IE 511: Integer Programming, Spring 2021 **20 Apr, 2021**

Lecture 24: Gomory's Cutting Plane Algorithm

Lecturer: Karthik Chandrasekaran *Scribe: Karthik*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

Recap

Consider the IP $z^* = \max\{c^T x : x \in P \cap \mathbb{Z}^n\} = \{c^T x : x \in P_I\}$.

Algorithm 1: Cutting Plane Algorithmic Approach

Input: A, b, c such that $P = \{x : Ax \leq b\}$
Output: $\bar{x} = \arg\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$

Initialize $Q = P$
repeat
    Solve $\max\{c^T x : x \in Q\}$ to find an extreme point optimum $\bar{x}$
    if $\bar{x} \in \mathbb{Z}^n$ **then**
        STOP and return $\bar{x}$
    else
        find a valid inequality $w^T x \leq \delta$ for P_I such that $w^T \bar{x} > \delta$
        $Q \leftarrow Q \cap \{x : w^T x \leq \delta\}$

We began studying the cutting plane algorithmic approach as a solving approach for unstructured IPs. We will discuss the use of this approach for structured IPs followed by unstructured IPs today.

24.1 Structured IPs

Structured IPs typically have a family F of valid inequalities for P_I. Note that the family F may not completely describe P_I. We use such a family in the cutting plane algorithm as described in Algorithm 2.

Algorithm 2: Cutting Plane Algorithmic Approach for Structured IPs

Input: A, b, c such that $P = \{x : Ax \leq b\}$

repeat
- Solve LP $\max\{c^T x : x \in P\}$ to find an extreme point optimum $\bar{x}$
- **if** *$\bar{x}$ is integral* **then**
  - STOP and return $\bar{x}$
- **else**
  - find a valid inequality $w^T x \leq \delta$ in F such that $w^T \bar{x} > \delta$ (separation problem for F)
  - **if** *such an inequality was found* **then**
    - $P \leftarrow P \cap \{x : w^T x \leq \delta\}$
  - **else**
    - we have a better solution than the optimum to the original LP-relaxation, STOP

An Example. Consider the maximum weight matching problem in non-bipartite graphs. For a graph $G = (V, E)$, the associated IP is:

$$\max\{\sum_{e \in E} w_e x_e : x \in P \cap \mathbb{Z}^E\}$$

where

$$P = \{x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \ \forall \ v \in V, \ x \geq 0\}.$$

Recall from the previous lecture that the following family of odd-set inequalities is valid for P_I:

$$F_{\text{odd}} := \{ \sum_{e \in E[S]} x_e \leq (|S| - 1)/2 \ \forall \ S \subseteq V : |S| \text{ odd}\}.$$

Thus, we can execute Algorithm 2 using F_{odd}. We mentioned a theorem of Edmonds' in the previous lecture—namely, $P \cap \{x : x \text{ satisfies all inequalities in } F_{\text{odd}}\}$ is exactly the convex hull of indicator vectors of perfect matching in G. So, when Algorithm 2 terminates, the objective value of the solution that it terminates with will correspond to the maximum weight of a matching in G.

Other considerations. It is possible for Algorithm 2 to terminate without an integral optimum if the family F does not describe P_I. If Algorithm 2 terminates without an integral optimum, then we will get a better upper bound for the optimum objective value (at least no worse than the initial one). Such upper bounds can be used in the branch and bound method.

In order to implement Algorithm 2, note that we need an efficient procedure to find a valid inequality for P_I from the family F that is violated by $\bar{x}$. This is usually possible for structured IPs/combinatorial optimization problems (e.g., for matchings).

Note that Algorithm 2 will always terminate in finite number of iterations if the family F is finite. Would it terminate in polynomial number of iterations? The answer is YES for the max weight matching problem in non-bipartite graphs via careful selection of cuts from F_{odd}. For other combinatorial optimization problems, this is an interesting direction of research.

24.2 Unstructured IPs

For unstructured IPs, in order to implement the cutting plane algorithm, we need an efficient procedure to generate valid inequalities for P_I but violated by the current extreme point optimum

$\bar{x}$. This will be our next goal.

24.2.1 Gomory's cut generation procedure

Recall that CG-cuts give us valid inequalities for P_I.

CG-Cut: If $w^T x \le \delta$ is valid for P and w is integral then $w^T x \le \lfloor \delta \rfloor$ is valid for P_I.

So, we will try to find a CG-cut that is violated by $\bar{x}$.

Let $P := \{x \in \mathbb{R}^n : Ax = b, x \ge 0\}$, $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $\bar{x}$ be an extreme point optimum to $\max\{c^T x : x \in P\}$.

Recall

- An extreme point solution is defined by a *basis* B.
- A *basis* is an index set $B \subseteq [n]$ with $|B| = m$ such that the corresponding sub-matrix A_B of A (using columns indexed by B) is non-singular.
- A basis is *feasible* if $\bar{x}_B := A^{-1}b \ge 0$. Then $\bar{x} := (\bar{x}_B, \bar{x}_N = 0)$ is a *basic feasible solution.*
- For a fixed basis, we can write $Ax = b$ as $[A_B\ A_N]x = b$.

Given: An optimal basis B and an optimal basic feasible solution $\bar{x} = (\bar{x}_B, \bar{x}_N = 0) \notin \mathbb{Z}^n$ where $\bar{x}_B = A_B^{-1} b$.

Goal: Find a valid inequality for P_I violated by $\bar{x}$.

With the notation $\bar{b} := A_B^{-1} b$, $\bar{A}_N := A_B^{-1} A_N$, we note that the equations

$$x_B + \bar{A}_N x_N = \bar{b} \tag{24.1}$$

are valid for P (since it is obtained by pre-multiplying $[A_B A_N]x = b$ by A_B^{-1}). If $\bar{x} \notin \mathbb{Z}^n$, then there exists $i \in B$ such that $\bar{x}_i = \bar{b}_i \notin \mathbb{Z}$.

