IE 511: Integer Programming, Spring 2021	**4 Mar, 2021**

Lecture 12: Matroids

Lecturer: Karthik Chandrasekaran — *Scribe: Karthik*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

Matroids are structures that can be used to model the feasible set of several combinatorial optimization problems. In this lecture, we will define matroids, consider some examples, introduce the matroid optimization problem and see its generality through some concrete optimization problems that it formulates, and see a BIP formulation of the matroid optimization problem. We will subsequently see that the LP-relaxation of the BIP formulation has an integral optimal solution (via the concept of TDI that we learnt in the previous lecture). This will ultimately broaden the family of IPs that can be solved by simply solving the LP relaxation.

12.1 Matroids: Definition and Examples

Definition 1. Let N be a finite set and $\mathcal{I} \subseteq 2^N$ (recall that 2^N denotes the collection of all subsets of N, i.e., $2^N := \{S : S \subseteq N\}$).

1. The pair $(N, \mathcal{I})$ is an *independence system* if

   (i) $\emptyset \in \mathcal{I}$ and

   (ii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (i.e., the set $\mathcal{I}$ has the hereditary property–see Figure 12.1).

   The sets in $\mathcal{I}$ are called *independent sets.*

Figure 12.1: Hereditary property of independent sets: If A is an independent set and B is a subset of A, then B should also be an independent set.

2. An independence system $(N, \mathcal{I})$ is a *matroid* if

   (iii) for all $A, B \in \mathcal{I}$ with $|B| > |A|$ there exists $e \in B \setminus A$ such that $A \cup \{e\} \in \mathcal{I}$.

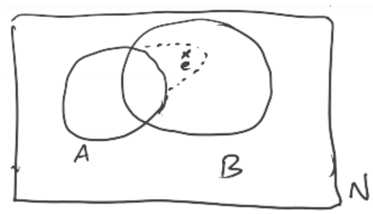


Figure 12.2: Independent set exchange property

The three conditions in the above definitions are together known as *matroid axioms.* We will now see two examples of matroids—we will use these two as running examples as we study matroids. These example matroids also explain the nomenclature for several matroid terminologies.

1. **Matrix matroid.**

   Definition 2. Let $M \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix with columns $M^1, \ldots, M^n$ i.e.,

   $$M = \begin{bmatrix} M^1 & M^2 & \ldots & M^n \end{bmatrix}.$$

   Consider $N := [n]$ and $\mathcal{I} := \{S \subseteq N : \text{vectors } \{M^i : i \in S\} \text{ are linearly independent}\}$. The pair $(N, \mathcal{I})$ is called a *matrix matroid.*

   We now verify that $(N, \mathcal{I})$ satisfies the matroid axioms. The following observation is immediate from definition:

   Observation. $(N, \mathcal{I})$ is an independence system.

   We now verify the third matroid axiom.

   Proposition 3. *$(N, \mathcal{I})$ is a matroid.*

   Proof. Let $A = \{i_1, \ldots, i_r\} \subseteq N$ such that $M^{i_1}, \ldots, M^{i_r}$ are linearly independent and $B = \{j_1, \ldots, j_t\} \subseteq N$ such that $M^{j_1}, \ldots, M^{j_r}$ are linearly independent. Then, $\text{span}\{M^{i_1}, \ldots, M^{i_r}\}$ is a subspace of dimension r and $\text{span}\{M^{j_1}, \ldots, M^{j_r}\}$ is a subspace of dimension t. Since $|A| < |B|$, we have $r < t$. So, there exists a point M^{j_k} such that $M^{j_k} \notin \text{span}\{M^{i_1}, \ldots, M^{i_r}\}$. Therefore, the vectors $M^{i_1}, \ldots, M^{i_r}, M^{j_k}$ are linearly independent. Thus, we have $j_k \in B \setminus A$ with $A \cup \{j_k\} \in \mathcal{I}$. $\square$

2. **Graphic matroid.**

   Definition 4. Let $G = (V, E)$ be a graph, $N := E$ and $\mathcal{I} := \{T \subseteq E : (V, T) \text{ is a forest}\}$. The pair $(N, \mathcal{I})$ is called a *graphic matroid.*

   Recall that a forest is a graph with no cycles. A tree is a connected forest. We now verify that $(N, \mathcal{I})$ is a matroid. Once again, the following observation is immediate from definition:

   Observation. $(N, \mathcal{I})$ is an independence system.

   We now verify the third matroid axiom.

   Proposition 5. *$(N, \mathcal{I})$ is a matroid.*

Proof. We will prove the third matroid axiom. Note that the number of connected components in a forest (V, R) is $|V| - |R|$ (**Exercise**. Prove this by induction on $|R|$).

