Disclaimer: These notes have not been subjected to the usual scrutiny reserved for formal publications.

We were studying the cutting plane algorithmic approach. In particular, we were studying how to obtain valid inequalities for $P_I$ given the inequality description $Ax \leq b$ of $P$. We saw the notion of CG-cuts and Chvátal closure. We will explore further aspects of CG-cuts and Chvátal closure today.

Recap

Algorithm 1: Cutting Plane Algorithmic Approach

Input: $A, b, c$ where $P = \{x : Ax \leq b\}$
Output: $\bar{x} = \arg \max \{c^T x : x \in P \cap \mathbb{Z}^n\}$

Initialize $Q = P$

repeat

Solve $\max \{c^T x : x \in Q\}$ to find an extreme point optimum $\bar{x}$

if $\bar{x} \in \mathbb{Z}^n$ then

STOP and return $\bar{x}$

else

find a valid inequality $w^T x \leq \delta$ for $P_I$ such that $w^T \bar{x} > \delta$

$Q \leftarrow Q \cap \{x : w^T x \leq \delta\}$

Definition 1. Let $P = \{x : Ax \leq b\}$ where $A, b$ are integral and let $w$ be integral and $w^T x \leq \delta$ be valid for $P$. Then $w^T x \leq \lfloor \delta \rfloor$ is valid for $P_I$ and is known as a Chvátal-Gomory cut (CG-cut) for $P$.

Definition 2. $P' := \{x \in P : x$ satisfies all CG-cuts for $P\}$ is the first Chvátal closure of $P$.

Observation. $P_I \subseteq P' \subseteq P$ and $(P_I)' = P_I$.

Theorem 3 (Schrijver(1980)). If $P$ is a rational polyhedron, then $P'$ is a rational polyhedron.

23.1 Optimization over the first closure

We now address optimization over the first Chvátal closure. We know that $P'$ is possibly closer to $P_I$ than $P$. Can we optimize over $P'$ efficiently?

Given: $A, b, c$ where $P = \{x : Ax \leq b\}$
Goal: $\max \{c^T x : x \in P'\}$

Theorem 4 (Eisenbrand 2000). Optimization over first closure is NP-hard.
23.2 Chvátal rank

We saw that $P'$ is possibly closer to $P_I$ than $P$. So, why stop with $P'$? It is possible that $(P')'$ is even closer to $P_I$ than $P'$ itself. This motivates the following definition.

**Definition 5.** Let $P^{(0)} := P, P^{(1)} := (P^{(0)})', P^{(2)} := (P^{(1)})', \ldots, P^{(i+1)} := (P^{(i)})'$ be a sequence of polyhedra obtained by taking Chvátal closure repeatedly.

We have the following immediate observations:

**Proposition 6.**
1. $P^{(0)} = P \supseteq P^{(1)} \supseteq P^{(2)} \supseteq \ldots$
2. $P^{(t)} \supseteq P_I \quad \forall t = 0, 1, 2, \ldots$

But is the sequence finite? Will it converge to $P_I$?

**Theorem 7** (Chvátal, Schrijver). *For every rational polyhedron $P$, there exists a finite number $t$ for which $P^{(t)} = P_I$.***

This theorem is significant for the following reasons:

1. It implies that every valid inequality for $P_I$ can be obtained using a sequence of CG-Cuts derived from existing inequalities.
2. It suggests that using CG-cuts in the cutting plane algorithm could possibly lead to finite termination.

**Definition 8.** The smallest number $t$ for which $P^{(t)} = P_I$ is the Chvátal rank of $P$.

Note that Chvátal rank is a crude estimate for the number of rounds of CG-cuts that we may have to add to arrive at the IP optimum via the cutting plane algorithmic approach. So, small rank would be nice.

**Exercise.** There exist polyhedra $P$ whose Chvátal rank is not bounded by a polynomial in the size of the description of $P$.

However, several discrete optimization problems use binary variables. In which case, the associated polyhedron is within $[0, 1]^n$. We have the following bounds on the Chvátal rank of such polytopes.

