IE 511: Integer Programming, Spring 2021 **2 Mar, 2021**

Lecture 11: TU application: Max flow-Min cut, Total Dual Integrality

Lecturer: Karthik Chandrasekaran *Scribe: Karthik*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications.*

In the previous lecture, we saw two applications of TU matrices. In this lecture we will see a third application.

Recap

Definition 1. A polyhedron P is *integral* if $P = P_I$ where $P_I := \text{conv-hull}(P \cap \mathbb{Z}^n)$.

Definition 2. A matrix A is *totally unimodular* (TU) if every square submatrix of A has determinant 0 or 1 or -1.

Lemma 2.1 (Hoffman-Kruskal). *If the constraint matrix A is TU and the RHS vector b is integral then the polyhedron $P = \{x : Ax \le b\}$ is integral.*

Theorem 3 (Sufficient conditions for TU). *Let $A \in \{0, 1, -1\}^{m\times n}$ be a matrix such that*

(i) Each column of A contains at most two non-zeros and

(ii) There exists a partition $M_1 \cup M_2 = [m]$ of the rows such that every column j with two non-zero entries satisfies

$$\sum_{i\in M_1} A_{ij} = \sum_{i\in M_2} A_{ij}.$$

Then A is TU.

Before seeing the third application, let us see another sufficient condition to guarantee the TU property.

Theorem 4. *Let $A \in \{0, 1, -1\}^{m\times n}$. If each column of A contains at most one $+1$ and at most one -1, then A is TU.*

Proof. The theorem can be viewed as a corollary of Theorem 3. I.e., it can be proved by the same approach as the proof of Theorem 3. We saw a proof of Theorem 3 by picking a smallest counterexample and arriving at a contradiction—I mentioned that this proof technique is essentially an inductive proof; to clarify this, we will see an inductive proof of the current theorem.

We will do induction on $m + n$. The base case of $m = 1 = n$ is immediate. Let us do the induction step. Induction hypothesis: Suppose that all $m' \times n'$ matrices that contain at most one $+1$ and at most one -1 are TU for $m' + n' \le m + n - 1$.

Let A be a $m \times n$ matrix that contains at most one $+1$ and at most one -1 in each column. If $m \ne n$, then every square submatrix M of A has number of rows plus number of cols at most $m + n - 1$ and consequently, M has determinant in $\{0, \pm 1\}$ by induction hypothesis and hence,

A is also TU. So, we may assume that $m = n$. Consider a submatrix M of A. If M is a proper submatrix of A (i.e., M has strictly fewer rows/columns than A), then M has at most one $+1$ and at most one -1 in each column and hence, by induction hypothesis, the determinant of M is in $\{0, \pm 1\}$. So, we only need to show that the square matrix A itself has determinant in $\{0, \pm 1\}$.

If there exists a column of A with exactly one non-zero, then expanding along that column and using induction hypothesis implies that determinant of A is in $\{0, \pm 1\}$. Otherwise, all columns of A have exactly one $+1$ and exactly one -1. This implies that the sum of the rows of A is zero. Consequently, the rows of A are linearly dependent and hence, determinant of A is 0.

$\square$

11.1 TU Application: Max $s \to t$ flow in directed graphs

Given a digraph $D = (V, E)$, nodes $s, t \in V$ and non-negative arc capacities $h_{ij} \in \mathbb{R}_{\geq 0}\ \forall ij \in E$, the goal is to find a max $s \to t$ flow. We recall the definition of a $s \to t$ flow and its value below.

Definition 5. A $s \to t$ *flow* is a function $f : E \to \mathbb{R}$ such that

1. the incoming flow at v is equal to the outgoing flow at v for every vertex $v \in V \setminus \{s, t\}$ and
2. the flow on each arc does not exceed the capacity of that arc.

The value of a $s \to t$ flow f is $\sum_{e \in \delta^{\text{out}}(s)} f_e$.

Formulating the max $s \to t$ flow: We may assume that there is no incoming arc into s and no outgoing arc from t (even if such arcs exist, we may remove them without changing the optimum since the optimum will never use these arcs). Now, we add an extra arc from t to s to the graph with arc capacity infinity (see Figure 11.1). Let $D = (V, E)$ be the resulting digraph with arc capacities $h : E \to \mathbb{R}_{\geq 0}$.

