In today’s lecture, we will touch upon some considerations while solving IPs. Recall that a discrete optimization problem can be specified as \(\max \{ c^T x : x \in X \subseteq \mathbb{Z}^n \} \) for some set \( X \). Here, the set \( X \) encodes the collection of feasible solutions of the problem. For today’s lecture we will focus on the objective being maximization. In order to solve the problem, we need:

1. a procedure to find feasible solutions and
2. a procedure to verify if a given feasible solution is optimal.

These are two essential aspects of solving any optimization problem (not just IPs). If we have efficient procedures as above, then it suggests the following algorithmic approach to solving the optimization problem:

\[
\text{Algorithmic Approach:} \\
\begin{align*}
\text{Start with a feasible solution} \\
\text{While the current solution is not optimal:} \\
& \quad \text{Improve the current solution.}
\end{align*}
\]

The approach repeatedly finds a feasible solution with non-decreasing objective value. To implement this approach, we need to know: (1) How to find a feasible solution? (2) How to verify optimality? and (3) How to improve the current solution? In this lecture, we will touch upon the first two questions.

### 2.1 Feasible Solutions

An obvious approach to obtain a feasible solution is the greedy method. Here, we construct a solution from scratch by repeatedly setting a variable that gives the best improvement in objective value without violating any constraints. For instance, consider the 0 \(-\) 1 Knapsack Problem:

\[
\max \sum_{j=1}^{n} p_j x_j \\
\sum_{j=1}^{n} w_j x_j \leq W \\
x_j \in \{0,1\} \ \forall j \in [n]
\]

To obtain a feasible solution for this, we consider the greedy procedure:

**Proposition 1.** Greedy procedure gives a feasible solution for the 0 \(-\) 1 Knapsack Problem.
**Greedy:**
Order the variables such that $\frac{p_i}{w_i} \geq \frac{p_{i+1}}{w_{i+1}}$ for every $i = 1, \ldots, n$.
For $i = 1, \ldots, n$:
Set $x_i = \begin{cases} 1, & \text{if } \sum_{j=1}^{i-1} w_j x_j + w_i \leq W \\ 0, & \text{otherwise} \end{cases}$

Proof. By induction on $n$.

While the greedy approach gives a feasible solution for the 0–1 Knapsack Problem, it may not be optimal. The main advantage of constructing a feasible solution (even if it may not be optimal) for a discrete optimization problem is that it gives a lower bound on the optimal objective value.

### 2.2 Optimality Conditions

Optimality conditions help us determine whether a given solution is optimal. They will also tell us the stopping condition in the above algorithmic approach for solving IPs. Let us first recall the optimality condition for LPs.

**Optimality conditions for LPs.** Recall the duality theorem and weak duality. Informally, they are stated as

- Duality theorem: $\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c^T, y \geq 0\}$
- Weak Duality: $c^T x^* \leq y^T b$ for all dual feasible solution $y$.

Thus, duality theorem gives the optimality condition for LPs. If we find a dual feasible solution $y$ such that $y^T b = c^T x$ for a primal feasible solution $x$, then $x$ is an optimum for the primal. In particular, duality theorem can be seen as a way to derive tight upper bounds for the maximization LP. Similarly, we can try to derive tight upper bounds for IPs which would help us in verifying optimality. The standard approach to derive upper bounds for a maximization problem is through relaxations.

#### 2.2.1 Relaxation

The main idea behind relaxations is to replace the given problem by a simpler optimization problem whose optimal value is at least the optimal value of the given problem. While relaxing the problem, we have two possibilities: (1) Enlarge the set of feasible solutions and (2) Replace the objective function by a function that has the same/larger value for every feasible solution.

**Definition 2.** A problem $\max\{f(x) : x \in \mathcal{T} \subseteq \mathbb{R}^n\}$ is a relaxation of the IP $\max\{c^T x : x \in \mathcal{X} \subseteq \mathbb{R}^n\}$ if

1. $\mathcal{X} \subseteq \mathcal{T}$ and
2. $f(x) \geq c^T x \ \forall \ x \in \mathcal{X}$.

**Proposition 3.** If $z_{\text{rel}} := \max\{f(x) : x \in \mathcal{T} \subseteq \mathbb{R}^n\}$ is a relaxation of $z := \max\{c^T x : x \in \mathcal{X} \subseteq \mathbb{R}^n\}$, then $z \leq z_{\text{rel}}$. 