Definition 1. Fix an i such that $\bar{b}_i \notin \mathbb{Z}$. Consider constraint for row i which is of the form $x_i = \bar{b}_i - \sum_{j \in N} \bar{a}_{ij} x_j$. The Gomory cut for row i is

$$\sum_{j \in N} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \ge \bar{b}_i - \lfloor \bar{b}_i \rfloor. \tag{24.2}$$

We will now show that inequality (24.2) is valid for P_I and is violated by $\bar{x}$.

Claim 1.1. *Inequality (24.2) is violated by $\bar{x}$.*

Proof. LHS evaluated at $\bar{x}$ is 0 but the RHS is strictly greater than 0 by the choice of i. □

Claim 1.2. *Inequality (24.2) is valid for P_I.*

Proof. The equation $x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i$ is valid for P and all variables are non-negative. Therefore,

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \le x_i + \sum_{j \in N} \bar{a}_{ij} x_j \quad \forall x \in P$$

Hence,

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \bar{b}_i \text{ is valid for } P.$$

The LHS coefficients are integral. It implies that

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor \text{ is a CG-cut for } P$$

and hence, is valid for P_I. Also,

$$x_i + \sum_{j \in N} a_{ij} x_j = b_i \text{ is valid for } P_I.$$

The above two inequalities together imply that

$$\sum_{j \in N} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor \text{ is valid for } P_I.$$

□

We note that the above proof also reveals that a Gomory cut is in fact a CG-cut. With this choice of cuts, Gomory gave the cutting plane algorithm described in Algorithm 3.

Algorithm 3: Gomory's cutting plane algorithm

Input: A, b, c such that $P = \{x : Ax = b, x \geq 0\}$
Output: $\bar{x} = \arg\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$

repeat
    Solve LP $\max\{c^T x : Ax = b, x \geq 0\}$ to find an optimal basic feasible solution $\bar{x}$
    if $\bar{x}$ *is integral* **then**
        STOP and return $\bar{x}$
    else
        Choose a basic variable x_i such that $\bar{x}_i = \bar{b}_i \notin \mathbb{Z}$
        Generate a Gomory cut and add it to the LP formulation

With careful choice of LP solvers and cut generation the algorithm can be shown to terminate in finite time.

Theorem 2 (Gomory)**.** *Suppose Gomory's algorithm is implemented by*

1. *using the lexicographic simplex algorithm for LP solving and*

2. *deriving Gomory cut from the fractional variable with the smallest index.*

Then the algorithm will terminate in finite numbers of iterations.

24.2.2 Disjunctive cuts

We have alternative ways to generate valid inequalities for P_I violated by $\bar{x}$ besides CG cuts. One such way is disjunctive cuts. Here, we represent a feasible region as a union of two sub-regions, generate valid inequalities for the sub-regions and use these valid inequalities to generate a valid inequality for the feasible region. See Figure 24.1 for an example.

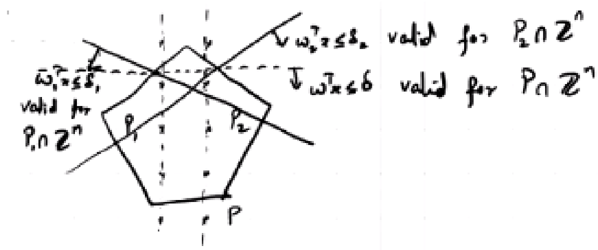

Figure 24.1: Disjunctive cuts

Definition 3. A set $S = S_1 \cup S_2$ with $S_1, S_2 \subseteq \mathbb{R}^n_+$ is a disjunction of two sets.

Example. Binary IPs have this disjunctive structure: Let P be a polyhedron in $[0,1]^n$ and

$$S = P \cap \{0,1\}^n.$$

Then $S = S_1 \cup S_2$ where

$$S_1 := \{x \in P \cap \{0,1\}^n : x_1 = 0\} \text{ and}$$
$$S_2 := \{x \in P \cap \{0,1\}^n : x_1 = 1\}.$$

In particular, $S_1 = P_1 \cap \{0,1\}^n$ and $S_2 = P_2 \cap \{0,1\}^n$ where $P_1 := \{x \in P : x_1 = 0\}$ and $P_2 := \{x \in P : x_1 = 1\}$.

We will use valid inequalities for S_1 and S_2 to generate valid inequalities for S.

Lemma 3.1. *Suppose* $\sum_{j=1}^n w_j^1 x_j \le \delta^1$ *is valid for* S_1 *and* $\sum_{j=1}^n w_j^2 x_j \le \delta^2$ *is valid for* S_2*. Then* $\sum_{j=1}^n w_j x_j \le \delta$ *is valid for* S *if* $w_j \le \min\{w_j^1, w_j^2\}$ *for all* $j \in [n]$ *and* $\delta \ge \max\{\delta^1, \delta^2\}$.

Proof. Suppose $x \in S$. It implies that either $x \in S_1$ or $x \in S_2$. Without loss of generality, say $x \in S_1$. Then since $x \ge 0$ we have that

$$\sum_{j=1}^n w_j x_j \le \sum_{j=1}^n w_j^1 x_j \le \delta^1 \le \delta.$$

□

The above lemma tells us how to obtain valid inequalities for P_I if we can write $P = P_1 \cup P_2$ and have inequalities for $(P_1)_I$ and $(P_2)_I$. Disjunctive cuts are also used to get better bounds in branch and bound. The theory of disjunctive cuts (similar to the notions of Chvátal closure and Chvátal rank for CG-cuts) is also well-understood.