IE 511: Integer Programming, Spring 2021 **15 Apr, 2021**

Lecture 23: CG-cuts and Chvatal closure

Lecturer: Karthik Chandrasekaran *Scribe: Karthik*

We were studying the cutting plane algorithmic approach. In particular, we were studying how to obtain valid inequalities for P_I given the inequality description $Ax \le b$ of P. We saw the notion of CG-cuts and Chvátal closure. We will explore further aspects of CG-cuts and Chvátal closure today.

Recap

Algorithm 1: Cutting Plane Algorithmic Approach

Input: A, b, c where $P = \{x : Ax \le b\}$
Output: $\bar{x} = \arg\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$

Initialize $Q = P$
repeat
  Solve $\max\{c^T x : x \in Q\}$ to find an extreme point optimum $\bar{x}$
  if $\bar{x} \in \mathbb{Z}^n$ **then**
    STOP and return $\bar{x}$
  else
    find a valid inequality $w^T x \le \delta$ for P_I such that $w^T \bar{x} > \delta$
    $Q \leftarrow Q \cap \{x : w^T x \le \delta\}$

Definition 1. Let $P = \{x : Ax \le b\}$ where A, b are integral and let w be integral and $w^T x \le \delta$ be valid for P. Then $w^T x \le \lfloor \delta \rfloor$ is valid for P_I and is known as a *Chvátal-Gomory cut* (CG-cut) for P.

Definition 2. $P' := \{x \in P : x$ satisfies all CG-cuts for $P\}$ is the *first Chvátal closure* of P.

Observation. $P_I \subseteq P' \subseteq P$ and $(P_I)' = P_I$.

Theorem 3 (Schrijver(1980))**.** *If P is a rational polyhedron, then P' is a rational polyhedron.*

23.1 Optimization over the first closure

We now address optimization over the first Chvátal closure. We know that P' is possibly closer to P_I than P. Can we optimize over P' efficiently?

Given: A, b, c where $P = \{x : Ax \le b\}$
Goal: $\max\{c^T x : x \in P'\}$

Theorem 4 (Eisenbrand 2000)**.** *Optimization over first closure is NP-hard.*

23.2 Chvátal rank

We saw that P' is possibly closer to P_I than P. So, why stop with P'? It is possible that $(P')'$ is even closer to P_I than P' itself. This motivates the following definition.

Definition 5. Let $P^{(0)} := P, P^{(1)} := (P^{(0)})', P^{(2)} := \left(P^{(1)}\right)', \ldots, P^{(i+1)} := \left(P^{(i)}\right)'$ be a sequence of polyhedra obtained by taking Chvátal closure repeatedly.

We have the following immediate observations:

Proposition 6. *1.* $P^{(0)} = P \supseteq P^{(1)} \supseteq P^{(2)} \supseteq \ldots$

2. $P^{(t)} \supseteq P_I \quad \forall t = 0, 1, 2, \ldots$

But is the sequence finite? Will it converge to P_I?

Theorem 7 (Chvátal, Schrijver)**.** *For every rational polyhedron P, there exists a finite number t for which $P^{(t)} = P_I$.*

This theorem is significant for the following reasons:

1. It implies that every valid inequality for P_I can be obtained using a sequence of CG-Cuts derived from existing inequalities.
2. It suggests that using CG-cuts in the cutting plane algorithm *could possibly* lead to finite termination.

Definition 8. The smallest number t for which $P^{(t)} = P_I$ is the *Chvátal rank* of P.

Note that Chvátal rank is a crude estimate for the number of rounds of CG-cuts that we may have to add to arrive at the IP optimum via the cutting plane algorithmic approach. So, small rank would be nice.

Exercise. There exist polyhedra P whose Chvátal rank is not bounded by a polynomial in the size of the description of P.

However, several discrete optimization problems use binary variables. In which case, the associated polyhedron is within $[0,1]^n$. We have the following bounds on the Chvátal rank of such polytopes.

Theorem 9. *If a polyhedron P is such that $P \subseteq [0,1]^n$, then Chvátal rank of P is at most $n^2(1 + \log n)$.*

Theorem 10. *There exist a polyhedron $P \subseteq [0,1]^n$ with Chvátal rank at least cn^2 for some fixed constant c.*

We emphasize that Chvátal rank is not necessarily an upper bound on the number of rounds to termination in the cutting plane algorithmic approach, but finite Chvátal rank suggests using CG-cuts in the cutting plane approach if we hope to have finite termination.

23.3 Cutting Plane Proof

Cutting plane proof is a method for demonstrating that $c^T x \le \delta$ is valid for P_I when we are given constraint matrix A and RHS vector b such that $P = \{x : Ax \le b\}$. Equivalently, it is a method for demonstrating that every integral solution of $Ax \le b$ satisfies a specified inequality $c^T x \le \delta$. We saw an example of the cutting plane proof in the previous lecture. Let us formalize this now.

