Disclaimer: These notes have not been subjected to the usual scrutiny reserved for formal publications.

26.1 Mixed IPs

MIP: \( \max \{ c^T x : Ax = b, x \geq 0, x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \} \).

Example: \( S = \{ (x, y) : x, y \geq 0, x + y \geq 5/2, y \in \mathbb{Z} \} \)

Gomory’s approach for generating cuts for IPs does not give valid cuts for MIPs. Recall that Gomory’s cuts for IPs are CG-cuts.

Recap

CG-cut: \( w^T x \leq \delta \) is valid for \( P \) and \( w \) is integral. If \( x \) is integral then \( w^T x \leq \lfloor \delta \rfloor \) is valid for \( P_I \).

But in a MIP, all variables of \( x \) are not integral. So we need an alternate approach to obtain cuts. We will see such an approach in this lecture. We start with two-dimensional cases.

**Lemma 0.1.** Let \( S^\geq := \{ (x, y) \in \mathbb{R} \times \mathbb{Z} : x + y \geq b, x \geq 0 \} \) and \( f := b - \lfloor b \rfloor > 0 \). Then the inequality \( \frac{x}{f} + y \geq \lfloor b \rfloor \) is valid for \( S^\geq \).

*Proof.* Let \((\bar{x}, \bar{y}) \in S^\geq\). If \( \bar{y} \geq [b] \), then \( \bar{x} \geq 0 \geq f ([b] - \bar{y}) \). If \( \bar{y} < [b] \), then \( \bar{x} \geq b - \bar{y} = f + ([b] - \bar{y}) \geq f + f([b] - \bar{y}) = f ([b] - \bar{y}) \) (the last inequality holds because \( [b] - \bar{y} \geq 0 \) and \( f \leq 1 \)). It implies that \( \frac{x}{f} + \bar{y} \geq \lfloor b \rfloor \). \( \square \)
In the above example, \( f = 1/2 \) and hence \( 2x + y \geq 3 \) is valid for \( S \).

**Corollary 0.1.** Let \( S^\leq := \{ (x,y) \in \mathbb{R} \times \mathbb{Z} : y \leq b + x, x \geq 0 \} \). Suppose \( f := b - \lfloor b \rfloor > 0 \). Then \( y \leq \lfloor b \rfloor + \frac{x}{1 - f} \) is valid for \( S^\leq \).

**Proof.** We have \( y \leq b + x \) iff \( x - y \geq -b \). Moreover, \( -b - \lfloor -b \rfloor = 1 - f \). By Lemma 0.1, \( \frac{x}{1 - f} - y \geq \lfloor -b \rfloor = -\lfloor b \rfloor \) is valid for \( S^\leq \). \( \square \)

Note that when \( x = 0 \), we obtain a CG-cut type inequality. Hence the above are generalizations of CG-cuts for mixed integer sets.

Let \( S^{\text{MIR}} := \{ (x,y) \in \mathbb{R} \times \mathbb{Z}^2 : x, y \geq 0, a_1y_1 + a_2y_2 - x \leq b \} \) with \( b \notin \mathbb{Z} \).

**Lemma 0.2** (Mixed Integer Rounding). Let \( f = b = \lfloor b \rfloor \) and \( f_i = a_i - \lfloor a_i \rfloor \) for \( i = 1,2 \). If \( f_1 \leq f \leq f_2 \), then

\[
[a_1]y_1 + \left( \lfloor a_2 \rfloor + \frac{f_2 - f}{1 - f} \right) y_2 \leq \lfloor b \rfloor + \frac{x}{1 - f}
\]

is valid for \( S^{\text{MIR}} \).

