**IE 511: Integer Programming, Spring 2021** **8 Apr, 2021**

# Lecture 22: Valid inequalities for the integral hull $P_I$

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**Disclaimer**: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

Recall that the integral hull $P_I$ of a polyhedron $P$ is the convex hull of integral points in $P$. Inequalities that are valid for $P_I$ but violated by extreme point optimum $\bar{x}$ (for $\max\{c^T x : x \in P\}$) are known as *cuts/cutting planes* (since they cut out some part of the polyhedron $P$). Cutting planes correspond to valid inequalities for $P_I$. In the next couple of lectures, we will focus on understanding cutting planes before returning to the cutting plane algorithmic approach.

Let $P := \{x \in \mathbb{R}^n : Ax \le b\}$. Recall that we are given the constraint matrix $A$ and the RHS vector $b$. We would like to derive valid inequalities for $P_I$. As a precursor to deriving valid inequalities for $P_I$ (that is violated by extreme point optimum $\bar{x}$), consider the following question: suppose a "little birdie" told us that an inequality $w^T x \le \delta$ is valid for $P_I$; can we verify if this is indeed a valid inequality for $P_I$? We will see the notion of *cutting plane proof* that will help us answer this question.

Formally, we would like to answer the following question:

**Question 0.** Given the constraint matrix $A$ and the RHS vector $b$ such that $P = \{x : Ax \le b\}$ and an inequality $w^T x \le \delta$, can we verify if the inequality $w^T x \le \delta$ is valid for $P_I$?

We start by addressing a simpler question:

**Question 1.** Given the constraint matrix $A$ and the RHS vector $b$ such that $P = \{x : Ax \le b\}$ and an inequality $w^T x \le \delta$, can we verify if the inequality $w^T x \le \delta$ is valid for $P$?

**Recap**

**Definition 1.** An inequality $w^T x \le \delta$ is *valid for a polyhedron* $Q$ if $w^T \bar{x} \le \delta \ \forall \bar{x} \in Q$.

## 22.1 Valid inequalities for $P$

To start off, let us see how to answer Question 1.

**Example:** Let

$$P := \left\{ \begin{array}{ll} x \in \mathbb{R}^3 : & 3x_1 + 5x_2 - 6x_3 \le 5 \\ & 4x_1 - 13x_2 + 7x_3 \le -6 \\ & -6x_1 + 4x_2 \le 7 \end{array} \right\}.$$

**Claim 1.1.** *The inequality $-8x_1 + 9x_2 - 5x_3 \le 25$ is valid for $P$.*

*Proof.* In order to prove this, we multiply the first inequality by 2, the second inequality by 1, and the third inequality by 3 and add them all up. Then we have

$$\begin{array}{rl} 6x_1 + 10x_2 - 12x_3 \leq 10 & \text{is valid for } P \\ 4x_1 - 13x_2 + 7x_3 \leq -6 & \text{is valid for } P \\ -18x_1 + 12x_2 \qquad\quad \leq 21 & \text{is valid for } P \\ \hline -8x_1 + 9x_2 - 5x_3 \leq 25 & \text{is valid for } P. \end{array}$$

Thus, we have proven validity of the inequality $-8x_1 + 9x_2 - 5x_3 \leq 25$ using multipliers $y_1 = 2, y_2 = 1, y_3 = 3$. □

Algebraically, a non-negative linear combination of existing inequalities will give a valid inequality for the polyhedron. The equivalent geometric viewpoint is that a non-negative linear combination of the facet-defining inequalities will give a valid inequality for the polyhedron.

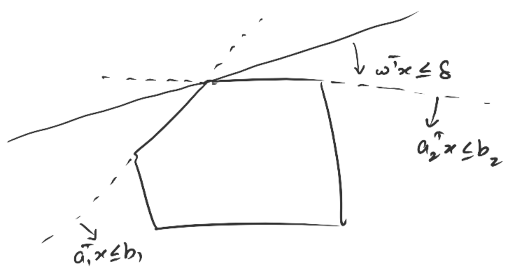



Figure 22.1: $w^T x \leq \delta$ is a positive linear combination of $a_1^T x \leq b_1$ and $a_2^T x \leq b_2$.

We now show that this is the only possible way to obtain valid inequalities for a polyhedron, thus answering Question 1.

