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# Lecture 2: Bounds, Relaxations, Optimality

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In today's lecture, we will touch upon how to obtain good bounds for an IP. Recall that a discrete optimization problem is specified as $z := \max\{c^T x : x \in \mathcal{X}\}$ for some set $\mathcal{X} \subseteq \mathbb{Z}^n$. Here, the set $\mathcal{X}$ encodes the collection of feasible solutions of the problem. For today's lecture we will focus on the objective being *maximization* for the sake of convention. In order to solve the IP, we need a lower bound and an upper bound on the optimal objective value and moreover, these bounds/values should coincide: i.e., if we know that

$$\begin{aligned} z &\geq z_L, \\ z &\leq z_U, \text{ and} \\ z_L &= z_U, \end{aligned}$$

then $z = z_L = z_U$. We will focus on how to obtain lower and upper bounds on the optimal objective value in this lecture. Note that we want largest possible lower bound and smallest possible upper bound.

## 2.1 Lower Bounds via Feasible Solutions

An obvious approach to obtain a lower bound on the optimal objective value is to identify a feasible solution. This is pretty much the only known approach to obtain lower bounds.

A classic approach to construct a feasible solution is the greedy method. In the greedy method, we construct a solution from scratch by repeatedly setting a variable that gives the best improvement in objective value without violating any constraints. For instance, consider the Binary Knapsack Problem:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} p_j x_j \\ & \sum_{j=1}^{n} w_j x_j \leq W \\ & x_j \in \{0,1\} \ \forall j \in [n] \end{aligned}$$

To obtain a feasible solution for this, we consider the greedy procedure given in the figure below:

**Proposition 1.** *Greedy procedure gives a feasible solution for the Binary Knapsack Problem.*

*Proof.* By induction on $n$. □

Although the greedy approach gives a feasible solution for the Binary Knapsack Problem, it may not be optimal.

GREEDY:

Order the variables such that $\frac{p_i}{w_i} \geq \frac{p_{i+1}}{w_{i+1}}$ for every $i = 1, \dots, n$.

For $i = 1, \dots, n$:

$$\text{Set } x_i = \begin{cases} 1, & \text{if } \sum_{j=1}^{i-1} w_j x_j + w_i \leq W \\ 0, & \text{otherwise} \end{cases}$$

## 2.2 Upper Bounds and Optimality Conditions

Good upper bounds tell us how close is the objective value of our solution (the feasible solution that we have constructed) relative to the optimum objective value. Recall that *optimality conditions* are those conditions that guarantee optimality of a given feasible solution. Optimality conditions help us determine whether a given feasible solution is optimal. Let us begin by recalling the optimality condition for LPs.

**Optimality condition for LPs.** For the primal problem $\max\{c^T x : Ax \leq b, x \geq 0\}$, we define the associated dual problem $\min\{y^T b : y^T A \geq c^T, y \geq 0\}$. Informally, weak duality and strong duality are stated as follows:

- Weak Duality: $c^T x \leq y^T b$ for all all primal feasible solutions $x$ and all dual feasible solutions $y$.
- Strong Duality: $\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c^T, y \geq 0\}$.

Weak duality gives an upper bound on the optimal objective value for the primal LP. Strong duality tells us that the "smallest upper bound via the dual problem" is the tightest possible upper bound for the primal LP. In particular, strong duality gives an optimality condition: A primal feasible solution $x$ that we have in hand is an optimum solution for the primal problem if and only if the optimum objective value of the dual problem coincides with the primal objective value of $x$. Stated alternatively, if we find a dual feasible solution $y$ such that the objective value $y^T b$ of the dual is equal to the objective value $c^T x$ of the primal feasible solution $x$ that we have in hand, then $x$ should be an optimum solution for the primal problem.

Similarly, we can try to derive tight upper bounds for IPs which might help us in verifying optimality. The standard approach to derive upper bounds for a maximization problem is through relaxations.