Definition 11. Let $Ax \le b$ be a system of inequalities. A *cutting plane proof of* $c^T x \le \delta$ is a sequence of inequalities

$$c_1^T x \le \delta_1$$

$$\vdots$$

$$c_M^T x \le \delta_M$$

where

1. $c_M = c, \delta_M = \delta$,
2. $c_1, \ldots, c_k$ are integral, and
3. $c_i^T x \le \delta_i'$ is a non-negative linear combination of the inequalities $Ax \le b$, $c_1^T x \le \delta_1$, $\ldots$, $c_{i-1}^T x \le \delta_{i-1}$ for some δ_i' such that $\lfloor \delta_i' \rfloor \le \delta_i$ (i.e., $c_i^T x \le \delta_i$ is a CG-cut that is derived from all previous inequalities).

Here, M is said to be the *length of the proof.*

A cutting plane proof can be viewed as a DAG by labeling each node by an inequality: Here, each node represents a CG-cut obtained using combinations of inequalities. The incoming edges into the node indicate the inequalities that contribute to the combination. See Figure 23.1.

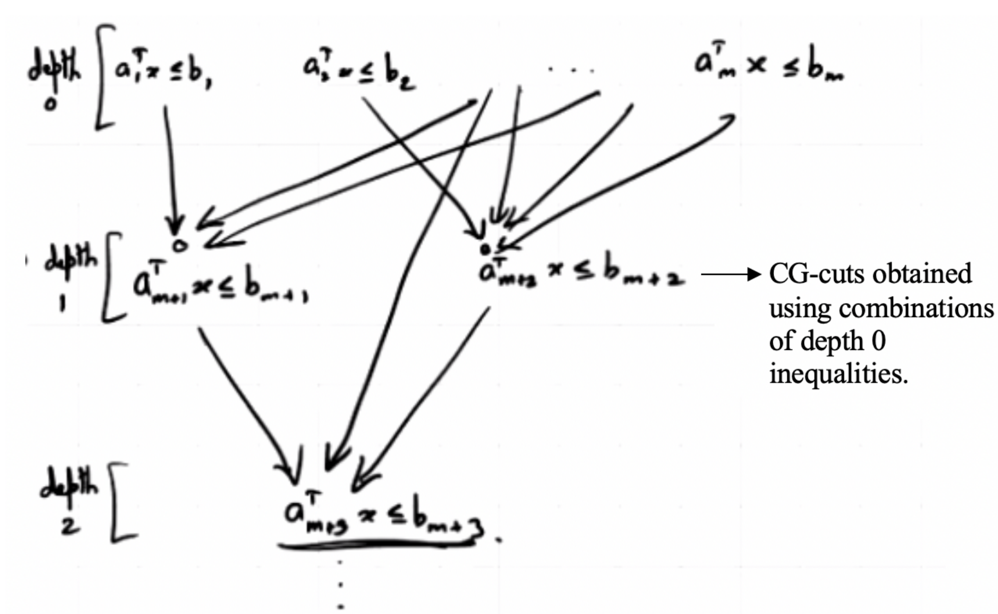


Figure 23.1: A cutting plane proof as a DAG.

Definition 12. The *depth of an inequality* is t if it can be obtained as a CG-cut of an inequality that is a combination of inequalities with depth at most $t-1$.

Every valid inequality for P_I has a cutting plane proof from the inequalities defining P. But, will the proof be of finite length? The following theorem preoves that there will always be a finite length cutting plane proof.

Theorem 13. *Let $P = \{x : Ax \leq b\}$ be a rational/bounded polyhedron. Let $w^T x \leq \beta$ be valid for P_I. Then there exists a finite depth cutting plane proof of $w^T x \leq \beta$.*

Proof. By bounded Chvátal rank theorem (i.e., Theorem 7). □

Note that if an IP is infeasible (e.g., SAT), then we could potentially demonstrate infeasibility by demonstrating the validity of the inequality $0^T x \leq -1$ for all integer solutions of the IP. This can be done via a cutting plane proof. Hence, we have the following corollary from Theorem 13.

Corollary 13.1. *Let $P = \{x : Ax \leq b\}$ be a rational polytope. If $P_I = \emptyset$, then there exists a cutting plane proof of $0^T x \leq -1$ from $Ax \leq b$.*

There has been a rich line of research trying to derive low depth cutting plane proofs of infeasibility of several natural constraint satisfaction problems.

23.4 Application of CG-cuts for structured IPs

Although we started studying the cutting plane algorithmic approach and CG-cuts for unstructured IPs, they have been valuable even for structured IPs. We illustrate an example here.

Max weight matching in non-bipartite graphs. We saw the max weight matching problem in bipartite graphs. How about non-bipartite graphs?

