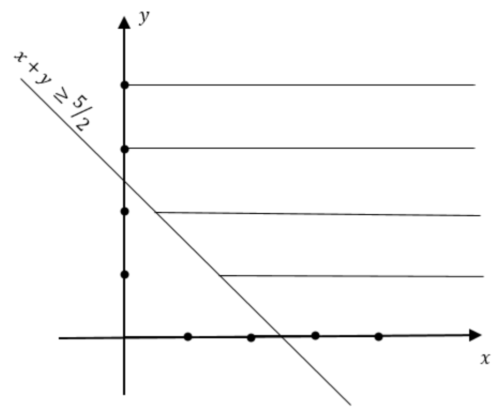

Figure 25.1: Feasible region of a Mixed IP

The cutting plane approach naturally extends to MIPs with two changes: (1) the stopping criterion now requires only certain variables to be integral and (2) we need an efficient procedure to generate cuts—i.e., valid inequalities for the mixed-integer region that are violated by the current optimum.

For IPs, we saw that Gomory's cut generation procedure is an efficient algorithm to generate cuts/valid inequalities that are violated by the current optimum. How about MIPs?

Unfortunately, Gomory's approach for generating cuts for IPs does not give valid cuts for MIPs. We discuss this issue first. Recall that Gomory's cuts for IPs are CG-cuts.

Recap

CG-cut: $w^T x \leq \delta$ is valid for P and w is integral. If x is integral, then $w^T x \leq \lfloor \delta \rfloor$ is valid for P_I.

Note that in a MIP, all variables of x are not integral. So we need an alternative approach to obtain cuts/valid inequalities for the convex-hull of feasible solutions to a MIP. We will see such an approach in this lecture. We start by understanding how to generate cuts for two-dimensional feasible regions. We begin with the special case of regions defined by $\geq$ constraint.

Lemma 0.1. *Let $S^{\geq} := \{(x, y) \in \mathbb{R} \times \mathbb{Z} : x + y \geq b, x \geq 0\}$ and $f := b - \lfloor b \rfloor > 0$. Then, the inequality*

$$\frac{x}{f} + y \geq \lceil b \rceil$$

is valid for $S^{\geq}$.

Proof. Let $(\bar{x}, \bar{y}) \in S^{\geq}$. We need to show that the point $(\bar{x}, \bar{y})$ satisfies the inequality in the lemma.

If $\bar{y} \geq \lceil b \rceil$, then $\bar{x} \geq 0$ and hence, $\bar{x} \geq f(\lceil b \rceil - \bar{y})$ (since the RHS is at most 0). Rewriting this last inequality shows that $(\bar{x}, \bar{y})$ satisfies the inequality in the lemma.

If $\bar{y} < \lceil b \rceil$, then $\bar{x} \geq b - \bar{y} = f + (\lfloor b \rfloor - \bar{y}) \geq f + f(\lfloor b \rfloor - \bar{y}) = f(\lceil b \rceil - \bar{y})$ (the last inequality holds because $\lfloor b \rfloor - \bar{y} \geq 0$ and $f < 1$). It implies that $\frac{\bar{x}}{f} + \bar{y} \geq \lceil b \rceil$. □

Considering Example 1 above, we note that $f = 1/2$ and hence $2x + y \geq 3$ is valid for S.

Next, we address regions defined by $\leq$ constraint.

Corollary 0.1. *Let $S^{\leq} := \{(x, y) \in \mathbb{R} \times \mathbb{Z} : y \leq b + x, x \geq 0\}$. Suppose $f := b - \lfloor b \rfloor > 0$. Then, the inequality*

$$y \leq \lfloor b \rfloor + \frac{x}{1 - f}$$

is valid for $S^{\leq}$.

Proof. We have $y \leq b + x$ iff $x - y \geq -b$. Moreover, $-b - \lfloor -b \rfloor = 1 - f$. By Lemma 0.1, $\frac{x}{1-f} - y \geq \lceil -b \rceil = -\lfloor b \rfloor$ is valid for $S^{\leq}$. □

Note that when $x = 0$, we obtain a CG-cut/Gomory cut type inequality from the above lemma and its corollary. Hence the above lemma and its corollary are generalizations of CG-cuts for mixed integer sets.

Next, let us consider slightly more general mixed integer region. Let

$$S^{\text{MIR}} := \{(x, y) \in \mathbb{R} \times \mathbb{Z}^2 : x, y \geq 0, a_1 y_1 + a_2 y_2 - x \leq b\}$$

with $b \notin \mathbb{Z}$.

Lemma 0.2 (Mixed Integer Rounding)**.** *Let $f = b - \lfloor b \rfloor$ and $f_i = a_i - \lfloor a_i \rfloor$ for $i = 1, 2$. If $f_1 \leq f \leq f_2$, then*

$$\lfloor a_1 \rfloor y_1 + \left(\lfloor a_2 \rfloor + \frac{f_2 - f}{1 - f} \right) y_2 \leq \lfloor b \rfloor + \frac{x}{1 - f}$$

is valid for S^{MIR}.

Proof. The inequality $\lfloor a_1 \rfloor y_1 + \lceil a_2 \rceil y_2 \leq b + x + (1 - f_2) y_2$ is valid for S^{MIR} (because $y_1 \geq 0$ and $a_2 = \lceil a_2 \rceil - (1 - f_2)$). By Corollary 0.1,

$$\lfloor a_1 \rfloor y_1 + \lceil a_2 \rceil y_2 \leq \lfloor b \rfloor + \frac{x + (1 - f_2) y_2}{1 - f}$$

is valid for S^{MIR}. It means that

$$\lfloor a_1 \rfloor y_1 + \left(\lceil a_2 \rceil + \frac{1 - f_2}{1 - f} \right) y_2 \leq \lfloor b \rfloor + \frac{x}{1 - f}$$

is valid for S^{MIR}. Using $\lceil a_2 \rceil = \lfloor a_2 \rfloor + 1$ gives the lemma. $\square$

Remark. Note that if $f_1 \leq f \leq f_2$ does not hold, then we can use one of the previous lemmas to get a cut.

