9.1 IPs with fast algorithms

Recap

Definition 1. A polyhedron $P$ is integral if $P = P_I$ where $P_I := \text{conv}(P \cap \mathbb{Z}^n)$.  

Example: See Figure 9.1.

![Figure 9.1: Integral polyhedron](image)

$P$ is not integral $\quad P$ is integral

Recall that if we can solve integral-optimization over the polyhedron $P$ if it is integral by simply solving the LP-relaxation.

Application: Perfect matching in bipartite graphs.

Lemma 1.1. A graph $G$ is bipartite iff $G$ does not contain a cycle with odd number of edges.

Definition 2. A perfect matching in $G$ is a set $M$ of edges such that each vertex is adjacent to exactly one edge in $M$.

Definition 3. $\text{PM}(G) := \{\chi^M : M$ is a perfect matching in $G\}$ where $\chi^M \in \{0,1\}^E$ is defined as

$$\chi^M(e) := \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise}. \end{cases}$$

We are interested in obtaining an inequality description of $\text{conv}(\text{PM}(G))$ in order to be able to solve the minimum cost perfect matching problem in bipartite graphs.

Theorem 4 (Birkhoff). Let $G$ be a bipartite graph. Let

$$P(G) := \left\{ x \in \mathbb{R}^E : \sum_{e \in E : e \text{ is incident to } v} x(e) = 1 \ \forall v \in V, x(e) \geq 0 \ \forall e \in E \right\}.$$
Then

1. \( \text{Conv}(\text{PM}(G)) = P(G) \), and

2. \( P(G) \) is integral.

Proof. First we will prove that conclusion 1 implies conclusion 2. Then we will show conclusion 1.

\[
(P(G))_I = \text{conv} (P(G) \cap \mathbb{Z}^E) \quad \text{(by definition of integral hull)}
\]

\[
= \text{conv} (P(G) \cap \text{PM}(G)) \quad \text{(integral points in } P(G) \text{ are indicator vectors of perfect matchings in } G)
\]

\[
= \text{conv} (\text{PM}(G))
\]

\[
= P(G) \quad \text{(from conclusion 1)}.
\]

Now we will prove conclusion 1 by proving that \( \text{Conv}(\text{PM}(G)) \subseteq P(G) \) and \( P(G) \subseteq \text{conv} (\text{PM}(G)) \).

1. \( \text{conv}(\text{PM}(G)) \subseteq P(G) \): Holds since \( \chi^M \in P(G) \) for all perfect matching \( M \) of \( G \). Therefore, since \( P(G) \) is convex, \( \text{conv}(\text{PM}(G)) \) is also in \( P(G) \).

2. \( P(G) \subseteq \text{conv}(\text{PM}(G)) \): Note that \( P(G) \) is a polytope. We will show this by showing that all extreme points of \( P(G) \) are integral. This is sufficient since each integral point in \( P(G) \) represents a perfect matching in \( G \). So \( P(G) = \text{conv}(\text{extreme points of } (P(G)) \subseteq \text{conv}(\text{PM}(G)) \).

Figure 9.2: Proof idea for \( P(G) \subseteq \text{conv}(\text{PM}(G)) \): Consider \( \bar{x} \) (the edges with no value shown have a value of zero) which is not integral. It can be expressed as a convex combination of points in \( P(G) \) and hence, \( \bar{x} \) is not an extreme point of \( P(G) \).

We now show that all extreme point of \( P(G) \) are integral (see Figure 9.2 for the approach.) Let \( \bar{x} \) be an extreme point of \( P(G) \). Suppose \( \bar{x} \) has fractional coordinates. Let \( F := \{ e \in E : 0 < x_e < 1 \} \) be the set of fractional edges. Consider \( H := G[F] \), the subgraph induced by the fractional edges. Degree of each node in \( H \) is at least two. Therefore, \( H \) contains a cycle, say \( C \). Also, \( H \) is a subgraph of a bipartite graph and hence, all cycles in \( H \) should have even number of edges. So, \( C \) has even number of edges. Order the edges of \( C \) as they appear in the cycle, say \( C = e_1 e_2 \ldots e_{2r-1} e_{2r} \).
Figure 9.3: An even cycle

Let

\[ y_1(e) := \begin{cases} 
\bar{x}(e) & \text{if } e \notin C \\
\bar{x}(e) + \epsilon & \text{if } e \in C \text{ and } e = e_{2i} \text{ for some } i \in [r] \text{ (e is an even edge in the cycle)} \\
\bar{x}(e) - \epsilon & \text{if } e \in C \text{ and } e = e_{2i+1} \text{ for some } i \in [r-1] \text{ (e is an odd edge in the cycle)} 
\end{cases} \]

and let

\[ y_2(e) := \begin{cases} 
\bar{x}(e) & \text{if } e \notin C \\
\bar{x}(e) - \epsilon & \text{if } e \in C \text{ and } e = e_{2i} \text{ for some } i \in [r] \text{ (e is an even edge in the cycle)} \\
\bar{x}(e) + \epsilon & \text{if } e \in C \text{ and } e = e_{2i+1} \text{ for some } i \in [r-1] \text{ (e is an odd edge in the cycle)} 
\end{cases} \]