**Proof.** The inequality \( [a_1]y_1 + [a_2]y_2 \leq b + x + (1 - f_2)y_2 \) is valid for \( S^{\text{MIR}} \) (because \( y_1 \geq 0 \) and \( a_2 = [a_2] - (1 - f_2) \)). By Corollary 0.1,

\[
[a_1]y_1 + [a_2]y_2 \leq \lfloor b \rfloor + \frac{x + (1 - f_2)y_2}{1 - f}
\]

is valid for \( S^{\text{MIR}} \). It means that

\[
[a_1]y_1 + \left( \lfloor a_2 \rfloor + \frac{1 - f_2}{1 - f} \right) y_2 \leq \lfloor b \rfloor + \frac{x}{1 - f}
\]

is valid for \( S^{\text{MIR}} \). Rewriting gives the lemma. \( \square \)

Note that if \( f_1 \leq f \leq f_2 \) does not hold, then we can use one of the previous lemmas to get a cut. We now have the ingredients to obtain a valid inequality for a mixed integer region that is violated by an extreme point optimum that is not integral.

### 26.1.1 Gomory Mixed Integer Cut

Let \( (\bar{x}, \bar{y}) \) be a basic feasible solution of the MIP \( \max \{ c_1^T x + c_2^T y : A_1 x + A_2 y = b, x, y \geq 0, y \in \mathbb{Z}^p, x \in \mathbb{R}^{n-p} \} \). Let \( y_i \) be the basic variable that has integer constraint but \( \bar{y}_i \notin \mathbb{Z} \). Consider the \( i \)th row of the optimal tableau:

\[
y_i + \sum_{j \in N_1} \bar{a}_{ij} y_j + \sum_{j \in N_2} \bar{a}_{ij} y_j = \bar{b}_i
\]

where \((y_i, y, x) \in \mathbb{Z} \times \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \) and \( y_i, y, x \geq 0 \). Let

\[
S^i := \{ (y_i, y, x) \in \mathbb{Z} \times \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} : y_i, y, x \geq 0, y_i + \sum_{j \in N_1} \bar{a}_{ij} y_j + \sum_{j \in N_2} \bar{a}_{ij} y_j = \bar{b}_i \}.
\]
Lemma 0.3. Let \( f_j := \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor \) for all \( j \in N_1 \cup N_2 \) and \( f_0 := \bar{b} - \lfloor \bar{b} \rfloor \). Then

\[
\sum_{j \in N_1 : f_j \leq f_0} f_j y_j + \sum_{j \in N_1 : f_j > f_0} \left( 1 - f_j \right) \frac{f_0}{1 - f_0} y_j + \sum_{j \in N_2 : a_{ij} > 0} \bar{a}_{ij} x_j + \sum_{j \in N_2 : a_{ij} < 0} \left( \frac{f_0}{1 - f_0} \right) \bar{a}_{ij} x_j \geq f_0 \tag{26.1}
\]

is a valid inequality for \( S^i \) that is violated by \((\bar{x}, \bar{y})\).

Inequality (26.1) is known as Gomory Mixed Integer Cut.

Proof. Violation follows as LHS at \((\bar{x}, \bar{y})\) is 0 while RHS > 0. We now show validity. The MIR inequality for \( S^i \) is

\[
y_i + \sum_{j \in N_1 : f_j \leq f_0} |\bar{a}_{ij}| y_j + \sum_{j \in N_1 : f_j > f_0} \left( |\bar{a}_{ij}| + \frac{f_j - f_0}{1 - f_0} \right) y_j + \sum_{j \in N_2 : a_{ij} < 0} \left( \frac{\bar{a}_{ij}}{1 - f_0} \right) x_j \leq \lfloor \bar{b}_i \rfloor .
\]

Substituting \( y_i \) gives the inequality (26.1). \( \square \)

Example:

\[
\begin{align*}
\text{max } & \quad 4y - x \\
7y - 2x & \leq 14 \quad (1) \\
x & \leq 3 \quad (2) \\
2y - 2x & \leq 3 \quad (3) \\
y, x & \geq 0 \\
y & \in \mathbb{Z}
\end{align*}
\]