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Proof. Let \( x^* \) be an optimum to the IP. Since \( x^* \in \mathcal{X} \subseteq \mathcal{T} \), we have that \( x^* \in \mathcal{T} \) and hence \( f(x^*) \leq z_{\text{rel}} \). But \( z = c^T x^* \leq f(x^*) \) and hence \( z \leq z_{\text{rel}} \).

In addition to giving upper bounds, relaxations are also helpful in identifying infeasibility and optimality.

**Proposition 4.** Consider the IP \( \max \{ c^T x : x \in \mathcal{X} \subseteq \mathbb{Z}^n \} \).

1. If a relaxation of the IP is infeasible, then the IP is infeasible.

2. Let \( x^*_{\text{rel}} \) be an optimum solution to the relaxation \( \max \{ f(x) : x \in \mathcal{T} \} \). If \( x^*_{\text{rel}} \in \mathcal{X} \) and \( f(x^*_{\text{rel}}) = c^T x^*_\text{rel} \), then \( x^*_{\text{rel}} \) is also an optimum solution to the IP.

Proof. 1. Since \( \mathcal{X} \subseteq \mathcal{T} \), it follows that if \( \mathcal{T} = \emptyset \), then \( \mathcal{X} = \emptyset \).

2. Since \( x^*_\text{rel} \in \mathcal{X} \), we have \( z = \max \{ c^T x : x \in \mathcal{X} \} \geq c^T x^*_{\text{rel}} = f(x^*_{\text{rel}}) = z_{\text{rel}} \). But \( z_{\text{rel}} \geq \max \{ c^T x : x \in \mathcal{X} \} \). Therefore, \( c^T x^*_{\text{rel}} = \max \{ c^T x : x \in \mathcal{X} \} \) and \( x^*_{\text{rel}} \in \mathcal{X} \). Consequently, \( x^*_{\text{rel}} \) is an optimum solution to the IP.

In order to derive tight upper bounds for our IP, it is sufficient to construct tight relaxations. We will see some possible ways to construct relaxations.

2.2.1.1 LP-relaxations

An obvious relaxation of an IP is obtained by dropping the integrality constraints.

**Definition 5.** Let \( P \) be a polyhedron. The LP-relaxation of the IP \( \max \{ c^T x : x \in \mathcal{X} \subseteq \mathbb{Z}^n \} \) is \( \max \{ c^T x : x \in P \} \).

Note that the LP-relaxation of an IP is indeed a relaxation. We would like to use relaxations to obtain tight upper bounds. Recall the definition of better formulations from the previous lecture. Better formulations give tighter upper bounds.

**Proposition 6.** Suppose \( P_1 \) and \( P_2 \) are two formulations for the IP \( z = \max \{ c^T x : x \in \mathcal{X} \subseteq \mathbb{Z}^n \} \). Suppose \( P_1 \) is a better formulation than \( P_2 \) (i.e., \( P_1 \subseteq P_2 \)). Let \( z_{b,\text{LP}} := \max \{ c^T x : x \in P_b \} \) for \( b = 1, 2 \). Then,

\[
 z \leq z_{1,\text{LP}} \leq z_{2,\text{LP}} \quad \text{for all objectives } c.
\]

Proof. Exercise.

Thus, better formulations provide tighter relaxations and consequently tighter upper bounds.
2.2.1.2 Combinatorial Relaxations

Sometimes, we can relax a difficult COP to an easy COP. The relaxed problem being easy can be solved quickly. The resulting solution gives an upper bound on the objective value of the difficult COP. Such relaxations arise by considering natural combinatorial interpretations.

For example, consider the TSP on the complete graph $G = (V, E)$. Here, we are given arc costs $c_{ij}$ for every $ij \in E$ and the objective is

$$z_{TSP} := \min \left\{ \sum_{ij \in T} c_{ij} : T \subseteq E, T \text{ forms a tour} \right\}.$$

A tour is a permutation and in particular, is an assignment. Recall that an “assignment” assigns exactly one edge per vertex. So, consider

$$z_{assign} := \min \left\{ \sum_{ij \in T} c_{ij} : T \subseteq E, T \text{ is an assignment} \right\}.$$

It is clear that $z_{assign}$ is a relaxation of $z_{TSP}$ as we have enlarged the collection of feasible solutions. It turns out that $z_{assign}$ can be solved efficiently. Note that the relaxation $z_{assign}$ is still a discrete optimization problem unlike LP-relaxations which are continuous optimization problems.