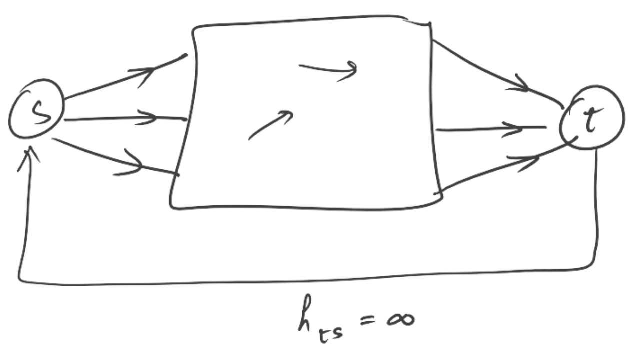

Figure 11.1: Flow problem formulation.

Further, we also define (see Figure 11.2)

$$V^{\text{in}}(i) := \{k : \overrightarrow{ki} \in E\}$$
$$V^{\text{out}}(i) := \{k : \overrightarrow{ik} \in E\}.$$

Note that $V^{\text{in}}(i) \cap V^{\text{out}}(i)$ could be non-empty.

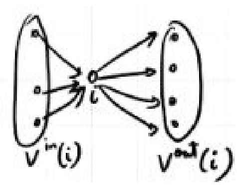

Figure 11.2: Incoming and Outgoing neighbors from a node i.

We can now formulate the max $s \to t$ flow as a linear program:

$$
\begin{aligned}
&\max \quad x_{ts} & & \\
&\sum_{k \in V^{\text{out}}(i)} x_{ik} - \sum_{k \in V^{\text{in}}(i)} x_{ki} = 0 \quad \forall i \in V & & \text{(Flow conservation constraints)} \\
&x_{ij} \leq h_{ij} \quad \forall ij \in E & & \text{(Capacity constraints)} \\
&x_{ij} \geq 0 \quad \forall ij \in E & & \text{(Non-negativity contraints)}
\end{aligned}
\tag{11.1}
$$

Proposition 6. *The constraint matrix A in Formulation (11.1) is TU.*

Proof. The constraint matrix A is of the form $\begin{bmatrix} C \\ -C \\ I \\ -I \end{bmatrix}$ where C is the constraint matrix for flow conservation. By properties of TU matrices, it is sufficient to show that C is TU. The rows of C are indexed by nodes and the columns of C are indexed by arcs with

$$
C[i, e] = \begin{cases} 1 & \text{if } e = ik \text{ for } k \in V^{\text{out}}(i), \\ -1 & \text{if } e = ki \text{ for } k \in V^{\text{in}}(i), \\ 0 & \text{otherwise.} \end{cases} .
$$

Thus, $C \in \{0, 1, -1\}^{V \times E}$ and every column of C has exactly one $+1$ and one -1. Therefore, by Theorem 4, the matrix C is TU. □

Note that Proposition 6 essentially shows that the vertex-arc incidence matrix of a digraph is TU.

Proposition 6 leads to the following corollary using Hoffman-Kruskal:

Corollary 6.1. *If all arc capacities are integral, then the LP (11.1) has an integral optimum solution.*

In particular, consider the max $s \to t$ *integral* flow problem: the input is the same as above, but the goal is to find a max *integer-valued* $s \to t$ flow, i.e., flow on every arc should be integer-valued. Note that we can solve it using the same LP if all arc capacities are integral. Let us focus on the strong dual problem to max $s \to t$ integral flow problem. We will show that the strong dual problem is indeed the min $s \to t$ cut problem and thereby derive the max flow-min cut theorem.

Definition 7. (i) A $s \to t$ *cut* in a directed graph D is a set U such that $s \in U \subseteq V - \{t\}$.

(ii) The *capacity* of a $s \to t$ cut U is $c(U) := \sum_{ij \in \delta^{out}(U)} h_{ij}$.

Example: In Figure 11.3, suppose all h_{ij}s are equal to one. Then $c(U) = 2$.

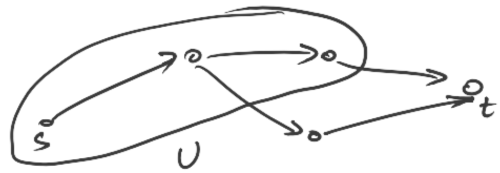

Figure 11.3: A $s \to t$ *cut*

Cuts are interesting from the perspective of a network attacker. The following problem gives a quantitative measure of the weakness in connectivity from node s to node t.

Min $s \to t$ cut problem: Given a digraph $D = (V, E)$, nodes $s, t \in V$, and non-negative capacities $h_{ij} \in \mathbb{R}_{\geq 0} \forall ij \in E$, the goal is to find a $s \to t$ cut of minimum capacity.