**Theorem 9.** *If a polyhedron $P$ is such that $P \subseteq [0, 1]^n$, then Chvátal rank of $P$ is at most $n^2(1 + \log n)$.***

**Theorem 10.** *There exist a polyhedron $P \subseteq [0, 1]^n$ with Chvátal rank at least $cn^2$ for some fixed constant $c$.***

We emphasize that Chvátal rank is not necessarily an upper bound on the number of rounds to termination in the cutting plane algorithmic approach, but finite Chvátal rank suggests using CG-cuts in the cutting plane approach if we hope to have finite termination.
23.3 Cutting Plane Proof

Cutting plane proof is a method for demonstrating that \( c^T x \leq \delta \) is valid for \( P_I \) when we are given constraint matrix \( A \) and RHS vector \( b \) such that \( P = \{ x : Ax \leq b \} \). Equivalently, it is a method for demonstrating that every integral solution of \( Ax \leq b \) satisfies a specified inequality \( c^T x \leq \delta \).

We saw an example of the cutting plane proof in the previous lecture. Let us formalize this now.

**Definition 11.** Let \( Ax \leq b \) be a system of inequalities. A *cutting plane proof* of \( c^T x \leq \delta \) is a sequence of inequalities

\[
\begin{align*}
    c_1^T x &\leq \delta_1 \\
    &\vdots \\
    c_M^T x &\leq \delta_M
\end{align*}
\]

where

1. \( c_M = c, \delta_M = \delta \),
2. \( c_1, \ldots, c_k \) are integral, and
3. \( c_i^T x \leq \delta_i' \) is a non-negative linear combination of the inequalities \( Ax \leq b, c_1^T x \leq \delta_1, \ldots, c_{i-1}^T x \leq \delta_{i-1} \) for some \( \delta_i' \) such that \( \lfloor \delta_i' \rfloor \leq \delta_i \) (i.e., \( c_i^T x \leq \delta_i \) is a CG-cut that is derived from all previous inequalities).

Here, \( M \) is said to be the *length of the proof*.

A cutting plane proof can be viewed as a DAG by labeling each node by an inequality: Here, each node represents a CG-cut obtained using combinations of inequalities. The incoming edges into the node indicate the inequalities that contribute to the combination. See Figure 23.1.

![Figure 23.1: A cutting plane proof as a DAG.](image-url)
**Definition 12.** The *depth of an inequality* is $t$ if it can be obtained as a CG-cut of an inequality that is a combination of inequalities with depth at most $t - 1$.

Every valid inequality for $P_I$ has a cutting plane proof from the inequalities defining $P$. But, will the proof be of finite length? The following theorem proves that there will always be a finite length cutting plane proof.

**Theorem 13.** Let $P = \{x : Ax \leq b\}$ be a rational/bounded polyhedron. Let $w^T x \leq \beta$ be valid for $P_I$. Then there exists a finite depth cutting plane proof of $w^T x \leq \beta$.

*Proof.* By bounded Chvátal rank theorem (i.e., Theorem 7).

Note that if an IP is infeasible (e.g., SAT), then we could potentially demonstrate infeasibility by demonstrating the validity of the inequality $0^T x \leq -1$ for all integer solutions of the IP. This can be done via a cutting plane proof. Hence, we have the following corollary from Theorem 13.

**Corollary 13.1.** Let $P = \{x : Ax \leq b\}$ be a rational polytope. If $P_I = \emptyset$, then there exists a cutting plane proof of $0^T x \leq -1$ from $Ax \leq b$.

There has been a rich line of research trying to derive low depth cutting plane proofs of infeasibility of several natural constraint satisfaction problems.

### 23.4 Application of CG-cuts for structured IPs

Although we started studying the cutting plane algorithmic approach and CG-cuts for unstructured IPs, they have been valuable even for structured IPs. We illustrate an example here.

**Max weight matching in non-bipartite graphs.** We saw the max weight matching problem in bipartite graphs. How about non-bipartite graphs?

- **Given:** $G = (V, E)$, $w : E \to \mathbb{R}$
- **Goal:** $\max \{ \sum_{e \in M} w_e : M \text{ is a matching in } G \}$

The IP is $\max \{ \sum_{e \in E} w_e x_e : X \in P \cap \mathbb{Z}^E \}$ where

$$P := \left\{ x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V, x_e \geq 0 \ \forall e \in E \right\}.$$

Then,

$$P_I := \text{Convex hull of indicator vectors of matchings in } G.$$

Recall that $P \neq P_I$ as illustrated by the following example:

**Example.** Let $G$ be the graph shown in Figure 23.2. For this graph, the solution given in Figure 23.3 is in $P$ but not in $P_I$.