We introduce slack variables \( s_1, s_2, s_3 \) for inequalities (1), (2), and (3) and solve the LP-relaxation. The optimal solution for the LP-relaxation is given by

\[
\begin{align*}
z = \max & \quad \frac{59}{7} - \frac{4}{7} s_1 - \frac{1}{7} s_2 \\
y + \frac{1}{7} s_1 - \frac{2}{7} s_2 & = \frac{20}{7} \\
x + s_2 & = 3 \\
- \frac{2}{7} s_1 + \frac{10}{7} s_2 + s_3 & = \frac{23}{7}
\end{align*}
\]

with the basic variables being \( \bar{x} = 3, \bar{y} = 20/7, \bar{s}_3 = 23/7 \). We note that \( \bar{y} \) is fractional. Therefore, the first row gives the MIR cut \( y \leq 2 \). Eliminating \( y \) gives \( \frac{1}{7} s_1 - \frac{2}{7} s_2 \geq \frac{6}{7} \). We add this cut and re-optimize to obtain \( y = 2, x = 1/2 \). Since \( y \) is integral, this is an optimal solution.

Similar to IPs, Gomory’s cutting plane algorithm for MIPs can also be shown to terminate in a finite numbers of steps using a careful choice of variable for cut generation and a careful choice of LP solving algorithm.

26-3
26.2 Speeding Up Technique 1: Lagrangian Duality

To speed up solution techniques, we will try to derive good upper bounds for a maximization IP. This will help in the branch and bound algorithm as it will enable the pruning of more nodes in the enumeration tree. Consider the IP $z = \max \{ c^T x : Ax \leq b, Dx \leq d, x \in \mathbb{Z}^n \}$. In several applications, we can partition the constraints of the IP as $Ax \leq b, Dx \leq d$ where $Ax \leq b$ are nice, (i.e., IP with just these constraints are easy to solve or have good approximation algorithms) and $Dx \leq d$ are complicating constraints. Examples of easy constraints are flow constraints, matching constraints, TU constraint matrix along with integral RHS, and matroid constraints. An examples of a collection of complicating constraints are degree bounds for the graphic matroid which we highlight below.

Example: Minimum cost degree bounded spanning tree problem.

Given: $G = (V,E), c : E \rightarrow \mathbb{R}_+, b \rightarrow \mathbb{Z}_+$
Goal: $\min \{ \sum_{e \in T} c(e) : T \text{ is a spanning tree with } \deg_T(u) \leq b(u) \ \forall u \in V \}$

IP:

$$\min \sum_{e \in T} c(e)$$

$$Ax \leq b \begin{bmatrix} \sum_{e \in F} x(f) \leq r(F) \\ \sum_{e \in E} x_e = |V| - 1 \\ x_e \in \{0, 1\} \ \forall e \in E \end{bmatrix}$$

$$Dx \leq d \begin{bmatrix} \sum_{e \in \delta(u)} x_e \leq b(u) \ \forall u \in V \end{bmatrix}$$

where $r(F)$ is the rank function of the graphic matroid over $G$. If we drop the complicating constraints then the IP can be solved. Many problems have such a structure. Dropping these complicating constraints and solving the resulting relaxation gives a bound on the optimum. But it could be weak since some constraints are ignored. One way to address these is by bringing these constraints into the objective with a penalty term. This leads us to the Lagrangian relaxation.

Definition 1. Let $z = \max \{ c^T x : x \in S, Dx \in d \}$ where $D \in \mathbb{R}^{m \times n}, d \in \mathbb{R}^m$. Let $u \in \mathbb{R}^m, u \geq 0$ and IP$(u)$ be

$$z(u) := \max \{ c^T x + u^T (d - Dx) : x \in S \}.$$ 

IP$(u)$ is the Lagrangian relaxation of IP with parameter $u$. Here, $u$ is the Lagrange multiplier associated with the constraints $Dx \leq d$.

IP$(u)$ handles the complicating constraints by having them in the objective with a penalty term $u^T (d - Dx)$. We will study the power of this relaxation in the next lecture.