### 2.2.1 Relaxation

The main idea behind relaxations is to replace the given problem by another optimization problem whose optimal value is at least the optimal value of the given problem. While relaxing the problem, we have two possibilities: (1) Enlarge the set of feasible solutions and (2) Replace the objective function by a function that has the same/larger value for every feasible solution.

**Definition 2.** Consider the IP $\max\{c^T x : x \in \mathcal{X}\}$ for some $\mathcal{X} \subseteq \mathbb{Z}^n$. A problem $\max\{f(x) : x \in \mathcal{T}\}$ for some $\mathcal{T} \subseteq \mathbb{R}^n$ is a relaxation of the IP if

1. $\mathcal{X} \subseteq \mathcal{T}$ and
2. $f(x) \geq c^T x \; \forall \; x \in \mathcal{X}$.

Note that the second condition above needs to hold only for points $x \in \mathcal{X}$ and need not necessarily hold for points $x \in \mathcal{T} \setminus \mathcal{X}$. The proposition below shows that relaxations can be used to get an upper bound on the optimal objective value.

**Proposition 3.** *If* $z_{rel} := \max\{f(x) : x \in \mathcal{T}\}$ *is a relaxation of* $z := \max\{c^T x : x \in \mathcal{X}\}$ *(for some subsets* $\mathcal{T} \subseteq \mathbb{R}^n$ *and* $\mathcal{X} \subseteq \mathbb{Z}^n$*), then* $z \leq z_{rel}$.

*Proof.* Let $x^*$ be an optimum to the IP. Since $x^* \in \mathcal{X} \subseteq \mathcal{T}$, we have that $x^* \in \mathcal{T}$ and hence $f(x^*) \leq z_{rel}$. But $z = c^T x^* \leq f(x^*)$ and hence $z \leq z_{rel}$. □

In addition to giving upper bounds, relaxations are also helpful in identifying infeasibility and optimality.

**Proposition 4.** *Consider the IP* $\max\{c^T x : x \in \mathcal{X}\}$ *for some set* $\mathcal{X} \subseteq \mathbb{Z}^n$.

1. *If a relaxation of the IP is infeasible, then the IP is infeasible.*
2. *Let* $x^*_{rel}$ *be an optimum solution to the relaxation* $\max\{f(x) : x \in \mathcal{T}\}$. *If* $x^*_{rel} \in \mathcal{X}$ *and* $f(x^*_{rel}) = c^T x^*_{rel}$, *then* $x^*_{rel}$ *is also an optimum solution to the IP.*

*Proof.*
1. Since $\mathcal{X} \subseteq \mathcal{T}$, it follows that if $\mathcal{T} = \emptyset$, then $\mathcal{X} = \emptyset$.
2. Since $x^*_{rel} \in \mathcal{X}$, we have $z = \max\{c^T x : x \in \mathcal{X}\} \geq c^T x^*_{rel} = f(x^*_{rel}) = z_{rel}$. But $z_{rel} \geq \max\{c^T x : x \in \mathcal{X}\}$. Therefore, $c^T x^*_{rel} = \max\{c^T x : x \in \mathcal{X}\}$ and $x^*_{rel} \in \mathcal{X}$. Consequently, $x^*_{rel}$ is an optimum solution to the IP.

□

Note that the second part of Proposition 4 gives us an optimality condition.

In order to derive smallest upper bounds for our IP, it is sufficient to construct "tight" relaxations. We will now see some possible ways to construct relaxations.

#### 2.2.1.1 LP-relaxations

An obvious relaxation of an IP is obtained by dropping the integrality constraints.

**Definition 5.** Let $P$ be a polyhedron. The *LP-relaxation* of the IP $\max\{c^T x : x \in P \cap \mathbb{Z}^n\}$ is $\max\{c^T x : x \in P\}$.

The LP-relaxation of an IP is indeed a relaxation (verify that it satisfies the definition of a "relaxation"). We would like to use relaxations to obtain tight/smallest upper bound. Recall the definition of "better formulations" from the previous lecture. Better formulations give tighter/smaller upper bounds.

