Disclaimer: These notes have not been subjected to the usual scrutiny reserved for formal publications.

Recall that a polyhedron is a Minkowski sum of a polytope and a cone. In this lecture we will focus on minimal description of a polyhedron.

**Minimal Description of a Polyhedron**

A minimal description of a polyhedron should have as few constraints as possible. There are two aspects of getting a minimal description:

1. We need minimal constraints to remove subspaces that do not intersect the polyhedron (see Example 1).
2. We need an irredundant system to describe the polyhedron (see Example 2).

**Example 1:** Consider the polyhedron

\[
P = \{(x, y, z) : x + y + z \leq 4, x + y \geq 3, 1 \leq y \leq 2, z \geq 1\}.
\]

It can be re-written as

\[
P = \{(x, y, z) : x + y = 3, 1 \leq y \leq 2, z = 1\}.
\]

We have thus obtained a description for \(P\) as a subset of \(\mathbb{R}^3\) for which \(z = 1\).

**Example 2:** Consider the polyhedron below for which we have redundant constraints. Dropping such redundant constraints does not change the polyhedron.

![Figure 5.1: A redundant constraint for a polyhedron](image)

To address the first aspect of getting a minimal description of a polyhedron we need to understand the subspace containing the polyhedron. To achieve this, we will understand the dimension of the subspace containing a polyhedron.
5.1 Dimension of a Polyhedron

Intuitively, the dimension of a polyhedron tells us the number of degrees of freedom. See the below example for intuition.

**Example:** Consider the numbers of degrees of freedom in the following figures as the notion of dimension.

(a) Number of degrees of freedom = 0
(b) Number of degrees of freedom = 1
(c) Number of degrees of freedom = 2
(d) Number of degrees of freedom = 2

![Figure 5.2: Dimension of a polyhedron](image)

Now we will formally define the dimension of a polyhedron. An important property of the dimension of a polyhedron is that it should be invariant under translation. To ensure this property, we will use the notion of affine independence. We will later see that affine independence is invariant under translation.

**Definition 1.** Vectors $x^1, \ldots, x^m \in \mathbb{R}^n$ are **affinely independent** if the only solution to the system

\[
\begin{align*}
\lambda_1 x^1 + \cdots + \lambda_m x^m &= 0 \\
\lambda_1 + \cdots + \lambda_m &= 0
\end{align*}
\]

is $\lambda_1 = \cdots = \lambda_m = 0$.

Linear independence implies affine independence by definition.

**Proposition 2.** $x^1, \ldots, x^m$ are linearly independent $\implies$ $x^1, \ldots, x^m$ are affinely independent.

Affine independence does not imply linear independence:

**Note:** $x^1, \ldots, x^m$ are affinely independent $\not\implies$ $x^1, \ldots, x^m$ are linearly independent.

**Example:** In Figure 5.4, $a_1$ and $a_2$ are affinely independent, i.e.,

\[
\begin{align*}
\lambda_1 + 2\lambda_2 &= 0, \quad \lambda_1 + \lambda_2 &= 0 \implies \lambda_1 = \lambda_2 = 0.
\end{align*}
\]

However, $a_1$ and $a_2$ are not linearly independent since $a_2 = 2a_1$.

![Figure 5.3: Affine independence does not imply linear independence](image)

Linear independence and affine independence are related:
Proposition 3. The following are equivalent:

1. \( x^1, \ldots, x^m \) are affinely independent.
2. \( x^2 - x^1, \ldots, x^m - x^1 \) are linearly independent.
3. \( (x^1_1), \ldots, (x^m_1) \) are linearly independent.

Proof. Exercise. □

Affine Independence is invariant under transition:

Proposition 4. Let \( x^1, \ldots, x^m, w \in \mathbb{R}^n \). If \( x^1, \ldots, x^m \) are affinely independent, then \( x^1 - \omega, \ldots, x^m - \omega \) are affinely independent.

Proof. Suppose \( \lambda_1(x^1 - \omega) + \cdots + \lambda_m(x^m - \omega) = 0 \) and \( \lambda_1 + \cdots + \lambda_m = 0 \). This implies that

\[
\begin{cases}
\lambda_1 x^1 + \cdots + \lambda_m x^m = (\sum_{i=1}^m \lambda_i) \cdot \omega \\
\lambda_1 + \cdots + \lambda_m = 0
\end{cases}
\]

\[\implies \begin{cases}
\lambda_1 x^1 + \cdots + \lambda_m x^m = 0 \\
\lambda_1 + \cdots + \lambda_m = 0
\end{cases}\]

\[\implies \lambda_1 = \cdots = \lambda_m = 0 \implies x^1 - \omega, \ldots, x^m - \omega \text{ are affinely independent.} \]

With this understanding of affine independence, we define the dimension of a set.

