Recap

**Definition 1.** Let $P^{(0)} := P, P^{(1)} := (P^{(0)})', P^{(2)} := (P^{(1)})', \ldots, P^{(i+1)} := (P^{(i)})'$ be a sequence of polyhedra obtained by taking Chvátal-Gomory closure repeatedly.

**Observation.** 1) $P^{(0)} \supseteq P^{(1)} \supseteq P^{(2)} \supseteq \ldots$, and 2) $P^{(t)} \supseteq P_I \forall t$.

**Theorem 2** (Chvátal, Schrijver). For every rational polyhedron $P$, there exists a finite number $t$ for which $P^{(t)} = P_I$.

**Definition 3.** The smallest number $t$ for which $P^{(t)} = P_I$ is the Chvátal rank of $P$.

Theorem 2 is false for irrational polyhedra as illustrated by the following example:

**Example** (Schrijver): $P = \{x \in \mathbb{R}^2 : x_2 - cx_1 = 0, x_1 \geq 0\}$ for some irrational $c$. Then $P_I = \{0\}$, but $P^{(k)} = P \forall k$.

Theorem 2 is true for irrational polytopes which we now prove.

**Theorem 4** (Chvátal (1973)). For every (possibly irrational) polytope $P$, there exists a finite number $t$ for which $P^{(t)} = P_I$.

**Proof.** Let $P \subseteq \mathbb{R}^n$ be a polytope. Since $P$ is bounded, $P \subseteq Q := \{x : -\Delta_i \leq x_i \leq \Delta_i \forall i \in [n]\}$. For every integral vector $z \in Q \setminus P$, there exists a rational half-space containing $P$ but not $z$. Let $S$ be the intersection of all such half-spaces. This means that $S$ is a rational polyhedron (finite number of inequalities since $Q$ is bounded). By Theorem 2 there exists a finite $t$ such that $S^{(t)} = S_I$. Therefore,

$$P_I \subseteq P^{(t)} \subseteq S^{(t)} = S_I = P_I \implies P^{(t)} = P_I.$$

\[\square\]

### 24.1 Application of CG-cuts for structured IPs

#### 24.1.1 Max weight matching in non-bipartite graphs

**Given:** $G = (V, E), w : E \to \mathbb{R}$

**Goal:** $\max\{\sum_{e \in M} w_e : M \text{ is a matching in } G\}$
The IP is \( \max \{ \sum_{e \in E} w_e x_e : X \in P \cap \mathbb{Z}^E \} \) where

\[
P := \left\{ x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V, x_e \geq 0, \ \forall e \in E \right\}.
\]

Then,

\[ P_I := \text{Convex hull of indicator vectors of matchings in } G. \]

Recall that \( P \neq P_I \) as illustrated by the following example:

**Example:** Let \( G \) be the graph shown in Figure 24.1 then the solution given in Figure 24.2 is in \( P \) but not in \( P_I \).

![Figure 24.1: A non-bipartite graph.](image)

Let us see some inequalities for the first Chvátal closure for \( P \). We want to derive valid inequalities for \( P_I \). Let \( S \subseteq V \).

![Figure 24.2: A point in \( P \) but not in \( P_I \).](image)

Then the inequalities

\[
\sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in S
\]

are valid for \( P \). If we add all the inequalities of (24.1) we obtain that the inequality

\[
2 \sum_{e \in E(S)} x_e + \sum_{e \in \delta(S)} x_e \leq |S|
\]

is valid for \( P \) where \( E(S) := \{ uv \in E : u, v \in S \} \) and \( \delta(S) := \{ uv \in E : |\{u, v\} \cap S| = 1 \} \). We also know that the inequalities

\[
-x_e \leq 0 \ \forall e \in \delta(S)
\]

are valid.
are valid for \( P \). By summing up inequalities in (24.2) and (24.3), we see that the inequality
\[
2 \sum_{e \in E(S)} x_e \leq |S| \quad \text{is valid for } P.
\]
i.e.,
\[
\sum_{e \in E(S)} x_e \leq \frac{|S|}{2} \quad \text{is valid for } P
\]
i.e.,
\[
\sum_{e \in E(S)} x_e \leq \left\lfloor \frac{|S|}{2} \right\rfloor \quad \text{is a CG-cut for } P.
\]

Therefore, if \(|S|\) is odd, then we obtain new valid inequalities \( \sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2} \) for \( P_I \) which are not valid for \( P \). Thus, we have the following observation:

**Observation.**
\[
P' \subseteq \left\{ x \in \mathbb{R}^E : \begin{array}{ll}
x \in \mathbb{R}^E : & \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \\
x \geq 0 & \forall v \in E \\
\sum_{e \in E(S)} x_e \leq \left\lfloor \frac{|S| - 1}{2} \right\rfloor & \forall S \subseteq V, |S| \text{ odd}
\end{array} \right\} =: Q
\]

The last set of inequalities above are known as *odd-set inequalities*. Edmonds showed that the odd-set inequalities are sufficient to describe the convex-hull of incidence vectors of matchings in \( G \).