**Lemma 1.1** (Farkas)**.** *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\} \neq \emptyset$. The following are equivalent.*

1. *The inequality $w^T x \leq \delta$ is valid for $P$.*
2. *There exists $y \geq 0$ such that $y^T A = w^T$ and $y^T b \leq \delta$ (and $y$ has at most $n$ positive components).*

*Proof.* $\max\{w^T x : x \in P\} \leq \delta$ iff $\min\{y^T b : y^T A = w^T, y \geq 0\} \leq \delta$. Therefore, $w^T x \leq \delta$ is valid for $P$ iff there exists $y \geq 0$ such that $y^T A = w^T$ and $y^T b \leq \delta$. (**Exercise.** Prove the extra structure on $y$ using Caratheodary's theorem.) □

By the above Farkas lemma, if $w^T x \leq \delta$ is valid for $P$, then we can derive/prove the validity of the inequality by showing a vector $y \geq 0$ such that $y^T A = w^T$ and $y^T b = \delta'$ for some $\delta' \leq \delta$. Therefore, we can derive all valid inequalities for $P = \{x : Ax \leq b\}$ by taking positive linear combinations of inequalities in the system $Ax \leq b$. As a consequence, note that there are infinitely many valid inequalities for a polyhedron.

## 22.2 Valid inequalities for $P_I$

We now begin addressing Question 0.

**Cutting Plane Proof to establish validity of an inequality for $P_I$.** We illustrate how to prove validity of an inequality for $P_I$ through an example.

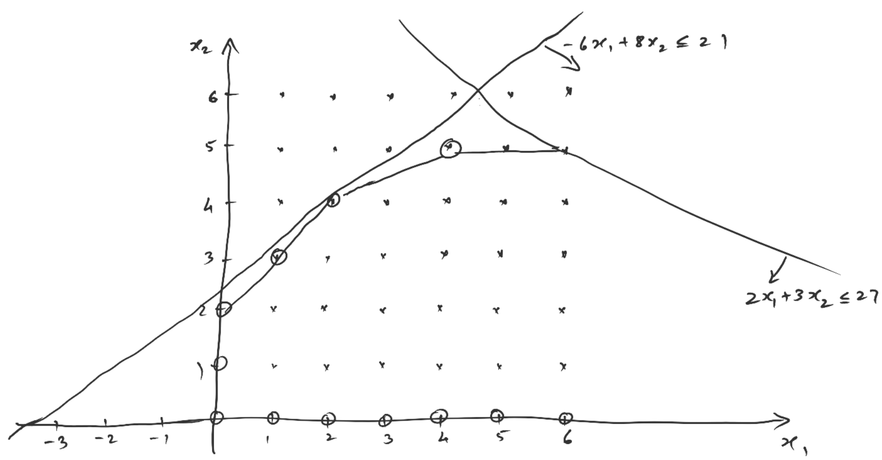
**Example:** Suppose

$$P := \left\{ \begin{array}{ll} x \in \mathbb{R}^2 : & -6x_1 + \ 8x_2 \leq 21 \\ & \ \ 2x_1 + 3x_2 \leq 27 \\ & \ \ \ x_1, x_2 \geq 0 \end{array} \right\}.$$

"Little birdie" tells us that $x_2 \leq 5$ is valid for $P_I$. How do we verify this?

**Claim 1.2.** *The inequality $x_2 \leq 5$ is valid for $P_I$.*

*Proof.* Since this is a two-variable polyhedron, we can plot and verify (see figure below).




From the above figure, it is clear that $x_2 \leq 5$ is valid for $P_I$. How can we prove this algebraically (i.e., without plotting as above)? Here is a proof:

$$
\begin{aligned}
& -6x_1 + 8x_2 \leq 21 \text{ is valid for } P \\
& \qquad \implies -3x_1 + 4x_2 \leq \frac{21}{2} \text{ is valid for } P \\
& \qquad \qquad \text{Note that if } (x_1, x_2) \in P \cap \mathbb{Z}^n \text{, then } -3x_1 + 4x_2 \text{ should be an integer.} \\
& \qquad \implies -3x_1 + 4x_2 \leq \lfloor \frac{21}{2} \rfloor = 10 \text{ is valid for } P_I. \\
& \qquad \implies -6x_1 + 8x_2 \leq 20 \text{ is valid for } P_I. \\
& \qquad \text{Also, } 6x_1 + 9x_2 \leq 81 \text{ is valid for } P_I \text{ since it is valid for } P.
\end{aligned}
$$

Adding these two inequalities, we obtain that $17x_2 \leq 101$ is valid for $P_I$, i.e., $x_2 \leq \frac{101}{17}$ is valid for $P_I$. Again, if $(x_1, x_2) \in P \cap \mathbb{Z}^n$, then $x_2$ is an integer which implies that $x_2 \leq \lfloor \frac{101}{17} \rfloor = 5$ is valid for $P_I$. □

The algebraic proof above is known as the *cutting plane proof* of validity of an inequality for $P_I$.