Let $A, B \in \mathcal{I}$ be forests with $|A| < |B|$. Let $C_1, \ldots, C_r$ be the connected components of (V, A). It implies that $r = |V| - |A|$. Since $|A| < |B|$, the number of connected components in (V, A) is more than the number of connected components in (V, B). Therefore, there exists an edge $e \in B \setminus A$ whose end nodes are in different components of (V, A). Adding e to A will not create any cycle. Thus, we have $e \in B \setminus A$ with $(V, A \cup \{e\})$ being a forest and consequently, $A \cup \{e\} \in \mathcal{I}$. $\square$

12.2 Matroid Optimization: Max weight independent set

We now consider the maximum weight independent set problem over matroids:

Given: a matroid $(N, \mathcal{I})$ and a weight function $w : N \to \mathbb{R}$
Goal: max $\{\sum_{e \in A} w_e : A \in \mathcal{I}\}$

Note that this is a COP formulation. Owing to the wide variety of combinatorial structures that satisfy the matroid axioms, several problems in discrete optimization can be formulated as a max weight independent set problem over some matroid. As special cases of the matroid optimization problem, we mention the following two discrete optimization problems:

1. **Max weight linearly independent set.** Given a matrix M with a weight for each column of M, find a linearly independent set of columns of M with maximum weight.
2. **Max weight forest.** Given a graph with a weight for each of its edge, find a subgraph that is a forest with maximum weight.

The second problem above arises in telecommunications and other network applications. In these applications, the predominant goal is to minimize link installation costs while ensuring that there is at least one path between every pair of nodes. In such scenarios, we are seeking a min cost tree (i.e., min cost spanning tree) as opposed to a max weight forest. These two problems are closely related (we will see the relation later).

12.3 Rank Function of a Matroid

To solve the max weight independent set problem over a matroid, we will formulate it as an IP. In order to formulate it as an IP, we need the rank function of a matroid.

Definition 6. The *rank function* of a matroid $(N, \mathcal{I})$ is the function $r : 2^N \to \mathbb{Z}_{\geq 0}$ given by

$$r(A) := \max\{|I| : I \subseteq A, I \in \mathcal{I}\} \ \forall A \subseteq N.$$

I.e., the rank of a set A is the size of a largest independent set contained in A.

What does the rank function of the matrix matroid/graphic matroid look like? Let us take a closer look.

1. **Rank function of the Matrix Matroid.** Let M be a $m \times n$ matrix and $(N = [n], \mathcal{I})$ be the associated matrix matroid. Then for $A \subseteq N$, we have

$$\begin{aligned} r(A) &= \max \text{ number of linearly independent columns in } \{M^i : i \in A\} \\ &= \text{rank of the matrix } \left[M^i : i \in A\right]. \end{aligned}$$

2. **Rank function of the Graphic Matroid.** Let $G = (V, E)$ be a graph and $(N = E, \mathcal{I})$ be the associated graphic matroid. Then for $A \subseteq E$, we have

$$r(A) = |V| - \text{ number of connected components in } (V, A).$$

Proof.

$$\begin{aligned} r(A) &= \sum_{\text{Components in } (V,A)} \left(\begin{array}{l} \text{Max number of edges that can be chosen from each} \\ \text{component while preserving the forest property} \end{array} \right) \\ &= \sum_{i=1}^{\text{number of components}} (|V_i| - 1) \\ &= |V| - \text{number of components in } (V, A). \end{aligned}$$

□

12.4 Rank function and max weight independent set problem

We next state properties of the rank function that will allow us to formulate the max weight independent set problem as a BIP. We have the following two properties by definition.

1. $r(A) \leq |A| \; \forall A \subseteq N$.
2. $r(A) = |A| \iff A \in \mathcal{I}$.

Using the above two properties, we have the following proposition:

Proposition 7. *Let $X \in \{0,1\}^N$ and let support$(X) := \{e : X_e = 1\}$. Then support$(X) \in \mathcal{I}$ iff $\sum_{e \in S} X_e \leq r(S) \; \forall S \subseteq N$.*

Proof. Let $T = \text{support}(X)$.

$\Longrightarrow$: Fix a subset $S \subseteq N$. Then $\sum_{e \in S} X_e = |T \cap S|$. If $T \in \mathcal{I}$ then $T \cap S \in \mathcal{I}$ (by hereditary axiom). Therefore, $|T \cap S| \leq r(S)$ by definition of the rank function.

$\Longleftarrow$: Suppose $T \notin \mathcal{I}$. Then $\sum_{j \in T} X_j = |\text{support}(X) \cap T| = |T| > r(T)$ which is a contradiction. The last inequality holds by properties 1 and 2 of the rank function.