For \( \epsilon := \min_{e \in C}\{\bar{x}(e), 1 - \bar{x}(e)\} \), the points \( y_1 \) and \( y_2 \) are in \( P(G) \) as they satisfy all the constraints. Moreover, \( y_1 \neq y_2 \) (because \( 0 < \bar{x}(e) < 1 \ \forall e \in E \) and hence \( \epsilon > 0 \)). Therefore, \( \bar{x} = \frac{y_1 + y_2}{2} \) where \( y_1 \neq y_2 \) and \( y_1, y_2 \in P(G) \). So, \( \bar{x} \) is not an extreme point of \( P(G) \) which is a contradiction.

\[ \square \]

**Corollary 4.1.** We can find a minimum cost perfect machining in bipartite graphs by solving

\[
\min \left\{ \sum_{e \in E} c_e x(e) : \sum_{e \in \delta(v)} x(e) = 1 \ \forall v \in V, \ x(e) \geq 0 \ \forall e \in E \right\}.
\]

Theorem 4 gives us an inequality description of the perfect matching polytope in bipartite graphs. We proved the theorem by showing that \( P(G) \) is integral. More generally, it would be helpful if we could recognize integral polyhedra from the inequality description. For this purpose, we will characterize integral polyhedra. Once we have such a characterization, Birkhoff’s theorem and integrality of several other polyhedra become much easy to show.

### 9.2 Integral Polyhedra

**Lemma 4.1.** Let \( P \) be a rational polyhedron. The following are equivalent:

1. \( P \) is integral i.e., \( P = P_1 \).
2. Each face of \( P \) contains an integral vector.
3. Each minimal face of \( P \) contains an integral vector.
4. \( \max\{c^T x : x \in P\} \) has an integral optimum solution \( \forall c \in \mathbb{R}^n \) for which the optimum objective value is finite.

Note: Suppose \( P \) is a polytope. Then minimal faces of \( P \) are extreme points of \( P \) therefore, \( 1 \iff 3 \) is equivalent to saying that \( P \) is an integral polytope iff all extreme points of \( P \) are integral.

Proof of lemma. We will prove the lemma by showing that \( 1 \implies 2 \implies 3 \implies 4 \implies 1 \).

1. \( 1 \implies 2 \):
   Let \( F \) be a face of \( P \). Then \( F = P \cap H \) for some supporting hyperplane \( H \) of \( P \). Let \( x \in F \). From 1, we have \( P = P_I = \text{conv}(P \cap \mathbb{Z}^n) \) which implies that \( x \) is a convex combination of integral points in \( P \). These integral points should also be in \( H \) since \( H \) is a supporting hyperplane of \( P \). So, we have integral points in \( P \cap H = F \).

2. \( 2 \implies 3 \): Holds since minimal faces are also faces.

3. \( 3 \implies 4 \): Let \( \delta := \max\{c^T x : x \in P\} \) be finite. Then \( F := \{x \in P : c^T x = \delta\} \) is a face of \( P \). From 3, we have that there exists an integral point in \( F \).

4. \( 4 \implies 1 \): In order to show that \( P = P_I \), we will show that \( P_I \subseteq P \) and \( P \subseteq P_I \).
   
   (a) \( P_I \subseteq P \): By definition of \( P_I \).
   
   (b) \( P \subseteq P_I \): Suppose \( P \not\subseteq P_I \). Then there exists \( y \in P \setminus P_I \) and an inequality \( w^T x \leq \delta \) that is valid for \( P_I \) but violated by \( y \) (see Figure 9.4). Therefore,
   \[
   \max\{w^T x : x \in P_I\} \leq \delta < \max\{w^T x : x \in P\}.
   \]

   Figure 9.4: \( P \not\subseteq P_I \)

   Now we will show that \( \max\{w^T x : x \in P\} \) is finite. Say not. Then there exists a rational \( z \in P \) such that \( w^T z > 0 \) and \( \alpha z \in P \forall \alpha > 0 \) (i.e. there exist an extreme ray...
in $P$ along the direction of $w$). Therefore, there exists an infinite sequence of $\alpha_1, \alpha_2, \ldots$ such that $\alpha_iz \in P \cap \mathbb{Z}^n$ and $w^T(\alpha_iz) > w^T(\alpha_{i-1}z)$ for every $i = 2, 3, \ldots$. Therefore, $\max\{w^T x : x \in P_I\}$ is unbounded. This contradicts $\max\{w^T x : x \in P_I\} < \delta$.

Thus, $\max\{w^T x : x \in P_I\} \leq \delta < \max\{w^T x : x \in P\}$ implies that $\max\{w^T x : x \in P\}$ is finite but without an optimum solution that is integral, thus contradicting 4.

\[\square\]

More characterization of integral polyhedra are also known. These characterizations are helpful to identify integral polyhedra.

**Lemma 4.2.** Let $P$ be a rational polyhedron. The following are equivalent.

1. $P$ is integral.
2. $\max\{c^T x : x \in P\}$ has an integral optimum solution $\forall c \in \mathbb{Z}^n$ for which the optimum objective value is finite.
3. The value $z = \max\{c^T x : x \in P\}$ is an integer $\forall c \in \mathbb{Z}^n$ for which the optimum objective value is finite.
4. Every rational supporting hyperplane of $P$ contains an integral vector.

In the next lecture, we will see a family of integral polyhedra.