We will now show that the LP-dual to Formulation (11.1) is in fact formulating the min $s \to t$ cut problem. The dual to LP (11.1) is

$$\begin{aligned} \min \sum_{ij \in E} & h_{ij} w_{ij} \\ \text{s.t. } u_i - u_j + w_{ij} &\geq 0 \quad \forall ij \in E \\ u_t - u_s &\geq 1 \\ w_{ij} &\geq 0 \quad \forall ij \in E \end{aligned}$$

The constraint matrix of the dual is the transpose of the constraint matrix of the primal. We already saw that the constraint matrix of the primal is TU. Hence, the constraint matrix of the dual is also TU. Therefore, the dual has an integral optimum solution (for all h_{ij}s which are not necessarily integral). Let (u, w) be a dual optimum solution that is integral. We may assume that $u_s = 0$ (otherwise, setting $u'_i = u_i - u_s \; \forall i \in V$ gives a dual feasible solution (u, w) with the same obj value). Let $S := \{j \in V : u_j \leq 0\}$, $\bar{S} := V \setminus S$.

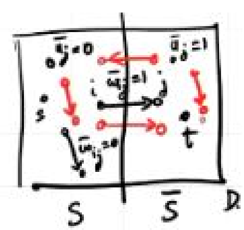

Figure 11.4: A cut and the related variables

Observe that $\bar{S} = \{j \in V : u_j \geq 1\}$ since u is integral. Consider

$$\bar{u}_j := \begin{cases} 0 & \text{if } j \in S \\ 1 & \text{if } j \in \bar{S} \end{cases} \quad \text{and} \quad \bar{w}_{ij} := \begin{cases} 1 & \text{if } i \in S, j \in \bar{S}, ij \in E \\ 0 & \text{otherwise} \end{cases}.$$

Observe that

1. $(\bar{u}, \bar{w})$ is a dual feasible solution: **exercise**: verify that the dual constraint for the arcs in all possible positions are satisfied (see the red arcs in Figure 11.4).

2. Moreover,
$$\sum_{ij \in E} h_{ij}\bar{w}_{ij} = \sum_{\substack{ij \in E \\ i \in S \\ j \in \bar{S}}} h_{ij} \leq \sum_{\substack{ij \in E \\ i \in S \\ j \in \bar{S}}} h_{ij}w_{ij} \leq \sum_{ij \in E} h_{ij}w_{ij}.$$

   The first equation is by definition of $\bar{w}$. The first inequality holds since $w_{ij} \geq u_j - u_i \geq 1$ for every $ij \in E$ with $i \in S, j \in \bar{S}$. Thus, the objective value of $(\bar{u}, \bar{w})$ is at most the objective value of the dual optimum solution (u, w).

Therefore, $(\bar{u}, \bar{w})$ is a dual optimum solution. We observe that $s \in S,\ t \in \bar{S}$, and $\delta^{out}(S) = \{ij : \bar{w}_{ij} = 1\}$. Hence, the set S is a $s \to t$ cut with capacity equal to the dual optimum value which is equal to the maximum $s \to t$ flow value in (D, h). We have thus shown the following theorem:

Theorem 8 (Max flow-Min cut). *Let D be a digraph with nodes s and t and non-negative arc capacities. Then the maximum $s \to t$ flow value is equal to the minimum $s \to t$ cut capacity.*

11.2 Total Dual Integrality

If $P = \{x : Ax \leq b\}$ is integral, then we know that the primal $\max\{c^T x : Ax \leq b\}$ always has an integral optimum solution along every direction c when the objective value is finite. How about the dual?

In particular, we saw that if A is TU and the objective vector c is integral, then the dual also has an integral optimum solution. If c is not necessarily integral, then even if we have a dual optimum solution that is integral, scaling c by $\frac{1}{k}$ for large k leads to a dual optimum solution which is not integral.

For several discrete optimization problems which are solvable efficiently, we have integral dual optimum solution if c is integral. In order to unify and describe discrete optimization problems with such dual problems (i.e., dual problems which have an integral optimum solution), Edmonds and Giles proposed the following definition.

Definition 9 (Edmonds-Giles). A rational linear system $Ax \leq b$ is *totally dual integral* (TDI) if for all integral c with $z^{LP} := \max\{c^T x : Ax \leq b\}$ finite, the dual $\min\{y^T b : y^T A = c^T, y \geq 0\}$ has an integral optimum solution.

On first glance, the TDI definition might seem to be unhelpful towards understanding whether the primal problem has an integral optimum solution—we usually care about whether the primal problem has an integral optimum solution, so how does knowing whether the dual problem has an integral optimum solution help us?! In fact, Edmonds and Giles developed the theory of TDI also as a tool towards proving integrality of the primal. Let us see how TDI can be used as a tool to show integrality of the primal.