□

Based on Proposition 7, we have the following BIP formulation of the max weight independent set problem :

$$\max \left\{ \sum_{e \in N} w_e x_e : \sum_{e \in S} x_e \leq r(S) \; \forall S \subseteq N, x_e \in \{0,1\} \; \forall e \in N \right\}.$$

The LP-relaxation of this BIP formulation is

$$\max\left\{\sum_{e\in N} w_e x_e : \sum_{e\in S} x_e \le r(S)\ \forall S\subseteq N, 0\le x_e\le 1\ \forall e\in N\right\}.$$

The polyhedron associated with the LP-relaxation is (for ease of notation, let us index the coordinates of x by elements of N)

$$P^{\text{ind}} := \left\{x\in\mathbb{R}^N : \sum_{e\in S} x_e \le r(S)\ \forall S\subseteq N, 0\le x_e\le 1\ \forall e\in N\right\}.$$

Note that $r(\{e\})\le 1\ \forall e\in N$ by definition. Therefore, we can drop $x_e\le 1$ constraints since they are implied by the rank constraints for singleton sets. Therefore, we have

$$P^{\text{ind}} = \left\{x\in\mathbb{R}^N : \sum_{e\in S} x_e \le r(S)\ \forall S\subseteq N, x_e\ge 0\ \forall e\in N\right\}.$$

To solve the max weight independent set problem, we are interested in the inequality description of $\left(P^{\text{ind}}\right)_I$. We will show that $P^{\text{ind}} = \left(P^{\text{ind}}\right)_I$ by showing that the following system is TDI:

$$\begin{aligned}&\sum_{e\in E} x_e \le r(S)\quad \forall S\subseteq N\\ &x_e\ge 0\quad \forall e\in N\end{aligned}\tag{12.1}$$

With this result, we may solve the max weight independent set problem by solving the LP $\max\{c^T x : x\in P^{\text{ind}}\}$. We need another property of the rank function to show that system (12.1) is TDI.

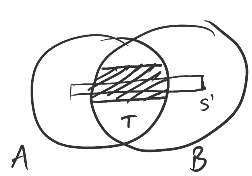
Lemma 7.1. *The rank function of a matroid $(N,\mathcal{I})$ satisfies the following two properties:*

1. $r(B)\le r(A)\ \forall B\subseteq A\subseteq N$ *(non-decreasing).*
2. $r(A)+r(B)\ge r(A\cup B)+r(A\cap B)\ \forall A,B\subseteq N$ *(submodularity).*

Proof. Non-decreasing property of the rank function follows by definition. We will show that $r(A)+r(B)\ge r(A\cup B)+r(A\cap B)\ \forall A,B\subseteq N$. Let $A,B\subseteq N$. Let T be a maximum size independent set in $A\cap B$. Then $r(A\cap B)=|T|$. We need the following claim.

Claim 7.1. *There exists a maximum size independent set S contained in $A\cup B$ such that $S\supseteq T$.*

Proof. Let S' be a maximum size independent set of $A\cup B$.



We perform the following procedure:

Procedure:

initialize $S \leftarrow T$;
while ($|S| < |S'|$) **do**
  By the third matroid axiom, there exists an element $e \in S' \setminus S$ such that $S \cup \{e\} \in \mathcal{I}$;
  Update $S \leftarrow S \cup \{e\}$
end

At the end of this procedure, we obtain a set $S \in \mathcal{I}$ with $|S| = |S'|$, $S \subseteq A \cup B$, and $S \supseteq T$. Hence, S is a maximum size independent set contained in $A \cup B$ such that $S \supseteq T$. □

We now have

$$\begin{aligned} r(A \cap B) + r(A \cup B) &= |T| + |S| \\ &= |S \cap (A \cap B)| + |S \cap (A \cup B)| \\ &= |S \cap A| + |S \cap B| \\ &\leq r(A) + r(B). \end{aligned}$$

The last inequality above holds since $S \in \mathcal{I}$ so by the hereditary axiom of matroids we have that $S \cap A \in \mathcal{I}$, $S \cap B \in \mathcal{I}$. Therefore, by the definition of rank function we have $|S \cap A| \leq r(A)$ and $|S \cap B| \leq r(B)$.

□

Set functions satisfying the properties mentioned in Lemma 7.1 are frequently encountered in the field of optimization. In the next lecture, we will briefly delve into these functions and prove a much more general result (that a linear inequality system associated with such functions is TDI). It would imply that system (12.1) is TDI and consequently, P^{ind} is integral.