**Theorem 5** (Edmonds). \( P_I = Q \).

**Corollary 5.1.** \( P' = P_I \) (\( P_I \subseteq P' \subseteq Q = P_I \)), i.e., Chvátal rank of \( P \) is one.

Thus, CG-cuts are a nice tool even for structured IPs:

- If we do not have an integral polyhedron for a structured IP then CG cuts give a systematic way to get closer to the integral hull.
- For some combinatorial optimization problems, like matchings, CG cuts *tell us how to strengthen the LP*.

### 24.2 More on Chvátal closure

The Chvátal rank may not be bounded by a polynomial in the size of the description of \( P \) (see HW for an example). Several discrete optimization problems use binary variables. In which case, the associated polyhedron is within \([0,1]^n\). We have the following bounds on the Chvátal rank of such polytopes.

**Theorem 6.** If \( P \subseteq [0,1]^n \), then Chvátal rank of \( P \) is at most \( n^2(1 + \log n) \).

**Theorem 7.** There exists \( P \subseteq [0,1]^n \) with Chvátal rank = \( \Omega(n^2) \).

We know that \( P' \) is possibly closer to \( P_I \) than \( P \). Can we optimize over \( P' \) efficiently?

**Given:** \( A, b, c \) where \( P = \{x : Ax \leq b\} \)

**Goal:** \( \max\{c^T x : x \in P'\} \)
Theorem 8 (Eisenbrand 2000). Optimization over first closure is NP-hard.

This raises the question of whether there are sufficient conditions on $A$ and $b$ to conclude that $P' = P_T$. Note that the polyhedron $P$ defined in Section 24.1.1 has this property. We define a more general family of matrices below:

**Definition 9** (Edmonds-Johnson (EJ) Matrices). A is an EJ matrix if $P(b,c,l,u) = \{x : b \leq Ax \leq c, l \leq x \leq u\}$ has Chvátal rank at most one for all integral $b,c,l,u$.

Recall that $A$ is a TU matrix if $P(b,c,l,u)$ has Chvátal rank 0 for all integral $b,c,l,u$. We have the following sufficient conditions for a matrix to be an EJ matrix:

1. (Edmonds-Johnson) If $A$ is integral and every column has $l_1$-norm $\leq 2$, then $A$ is an EJ matrix.
2. (Gerards-Schrijver) If $A$ is integral and every column has $l_1$-norm $\leq 2$ and every row has $l_1$-norm $\leq 2$ and $A$ has no odd-$k_4$ minor.

Identifying more general conditions remains an interesting research problem.

### 24.3 Cutting Plane Proof

Cutting plane proof is a method for demonstrating that every integral solution of $Ax \leq b$ satisfies a specified inequality $c^T x \leq \delta$. We could potentially show infeasibility of an IP (e.g., SAT) by deriving $0^T x \leq -1$ using CG-cuts.

**Definition 10.** Let $Ax \leq b$ be a system of inequalities. A cutting plane proof of $c^T x \leq \delta$ is a sequence of inequalities.

\[
\begin{align*}
c_1^T x &\leq \delta_1 \\
&\vdots \\
c_M^T x &\leq \delta_M
\end{align*}
\]

where

1. $c_M = c, \delta_M = \delta$.
2. $c_1, \ldots, c_k$ are integral.
3. $c_i^T x \leq \delta_i'$ is a non-negative linear combination of the inequalities $Ax \leq b, c_1^T x \leq \delta_1, \ldots, c_{i-1}^T x \leq \delta_{i-1}$ for some $\delta_i'$ such that $\lfloor \delta_i' \rfloor \leq \delta_i$.

Also, $M$ is the length of the proof.

A cutting plane proof can be viewed as a DAG by labeling each node by an inequality. See Figure 24.3.
Nodes represent CG-cuts obtained using combinations of inequalities.

**Definition 11.** The depth of an inequality is \( t \) if it can be obtained as a CG-cut of an inequality that is a combination of inequalities with depth at most \( t - 1 \).

**Theorem 12.** Let \( P = \{x : Ax \leq b\} \) be a rational/bounded polyhedron. Let \( w^T x \leq \beta \) be valid for \( P_I \). Then there exists a finite depth cutting plane proof of \( w^T x \leq \beta \).

**Proof.** By bounded Chvátal rank theorem. \( \square \)

**Corollary 12.1.** Let \( P = \{x : Ax \leq b\} \) be a rational polytope. If \( P_I = \emptyset \), then there exists a cutting plane proof of \( 0^T x \leq -1 \) from \( Ax \leq b \).