![Figure 23.2: A non-bipartite graph.](image)
Figure 23.3: A point in \( P \) but not in \( P_I \).

Since this is a structured polyhedron, we could try to derive \( P' \) and ask if \( P' = P_I \). Let us see some valid inequalities for the first Chvátal closure for \( P \): we will derive valid inequalities for \( P_I \) using CG-cuts for \( P \). Let \( S \subseteq V \).

Then the inequalities

\[
\sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in S
\]  

(23.1)

are valid for \( P \). By adding all the inequalities of (23.1), we obtain that the inequality

\[
2 \sum_{e \in E(S)} x_e + \sum_{e \in \delta(S)} x_e \leq |S| \quad \text{is valid for } P
\]  

(23.2)

where \( E(S) := \{uv \in E : u, v \in S\} \) and \( \delta(S) := \{uv \in E : |\{u, v\} \cap S| = 1\} \). We also know that the inequalities

\[
-x_e \leq 0 \quad \forall e \in \delta(S)
\]  

(23.3)

are valid for \( P \). By summing up inequalities in (23.2) and (23.3), we see that the inequality

\[
2 \sum_{e \in E(S)} x_e \leq |S| \quad \text{is valid for } P,
\]

i.e., \( \sum_{e \in E(S)} x_e \leq \frac{|S|}{2} \) is valid for \( P \),

i.e., \( \sum_{e \in E(S)} x_e \leq \left\lfloor \frac{|S|}{2} \right\rfloor \) is a CG-cut for \( P \).

Therefore, if \( |S| \) is odd, then we obtain a new valid inequality \( \sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2} \) for \( P_I \) (which is not valid for \( P \)). Thus, we have the following observation:

**Proposition 14.**

\[
P' \subseteq \left\{ \begin{array}{ll}
x \in \mathbb{R}^E : & \sum_{e \in \delta(v)} x_e \leq 1 \\
& \forall v \in V \\
x_e \geq 0 & \forall e \in E \\
& \forall S \subseteq V, |S| \text{ odd} \\
& \sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2} \\
& \forall S \subseteq V, |S| \text{ odd} \\
\end{array} \right\} =: Q
\]
The last set of inequalities above are known as *odd-set inequalities*. Edmonds showed that these additional odd-set inequalities are sufficient to describe the convex-hull of incidence vectors of matchings in \( G \).

**Theorem 15** (Edmonds). \( P_I = Q \).

**Corollary 15.1.** \( P' = P_I \) (\( P_I \subseteq P' \subseteq Q = P_I \)), i.e., Chvátal rank of \( P \) is one.

Thus, CG-cuts are a useful tool even for structured IPs for the following reasons:

- If we have a polyhedron for a structured IP that is not integral, then CG-cuts give a systematic way to get closer to the integral hull.

- For some combinatorial optimization problems (like matchings) CG-cuts give a systematic way to *strengthen the LP-relaxation* (i.e., a systematic way to obtain a tighter LP-relaxation for the integral-hull of the polyhedron). This is useful in approximation algorithms for certain combinatorial optimization problems.

**Sufficient conditions to guarantee** \( P' = P_I \). The corollary above raises the question of whether there are sufficient conditions on \( A \) and \( b \) to conclude that the first closure is equal to the integral hull of the polyhedron (i.e., \( P' = P_I \)). We define a more general family of matrices below:

**Definition 16** (Edmonds-Johnson (EJ) Matrices). Matrix \( A \) is an EJ matrix if \( P(b, c, l, u) = \{ x : b \leq Ax \leq c, l \leq x \leq u \} \) has Chvátal rank at most one for all integral \( b, c, l, u \).

Recall that a matrix \( A \) is a TU matrix if \( P(b, c, l, u) \) has Chvátal rank 0 for all integral \( b, c, l, u \). We have the following sufficient conditions for a matrix to be an EJ matrix:

1. (Edmonds-Johnson) If \( A \) is integral and every column has \( l_1 \)-norm \( \leq 2 \), then \( A \) is an EJ matrix.

2. (Gerards-Schrijver) If \( A \) is integral, every row has \( l_1 \)-norm \( \leq 2 \), and \( A \) has no odd-4 minor, then \( A \) is an EJ matrix.

Identifying general families of EJ matrices remains an interesting research direction.