**Proposition 6.** *Suppose* $P_1$ *and* $P_2$ *are two formulations for the IP* $z := \max\{c^T x : x \in \mathcal{X}\}$ *for some* $\mathcal{X} \subseteq \mathbb{Z}^n$. *Suppose* $P_1$ *is a better formulation than* $P_2$ *(i.e.,* $P_1 \subseteq P_2$*). Let* $z_b^{LP} := \max\{c^T x : x \in P_b\}$ *for* $b = 1, 2$. *Then,*

$$z \leq z_1^{LP} \leq z_2^{LP} \text{ for all objectives } c.$$

*Proof.* **Exercise.** □

Thus, better formulations provide tighter relaxations and consequently tighter/smaller upper bounds.

#### 2.2.1.2 Combinatorial Relaxations

Sometimes, we can relax a difficult COP to an easy COP. The relaxed problem being easy can be solved quickly. The resulting solution gives an upper bound on the objective value of the difficult COP. Such relaxations arise by considering natural combinatorial intepretations.

For example, consider the TSP on the complete graph $G = (V, E)$. Here, we are given arc costs $c_{ij}$ for every $ij \in E$ and the objective is

$$z^{TSP} := \min \left\{ \sum_{ij \in T} c_{ij} : T \subseteq E, T \text{ forms a tour} \right\}.$$

Let us define a subset $T$ of edges to be an *assignment* if there exists a mapping of edges in $T$ to vertices so that every vertex has exactly one edge of $T$ mapped to it. A tour is a permutation and in particular, is an assignment. So, consider

$$z^{assign} := \min \left\{ \sum_{ij \in T} c_{ij} : T \subseteq E, T \text{ is an assignment} \right\}.$$

It should be clear that $z^{assign}$ is a relaxation of $z^{TSP}$ (as we have enlarged the collection of feasible solutions). It turns out that $z^{assign}$ can be solved efficiently. Note that the relaxation $z^{assign}$ is still a discrete optimization problem unlike LP-relaxations which are continuous optimization problems.

#### 2.2.1.3 Lagrangian Relaxations

In LP-relaxations, we dropped the integrality constraints. More generally, we can also drop some linear constraints to obtain a relaxation. We can go a step further: instead of simply dropping linear constraints, we can also fold them into the objective.

**Proposition 7.** *Consider the IP* $z := \max\{c^T x : x \in \mathcal{X} \subseteq Z^n, Ax \leq b\}$. *Let*

$$z(\lambda) := \max\{c^T x + \lambda^T (b - Ax) : x \in \mathcal{X}\}.$$

*Then,* $z \leq z(\lambda)$ *for all* $\lambda \geq 0$.

*Proof.* Let $x^*$ be an optimum solution for the IP. Then $x^* \in \mathcal{X}$ and $Ax^* \leq b$. Therefore,

$$\begin{aligned} z = c^T x^* &\leq c^T x^* + \lambda^T (b - Ax^*) \text{ for all } \lambda \geq 0 \\ &\leq z(\lambda) \text{ for all } \lambda \geq 0. \end{aligned}$$

□

Thus, $z(\lambda)$ for every $\lambda \geq 0$ provides a relaxation. This is known as the *Lagrangian relaxation.* We will discuss more about the Lagrangian relaxation in the latter part of the course. In particular, we will address the question of which non-negative $\lambda$ gives the tightest/smallest upper bound.

### 2.2.2 Duality

Recall that our goal is to verify whether a given solution is optimal. We saw that duality gives a way to obtain tight upper bounds for LPs. Can we derive a similar duality theorem for IPs? Let us use the analogy with LPs to define a dual problem for IPs.

**Definition 8.** Let $z := \max\{c^T x : x \in \mathcal{X}\}$ and $w := \min\{w(u) : u \in \mathcal{U}\}$. The two problems form a *weak dual pair* if

$$c^T x \leq w(u) \ \forall x \in \mathcal{X}, u \in \mathcal{U}.$$

If $z = w$, then they form a strong dual pair.