Theorem 10 (Edmonds-Giles). *If $Ax \leq b$ is TDI and b is integral then $P = \{x : Ax \leq b\}$ is integral.*

Proof. Recall the characterization of integral polyhedron: the polyhedron $P = \{x : Ax \le b\}$ is integral iff $z^{LP} := \max\{c^T x : Ax \le b\}$ is integer-valued for all objective directions $c \in \mathbb{Z}^n$ for which the optimum objective value is finite.

Suppose $z^{LP} = \max\{c^T x : Ax \le b\}$ is finite. Then by duality,

$$z^{LP} = \min\{y^T b : y^T A = c^T, y \ge 0\}.$$

The right side has an integer optimum solution y^* by the TDI property of the system $Ax \le b$. Since b is integral $y^{*T} b \in \mathbb{Z}$ and hence z^{LP} is integer-valued. Therefore, for all $c \in \mathbb{Z}^n$, if Z^{LP} is finite then Z^{LP} is integer-valued. This implies that $P = P_I$. □

Significance of Edmonds-Giles' theorem. Recall that we have seen total unimodularity as one tool that allows us to confirm integrality of a polyhedron. Edmonds-Giles's theorem shows that TDI is another tool that allows us to confirm integrality of a polyhedron. We will see applications of TDI to confirm integrality of a polyhedron in the next few lectures.

Remark 1. We emphasize that TDI is a property of the system $Ax \le b$ and not a property of the polyhedron $P = \{x : Ax \le b\}$ that is defined by that system. See the example below.

TDI is a property of the inequality system

Example: Consider the following two systems.

System 1: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ System 2: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Both define the same polyhedron but we will see that system 1 is not TDI while system 2 is TDI. We write the two primal problems:
Primal 1: $\max\{c_1 x_1 + c_2 x_2 : x_1 + x_2 \le 0, x_1 - x_2 \le 0\}$ and
Primal 2: $\max\{c_1 x_1 + c_2 x_2 : x_1 + x_2 \le 0, x_1 - x_2 \le 0, x_1 \le 0\}$.
The corresponding dual problems are
Dual 1: $\min\{0 : y_1 + y_2 = c_1, y_1 - y_2 = c_2, y_1, y_2 \ge 0\}$ and
Dual 2: $\min\{0 : y_1 + y_2 + y_3 = c_1, y_1 - y_2 = c_2, y_1, y_2, y_3 \ge 0\}$
For instance, suppose $c_1 = 2, c_2 = 1$. Then
Dual 1: $y_1 + y_2 = 2, y_1 - y_2 = 1 \implies y_1 = \frac{3}{2}, y_2 = \frac{1}{2}$, i.e., it has a unique dual optimum solution.
Dual 2: $y_1 + y_2 + y_3 = 2, y_1 - y_2 = 1, y_1, y_2, y_3 \ge 0 \implies y_1 = \dfrac{3 - y_3}{2}, y_2 = \dfrac{1 - y_3}{2}$. For instance, $y_3 = 1, y_1 = 1, y_2 = 0$ is an integral dual optimum solution.
Indeed $y_1 + y_2 + y_3 = c_1, y_1 - y_2 = c_2, y_1, y_2, y_3 \ge 0$ has integral feasible solution for every $c_1, c_2 \in \mathbb{Z}$.

Remark 2. We also emphasize that the RHS vector b being integral is crucial to apply the Edmonds-Giles' theorem. In particular, the system $Ax \le b$ being TDI does not necessarily imply that $P = \{x : Ax \le b\}$ is integral.

Edmonds-Giles' theorem tells us that a TDI system with integral RHS describes an integral polyhedron. The following theorem gives a converse:

Theorem 11 (Giles-Pulleyblank)**.** *Let P be a rational polyhedron. Then*

1. *There exists a unique minimal TDI system $Ax \leq b$ with A integral such that $P = \{x : Ax \leq b\}$.*
2. *$P = P_I$ iff the RHS vector b (in the unique minimal TDI system $Ax \leq b$ with A integral) can be chosen to be integral.*

The first part of the theorem says that every rational polyhedron has a TDI description. However, the RHS vector b in this TDI description may not be integral. The second part of the theorem says that the RHS vector b is integral in this TDI description iff P is an integral polyhedron.

In the next couple of lectures, we will focus on matroids, and more generally submodular functions, and show that an inequality system associated with these structures is TDI. This will in turn allow us to solve numerous discrete optimization problems by simply solving an LP-relaxation of a suitable IP.