We now have the ingredients to obtain a valid inequality for a mixed integer region that is violated by an extreme point optimum that does not satisfy the integral requirements.

25.1 Gomory's Mixed Integer Cut

Now that we have seen how to obtain valid inequalities for a mixed-integer region, let us see how to generate a valid inequality for the mixed-integer region that is violated by the current extreme point.

Consider the standard form of MIP (where x variables are real and y variables are required to be integral):

$$\max\{c_1^T x + c_2^T y : A_1 x + A_2 y = b, x, y \geq 0, y \in \mathbb{Z}^p, x \in \mathbb{R}^{n-p}\}.$$

Let $(\bar{x}, \bar{y})$ be a basic feasible solution to the LP-relaxation of this MIP. If $\bar{y}$ is integral, then we are done with the cutting plane approach. Suppose $\bar{y}$ is not integral.

Let y_i be the basic variable with $\bar{y}_i \notin \mathbb{Z}$. Consider the i^{th} row of the optimal tableau (obtained with knowledge of the basis):

$$y_i + \sum_{j \in N_1} \bar{a}_{ij} y_j + \sum_{j \in N_2} \bar{a}_{ij} x_j = \bar{b}_i \tag{25.1}$$

where $(y_i, y, x) \in \mathbb{Z} \times \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$, where N_1 is the subset of non-basic variables among y and N_2 is the subset of non-basic variables among x. Recall that the non-basic variables are set to 0 in the solution $(\bar{x}, \bar{y})$. Consider the mixed-integer region

$$S^i := \left\{ (y_i, y, x) \in \mathbb{Z} \times \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} : y_i, y, x \geq 0, y_i + \sum_{j \in N_1} \bar{a}_{ij} y_j + \sum_{j \in N_2} \bar{a}_{ij} y_j = \bar{b}_i \right\}.$$

Lemma 0.3. *Let $f_j := \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$ for all $j \in N_1 \cup N_2$ and $f_0 := \bar{b}_i - \lfloor \bar{b}_i \rfloor$. Then*

$$\sum_{j \in N_1 : f_j \leq f_0} f_j y_j + \sum_{j \in N_1 : f_j > f_0} (1 - f_j) \frac{f_0}{1 - f_0} y_j + \sum_{j \in N_2 : \bar{a}_{ij} > 0} \bar{a}_{ij} x_j + \sum_{j \in N_2 : \bar{a}_{ij} < 0} \left(\frac{f_0}{1 - f_0} \right) \bar{a}_{ij} x_j \geq f_0 \tag{25.2}$$

is a valid inequality for S^i that is violated by $(\bar{x}, \bar{y})$.

Inequality (25.2) is known as *Gomory Mixed Integer Cut.*

Proof. Violation follows as LHS evaluated at $(\bar{x}, \bar{y})$ is 0 while RHS > 0. We now show validity. The MIR inequality for S^i (by Lemma 0.2) is

$$y_i + \sum_{j \in N_1 : f_j \le f_0} \lfloor \bar{a}_{ij} \rfloor y_j + \sum_{j \in N_1 : f_j > f_0} \left(\lfloor \bar{a}_{ij} \rfloor + \frac{f_j - f_0}{1 - f_0} \right) y_j + \sum_{j \in N_2 : \bar{a}_{ij} < 0} \left(\frac{\bar{a}_{ij}}{1 - f_0} \right) x_j \le \lfloor \bar{b}_i \rfloor.$$

Substituting y_i from equation (25.1) gives the inequality (25.2). □

Example. Consider the MIP

$$\begin{aligned} \max\ & 4y - x \\ & 7y - 2x \le 14 && (1) \\ & x \le 3 && (2) \\ & 2y - 2x \le 3 && (3) \\ & y, x \ge 0 \\ & y \in \mathbb{Z} \end{aligned}$$

We introduce slack variables s_1, s_2, s_3 for inequalities (1), (2), and (3) and solve the LP-relaxation. It turns out that the optimal solution for the LP-relaxation is the one in which x, y and s_3 are basic variables and the corresponding solution is $\bar{x} = 3, \bar{y} = 20/7, \bar{s}_3 = 23/7$ (recall that non-basic variables are set to 0). The optimal tableax (from the basis) is as follows:

$$\begin{aligned} z = \max\ & \frac{59}{7} - \frac{4}{7}s_1 - \frac{1}{7}s_2 \\ & y + \frac{1}{7}s_1 - \frac{2}{7}s_2 = \frac{20}{7} \\ & x + s_2 = 3 \\ & -\frac{2}{7}s_1 + \frac{10}{7}s_2 + s_3 = \frac{23}{7} \end{aligned}$$

We note that $\bar{y}$ is fractional. Therefore, the first row gives the MIR cut $y \le 2$. Substituting for y in the MIR cut using the first equation gives $\frac{1}{7}s_1 - \frac{2}{7}s_2 \ge \frac{6}{7}$. We add this cut and re-solve the new LP to obtain an LP optimal solution $y = 2, x = 1/2$. Since y is integral, this is an optimal solution for the MIP itself.

Similar to IPs, Gomory's cutting plane algorithm for MIPs can also be shown to terminate in a finite numbers of steps using a careful choice of variable for cut generation and a careful choice of LP solving algorithm.