Definition 5. The dimension of a set \( K \subseteq \mathbb{R}^n \) is one less than the maximum cardinality of an affinely independent subset of \( K \).

Example: See Figure 5.2 and note that the dimension (under this definition) and the intuitive notion of degrees of freedom coincide.

Note: Dimension is invariant under translation.

The above definition of dimension is not helpful to understand the dimension of a polyhedron as a polyhedron is given by its inequality description. Recall that a polyhedron is of the form \( P = \{ x : Ax \leq b \} \). It would be nice to be able to compute the dimension of a polyhedron from its inequality description. Let us see how to do this.

Definition 6. The affine-hull of a set \( S \subseteq \mathbb{R}^n \) is

\[
\text{affine-hull}(S) := \left\{ \lambda_1 a_1, \ldots, \lambda_t a_t : t \geq 1, \ a_1, \ldots, a_t \in S, \ \sum_{i=1}^t \lambda_i = 1 \right\}.
\]

Example: Suppose \( S = \{a_1, a_2\} \) as shown in Figure 5.4. The affine-hull(S) is shown in Figure 5.4.
Definition 7. An inequality $a_i^T x \leq b_i$ from $Ax \leq b$ is called an implicit equality if

$$a_i^T \bar{x} = b_i \forall \bar{x} \in \{ x : Ax \leq b \}.$$ 

Example: Suppose $P = \{(x, y, z) : x + y + z \leq 4, x + y \geq 3, z \geq 1, 1 \leq y \leq 2\}$ then $x + y + z \leq 4, x + y \geq 3, z \geq 1$ are implicit equalities.

Observation: There exists $\bar{x} \in P$ such that $A^=\bar{x} = b^=, A^+\bar{x} < b^+$. In other words, there is an interior point in $P$.

Proof idea. Assuming that the polyhedron is non-empty, we know that there exists $\bar{x}$ such that

$$A^=\bar{x} = b^=, A^+\bar{x} \leq b^+, a_i^T \bar{x} = b_i$$ for some $a_i^T x \leq b_i$ in $A^+x \leq b^+$.

Since $a_i^T x_0 \leq b_i$ is in $A^+x \leq b^+$, it follows that there exists $x_0 \in P$ for which $a_i^T x_0 < b_i$. Now move from $\bar{x}$ towards $x_0$ and repeat.

With this notation, we can obtain the affine-hull of a polyhedron from its inequality description.

Lemma 8. Let $P = \{ x : Ax \leq b \}$, then

$$\text{affine-hull}(P) = \{ x \in \mathbb{R}^n : A^=x = b^= \} = \{ x \in \mathbb{R}^n : A^=x \leq b^= \}.$$ 

Proof. We prove this lemma by proving three containments:

1. $\text{affine-hull}(P) \subseteq \{ x : A^=x \leq b^= \}$:
   By definition we have that $P \subseteq \{ x : A^=x \leq b^= \}$. Let $\bar{x} \in \text{affine-hull}(P)$. Then
   $$\bar{x} = \lambda_1 x^1 + \cdots + \lambda_t x^t$$ for some $x^1, \ldots, x^t \in P$, $\lambda_1, \ldots, \lambda_t \in \mathbb{R}$, $\sum_{i=1}^t \lambda_i = 1$
   $$\implies A^=\bar{x} = \lambda_1 A^=x^1 + \cdots + \lambda_t A^=x^t = \left( \sum_{i=1}^t \lambda_i \right) b^= = b^= $$
   $$\implies \bar{x} \in \{ x : A^=x = b^= \}.$$ 

2. $\{ x : A^=x = b^= \} \subseteq \{ x : A^=x \leq b^= \}$: This is immediate.
3. \( \{ x : A^=x \leq b^= \} \subseteq \text{affine-hull}(P) \):

Let \( \bar{x} \) satisfy \( A^=x \leq b^= \) and let \( x' \in P \) such that \( A^=x' = b^=, A^+x' < b^+ \). If \( \bar{x} = x' \) then \( \bar{x} \) is in \( P \) and hence \( \bar{x} \) is in \text{affine-hull}(P) \). So, we may assume that \( \bar{x} \neq x' \).