We explicitly emphasize one of the observations in the above proof:

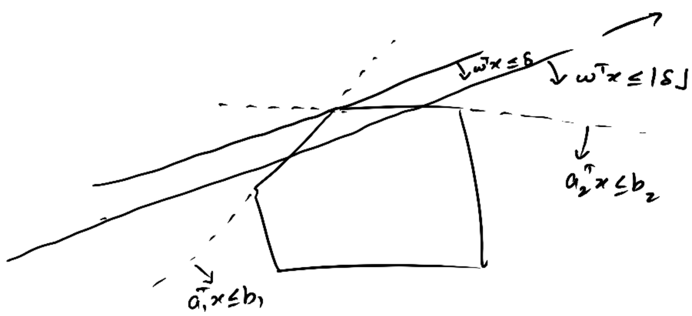
**Proposition 2.** *Suppose $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ for some constraint matrix $A$ and RHS vector $b$. Suppose $w^T x \leq b$ is valid for $P$ and $w$ is an integral vector. Then, $w^T x \leq \lfloor \delta \rfloor$ is valid for $P_I$.*

*Proof.* If $w^T$ is integral, then for all integral points $x \in P$, the inner product $w^T x$ should be an integer. This implies that every integral point $x \in P \cap \mathbb{Z}^n$ should satisfy $w^T x \leq \lfloor \delta \rfloor$. Hence, $w^T x \leq \lfloor \delta \rfloor$ is valid for $P_I$. $\square$

Note that the proposition gives a way to generate valid inequalities for $P_I$. This will be useful in the cutting plane algorithmic approach. We have a terminology for valid inequalities for $P_I$ of the form given in the proposition.

**Definition 3.** Let $P = \{x : Ax \leq b\}$ and $w^T x \leq \delta$ be valid for $P$. If $w$ is integral, then $w^T x \leq \lfloor \delta \rfloor$ is valid for $P_I$ and is known as a *Chvátal-Gomory cut* (CG-cut) for $P$.

**Geometric interpretation of CG-cuts.** The above definition is algebraic. CG-cuts have a nice geometric interpretation: a CG-cut pushes a hyperplane corresponding to a valid inequality of the polyhedron into the polyhedron until it hits an integer point (the integer point that it hits could be outside $P$). See figure below.




### 22.2.1 Chvátal Closure

Let us consider the set obtained by adding all such CG-cuts.

**Definition 4.** $P' := \{x \in P : x \text{ satisfies all CG cuts for } P\}$, i.e.,

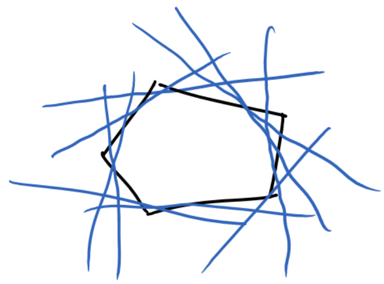
$$P' = \bigcap_{\substack{(w,\delta)\,:\, w \text{ is integral} \\ w^T x \leq \delta \text{ is valid for } P}} \{x \in \mathbb{R}^n : w^T x \leq \lfloor \delta \rfloor\}.$$

$P'$ is the *first Chvátal closure* of $P$.

By Lemma 1.1, we have that

$$P' = \left\{x : Ax \leq b, (y^T A)x \leq \lfloor y^T b \rfloor \;\; \forall y \text{ such that } y^T A \text{ is integral and } 0 \leq y \right\}. \tag{22.1}$$

See figure below for an example. Note that we add possibly infinitely many inequalities to obtain the first Chvátal closure of a polyhedron.



We have the following two observations from definition.

**Proposition 5.** *The following hold:*

1. $P_I \subseteq P' \subseteq P$.
2. $(P_I)' = P_I$.

We note that $P'$ is possibly closer to $P_I$ than $P$. So, the next natural questions are

1. Can we optimize over $P'$ given the inequality description of $P$? and
2. In particular, is $P'$ a polyhedron?

We now show that $P'$ is indeed a polyhedron. Our proof will be based on the lemma below. Note how the lemma improves upon the definition of $P'$ given in (22.1).

**Lemma 5.1.** *Let* $P := \{x : Ax \le b\}$*, where* $A \in \mathbb{Z}^{m\times n}$*,* $b \in \mathbb{Z}^n$*. Then,*

$$P' = \left\{x : Ax \le b, (y^T A)x \le \lfloor y^T b \rfloor \;\; \forall y \text{ such that } y^T A \text{ is integral and } 0 \le y \le 1\right\}.$$

*Proof.* <u>$P' \subseteq$ RHS</u>: This is because each inequality that we have in the set defining the RHS is a CG-cut for $P$.