*Remark.* Between relaxation and dual problems, which is preferable for obtaining tight upper bounds? Well, it is a trade-off. Note that any feasible solution to the dual problem gives an upper bound whereas only optimal solutions to the relaxation problem gives an upper bound. Thus, to get upper bounds using the dual problem, we only need to find feasible solutions to the dual problem while to get upper bounds using the relaxation problem, we need to solve it to optimality.

Now that we have defined the notion of a dual pair, the next natural questions are the following: Does every IP have a dual pair? Does every IP have a weak dual pair? Does every IP have a strong dual pair?

Note that the dual to the LP-relaxation is a weak dual to the IP as shown below.

**Proposition 9.** *Suppose $z := \max\{c^T x : Ax \leq b, x \geq 0, x \in \mathbb{Z}^n\}$. Consider the dual problem to the LP-relaxation, namely $w^{LP} := \min\{y^T b : y^T A \geq c^T, y \geq 0\}$. These two problems, i.e., $z$ and $w^{LP}$, form a weak dual pair.*

*Proof.* **Exercise.** ☐

Similar to relaxations, dual problems also tell us about infeasibility and optimality.

**Proposition 10.** *Let $z = \max\{c^T x : x \in \mathcal{X}\}$ and $w := \min\{w(u) : u \in \mathcal{U}\}$ be a weak dual pair.*

1. *If $w$ is unbounded, then $z$ is infeasible.*
2. *If $x^* \in \mathcal{X}$ and $u^* \in \mathcal{U}$ satisfy $c^T x^* = w(u^*)$, then $x^*$ is an optimum solution for $z$ and $u^*$ is an optimum solution for $w$.*

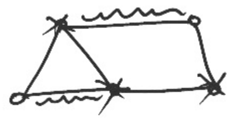
*Proof.* **Exercise.** ☐

#### 2.2.2.1 IP weak dual pair - Example

We will see a weak dual pair of *discrete* optimization problems—i.e., a weak dual pair that does NOT arise from the dual of the LP-relaxation of the IP. Let $G = (V, E)$ be a graph.

**Definition 11.** A *matching* is a set of vertex-disjoint edges.

For example, the squiggly edges in the figure below form a matching.



Every graph has a matching: for example, the empty set is a matching. A natural optimization problem associated with matching is the problem of finding a maximum cardinality matching:

$$(P_1)\ \text{Max cardinality matching} : \max\{|M| : M \subseteq E \text{ is a matching}\}.$$

The size of a max cardinality matching in the graph given in the above figure is 2. This can be shown by exhibiting tight lower and upper bounds: For the lower bound, note that the size of a max cardinality matching is at least 2 since the squiggly edges form a matching. For the upper bound, since we have 5 vertices in the graph, it follows that the maximum number of *vertex-disjoint pairs of vertices* from 5 vertices is at most 2.

**Definition 12.** A *vertex cover* is a set $S$ of vertices with each edge (of the graph) incident to at least one vertex in $S$.

Every graph has a vertex cover: for example, consider the set $S = V$. A natural optimization problem associated with vertex cover is the problem of finding a minimum cardinality vertex cover:

$$(P_2)\ \text{Min cardinality Vertex Cover} : \min\{|S| : S \subseteq V \text{ is a Vertex Cover}\}.$$

**Proposition 13.** *$P_1$ and $P_2$ form a weak dual pair.*

*Proof.* Let $M = \{u_1v_1, u_2v_2, \ldots, u_nv_n\}$ be a matching. Any vertex cover $S$ must contain at least one vertex from each pair $\{u_iv_i\}$ for all $i = 1, \ldots, k$ and they are all disjoint by definition of matching. Therefore, $|S| \geq |M|$. $\square$

**Exercise.** Construct a graph to illustrate that $P_1$ and $P_2$ do not form a strong dual pair.

We will later see that they will form a strong dual pair in certain families of graphs.