The line segment joining \( \bar{x} \) and \( x' \) contains more than one point in \( P \) since \( A^+x' \leq b^+ \) and \( A^-y \leq b^- \) for all \( y \) in the line segment.

![Figure 5.5: Line segment joining \( \bar{x} \) and \( x' \)](image)

Let \( x_0 \) be another point in this line segment in \( P \) (beside \( x' \)). Now \( \text{affine-hull}(P) \supseteq \text{affine-hull}\{x_0, x'\} \supseteq \bar{x} \). Therefore, \( \bar{x} \in \text{affine-hull}(P) \).

![Figure 5.6: Line segment joining \( \bar{x} \) and \( x' \) contains more than one point in \( P \)](image)

**Corollary 9.** Let \( P = \{ x : Ax \leq b \} \). Then

\[
\dim(P) = \dim(\text{affine-hull}(P)) = n - \text{rank}(A^=).
\]

**Proof.** The first equality holds since \( P \) and \( \text{affine-hull}(P) \) contain the same number of affinely independent vectors. For the second equality, by Lemma \( S \) we have:

\[
\text{affine-hull}(P) = \{ x \in \mathbb{R}^n : A^=x = b^= \}
\]

\[
\dim(\{ x \in \mathbb{R}^n : A^=x = b^= \}) = \dim(\{ x \in \mathbb{R}^n : A^-x = 0 \})
\]

\[
= \dim(\text{null space of columns of } A^-)
\]

\[
= n - \text{rank}(A^-)
\]

\[
\implies \dim(\text{affine-hull}(P)) = n - \text{rank}(A^=).
\]

\( \square \)
Corollary 9 allows us to use the inequality description of the polyhedron to understand its dimension.

Now that we understand the dimension of a polyhedron, we can replace \( A^=x \leq b^= \) in the polyhedron description \( Ax \leq b \) by a minimal description \( A'x \leq b' \) with the property

\[
\text{affine-hull}\{x : A'x \leq b'\} = \text{affine-hull}\{x : A^=x \leq b^=\}.
\]

**Example:**

\[
P = \{(x, y, z) : x + y + z \leq 4, x + y \geq 3, 1 \leq y \leq 2, z \geq 1\} = \{(x, y, z) : x + y = 3, 1 \leq y \leq 2, z = 1\}.
\]

We now see some more implications of the above results.

**Corollary 10.** If \( a_i^Tx \leq b_i \) is an implicit equality in \( Ax \leq b \), then \( a_i^Tx \leq b_i \) is also an implicit equality in the system \( A^=x \leq b^= \).

**Definition 11.** A polyhedron \( P \subseteq \mathbb{R}^n \) has full dimension if \( \dim(P) = n \).

**Corollary 12.** A polyhedron \( \{x : Ax \leq b\} \) has full dimension iff there are no implicit equalities in \( Ax \leq b \).

### 5.2 Redundant Inequalities

Next we want to understand which inequalities are necessary to describe a polyhedron.

**Definition 13.** Consider \( Ax \leq b \). An inequality is redundant if it is implied by other inequalities in \( Ax \leq b \). An irredundant system has no redundant inequality.

We seek an irredundant system to describe a polyhedron. There are two kinds of redundant inequalities as seen in the figure below.

It is helpful to distinguish between the two.

**Definition 14.** Let \( P = \{x : Ax \leq b\} \):

1. An inequality \( c^Tx \leq \delta \) is valid for \( P \) if \( c^T\bar{x} \leq \delta, \forall \bar{x} \in P \).
2. \( \{x : c^Tx = \delta\} \) is a supporting hyperplane of \( P \) if \( \delta = \max{c^Tx : Ax \leq b} \) and \( c \neq 0 \). i.e. \( c^Tx \leq \delta \) is valid for \( P \) and \( \{x : c^Tx = \delta\} \) intersects \( P \).

Note that both kind 1 and kind 2 in the figure are valid, but kind 1 gives a supporting hyperplane while kind 2 does not.

Using these tools, we will define a face of a polyhedron. We will see that Maximal faces give a minimal inequality description for the polyhedron and Minimal faces give the vertices of the polyhedron.