<u>RHS $\subseteq P'$</u>: Let $w^T x \le \lfloor \delta \rfloor$ be a CG-cut for $P$. We will show that it is implied by inequalities in the following system:

$$(y^T A)x \le \lfloor y^T b \rfloor \;\; \forall y \text{ such that } y^T A \text{ is integral and } 0 \le y \le 1. \tag{22.2}$$

We begin with the following claim.

**Claim 5.1.** *There exists* $y \ge 0$ *with* $y^T A = w^T$ *and* $y^T b \le \delta$*.*

*Proof.* The inequality $w^T x \le \lfloor \delta \rfloor$ is a CG cut for $P$. Therefore, $w$ is integral, and $w^T x \le \delta$ is valid for $P$ which immediately implies that there exists $y \ge 0$ such that $y^T A = w^T, y^T b \le \delta$ (by Farkas lemma above). □

Fix $y$ as in the claim. If all coordinates of $y$ are strictly less than 1, then the CG-cut $w^T x \le \lfloor \delta \rfloor$ is implied by inequalities in (22.2) and we are done. So, assume without loss of generality that $y_1 \ge 1$. Let

$$\bar{y}_i := \begin{cases} y_i - 1 & \text{if } i = 1 \\ y_i & \text{otherwise,} \end{cases}$$

and let $\bar{w} := \sum_{i=1}^m \bar{y}_i a_i^T$ and $\bar{\delta} := \sum_{i=1}^m \bar{y}_i b_i$, where $a_1^T, \ldots, a_m^T$ are the rows of the constraint matrix $A$. This means that $\bar{w}$ is integral (since $w$ was integral and $a_1$ is integral) and $\bar{w}^T x \leq \bar{\delta}$ is valid for $P$ (since it is a non-negative linear combination of inequalities in $Ax \leq b$). Consequently, $\bar{w}^T x \leq \lfloor \bar{\delta} \rfloor$ is valid for $P_I$ and it is a CG-cut for $P$.

**Claim 5.2.** *The CG-cut $w^T x \leq \lfloor \delta \rfloor$ is a non-negative linear combination of inequalities $\bar{w}^T x \leq \lfloor \bar{\delta} \rfloor$ and $a_1^T x \leq b_1$.*

*Proof.* The inequality $\bar{w}^T x \leq \lfloor \bar{\delta} \rfloor$ can equivalently be written as

$$
\begin{aligned}
(y_1 - 1)a_1^T x + \left(\sum_{i=2}^m y_i a_i^T\right) x \leq \left\lfloor (y_1 - 1)b_1 + \sum_{i=2}^m y_i b_i \right\rfloor &= \left\lfloor \sum_{i=1}^m y_i b_i - b_1 \right\rfloor \\
&= \left\lfloor \sum_{i=1}^m y_i b_i \right\rfloor - b_1. \quad \text{(because } b_1 \in \mathbb{Z})
\end{aligned}
$$

Hence, adding the two inequalities

$$
\begin{gathered}
(y_1 - 1)a_1^T x + \left(\sum_{i=2}^m y_i a_i^T\right) x \leq \left\lfloor \sum_{i=1}^m y_i b_i \right\rfloor - b_1 \\
a_1^T x \leq b_1
\end{gathered}
$$

we obtain that $\left(\sum_{i=1}^m y_i a_i^T\right) x \leq \lfloor \sum_{i=1}^m y_i b_i \rfloor$, i.e., we obtain the inequality $w^T x \leq \lfloor \delta \rfloor$. □

The above claim shows that the CG-cut $w^T x \leq \lfloor \delta \rfloor$ is implied by CG-cuts $\bar{w}^T x \leq \lfloor \bar{\delta} \rfloor$ and $a_1^T x \leq b_1$. Claim 5.2 can be applied repeatedly for all coordinates of $y$ which are greater than 1 to conclude that all CG-cuts are implied by the inequalities in (22.2). This proves Lemma 5.1. □

We now show that $P'$ is a polyhedron.

**Theorem 6** (Schrijver(1980))**.** *If $P$ is a rational polyhedron, then $P'$ is a rational polyhedron.*

*Proof.* Let $P := \{x : Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^n$. By Lemma 5.1, we have that

$$
P' = \left\{x : Ax \leq b, (y^T A)x \leq \lfloor y^T b \rfloor \;\; \forall y \text{ such that } y^T A \text{ is integral and } 0 \leq y \leq 1\right\}.
$$

Let $T := \{w : w = y^T A \text{ for some } 0 \leq y < 1\}$. Then, $T$ is bounded. Therefore, $T \cap \mathbb{Z}^m$ is finite. This implies that the number of inequalities (apart from $Ax \leq b$) that are added in the above description of $P'$ is finite. Thus, $P'$ is described by a finite number of inequalities and hence, $P'$ is a polyhedron. Rationality follows by the same argument. □