Given: $G = (V, E)$, $w : E \to \mathbb{R}$
Goal: $\max\{\sum_{e \in M} w_e : M \text{ is a matching in } G\}$

The IP is $\max\{\sum_{e \in E} w_e x_e : X \in P \cap \mathbb{Z}^E\}$ where

$$P := \left\{ x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V, x_e \geq 0 \ \forall e \in E \right\}.$$

Then,

$$P_I := \text{ Convex hull of indicator vectors of matchings in } G.$$

Recall that $P \neq P_I$ as illustrated by the following example:

Example. Let G be the graph shown in Figure 23.2. For this graph, the solution given in Figure 23.3 is in P but not in P_I.

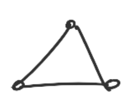


Figure 23.2: A non-bipartite graph.

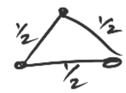
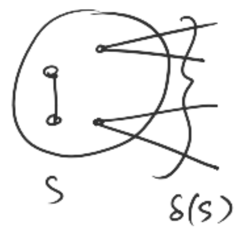


Figure 23.3: A point in P but not in P_I.

Since this is a structured polyhedron, we could try to derive P' and ask if $P' = P_I$. Let us see some valid inequalities for the first Chvátal closure for P: we will derive valid inequalities for P_I using CG-cuts for P. Let $S \subseteq V$.



Then the inequalities

$$\sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in S \tag{23.1}$$

are valid for P. By adding all the inequalities of (23.1), we obtain that the inequality

$$2 \sum_{e \in E(S)} x_e + \sum_{e \in \delta(S)} x_e \leq |S| \text{ is valid for } P \tag{23.2}$$

where $E(S) := \{uv \in E : u, v \in S\}$ and $\delta(S) := \{uv \in E : |\{u, v\} \cap S| = 1\}$. We also know that the inequalities

$$-x_e \leq 0 \quad \forall e \in \delta(S) \tag{23.3}$$

are valid for P. By summing up inequalities in (23.2) and (23.3), we see that the inequality

$$2 \sum_{e \in E(S)} x_e \leq |S| \text{ is valid for } P,$$

$$\text{i.e., } \sum_{e \in E(S)} x_e \leq \frac{|S|}{2} \text{ is valid for } P,$$

$$\text{i.e., } \sum_{e \in E(S)} x_e \leq \left\lfloor \frac{|S|}{2} \right\rfloor \text{ is a CG-cut for } P.$$

Therefore, if $|S|$ is odd, then we obtain a *new* valid inequality $\sum_{e \in E(S)} x_e \leq \frac{|S|-1}{2}$ for P_I (which is not valid for P). Thus, we have the following observation:

Proposition 14.

$$P' \subseteq \left\{ \begin{array}{lll} x \in \mathbb{R}^E : & \sum_{e \in \delta(v)} x_e \leq 1 & \forall v \in V \\ & x_e \geq 0 & \forall e \in E \\ & \sum_{e \in E(S)} x_e \leq \frac{|S|-1}{2} & \forall S \subseteq V, |S| \ odd \end{array} \right\} =: Q$$

The last set of inequalities above are known as *odd-set inequalities*. Edmonds showed that these additional odd-set inequalities are sufficient to describe the convex-hull of incidence vectors of matchings in G.

Theorem 15 (Edmonds)**.** *$P_I = Q$.*

Corollary 15.1. *$P' = P_I$ ($P_I \subseteq P' \subseteq Q = P_I$), i.e., Chvátal rank of P is one.*

Thus, CG-cuts are a useful tool even for structured IPs for the following reasons:

- If we have a polyhedron for a structured IP that is not integral, then CG-cuts give a systematic way to get closer to the integral hull.
- For some combinatorial optimization problems (like matchings) CG-cuts give a systematic way to *strengthen the LP-relaxation* (i.e., a systematic way to obtain a tighter LP-relaxation for the integral-hull of the polyhedron). This is useful in approximation algorithms for certain combinatorial optimization problems.

Sufficient conditions to guarantee $P' = P_I$. The corollary above raises the question of whether there are sufficient conditions on A and b to conclude that the first closure is equal to the integral hull of the polyhedron (i.e., $P' = P_I$). We define a more general family of matrices below:

Definition 16 (Edmonds-Johnson (EJ) Matrices)**.** Matrix A is an EJ matrix if $P(b, c, l, u) = \{x : b \leq Ax \leq c, l \leq x \leq u\}$ has Chvátal rank at most one for all integral b, c, l, u.

Recall that a matrix A is a TU matrix if $P(b, c, l, u)$ has Chvátal rank 0 for all integral b, c, l, u. We have the following sufficient conditions for a matrix to be an EJ matrix:

1. (Edmonds-Johnson) If A is integral and every column has l_1-norm ≤ 2, then A is an EJ matrix.
2. (Gerards-Schrijver) If A is integral, every row has l_1-norm ≤ 2, and A has no odd-k_4 minor, then A is an EJ matrix.

Identifying general families of EJ matrices remains an interesting research direction.