2.2.1.3 Lagrangian Relaxations

In LP-relaxations, we dropped the integrality constraints. More generally, we can also drop some linear constraints to obtain a relaxation. Further, instead of dropping these constraints, we can also fold them into the objective.

**Proposition 7.** Consider the IP $z := \max\{c^T x : x \in \mathcal{X} \subseteq \mathbb{Z}^n, Ax \leq b\}$. Let

$$z(\lambda) := \max\{c^T x + \lambda^T (b - Ax) : x \in \mathcal{X}\}.$$

Then, $z \leq z(\lambda)$ for all $\lambda \geq 0$.

**Proof.** Let $x^*$ be an optimum solution for the IP. Then $x^* \in \mathcal{X}$ and $Ax^* \leq b$. Therefore,

$$z = c^T x^* \leq c^T x^* + \lambda^T (b - Ax^*) \text{ for all } \lambda \geq 0 \leq z(\lambda).$$

Thus, $z(\lambda)$ for every $\lambda \geq 0$ provides a relaxation. We will see more about the Lagrangian relaxation in the latter part of the course. In particular, we will address the question of which non-negative $\lambda$ gives the tightest relaxation.
2.2.2 Duality

Recall that our goal is to verify whether a given solution is optimal. We saw that for LPs, duality gives a way to obtain tight upper bounds for LPs. Can we derive a similar duality theorem for LP?

Let us use the analogy with LPs to define a dual problem.

**Definition 8.** Let \( z := \max \{c^T x : x \in \mathcal{X} \} \) and \( w := \min \{w(u) : u \in \mathcal{U} \} \). The two problems form a weak dual pair if

\[
c^T x \leq w(u) \quad \forall x \in \mathcal{X}, u \in \mathcal{U}.
\]

If \( z = w \), then they form a strong dual pair.

**Remark.** Between relaxation and dual problems, which is preferable for obtaining tight upper bounds? Well, it is a trade-off. Note that any feasible solution to the dual problem gives an upper bound whereas only optimal solutions to the relaxation problem gives an upper bound. Thus, to get upper bounds using the dual problem, we only need to find feasible solutions to the dual problem while to get upper bounds using the relaxation problem, we need to solve it to optimality.

The next natural question is, do dual problems exist for IPs? In fact, the dual to the LP-relaxation is a weak dual to the IP.

**Proposition 9.** Suppose \( z := \max \{c^T x : Ax \leq b, x \geq 0, x \in \mathbb{Z}^n \} \). Consider the dual problem to the LP-relaxation, namely \( w^{LP} := \min \{y^T b : y^T A \geq c^T, y \geq 0 \} \). These two problems form a weak dual pair.

**Proof.** Exercise.

Similar to relaxations, dual problems also tell us about infeasibility and optimality.

**Proposition 10.** Let \( z = \max \{c^T x : x \in \mathcal{X} \} \) and \( w := \min \{w(u) : u \in \mathcal{U} \} \) be a weak-dual pair.

1. If \( w \) is unbounded, then \( z \) is infeasible.
2. If \( x^* \in \mathcal{X} \) and \( u^* \in \mathcal{U} \) satisfy \( c^T x^* = w(u^*) \), then \( x^* \) is an optimum solution for \( z \) and \( u^* \) is an optimum solution for \( w \).

**Proof.** Exercise.

2.2.2.1 Example

We will see a weak-dual pair of discrete optimization problems. Let \( G = (V, E) \) be a graph.

**Definition 11.** A matching is a set of vertex-disjoint edges.

For example, the squiggly edges in the figure below form a matching.
Every graph has a matching: for example, the empty set is a matching. A natural optimization problem associated with these structures is the problem of finding a maximum cardinality matching:

\[(P_1) \text{ MAX cardinality matching : } \max\{|M| : M \subseteq E \text{ is a matching}\} .\]

The size of a max cardinality matching in the above example graph is 2. In fact, it is at most 2 since we have 5 vertices in the graph and the maximum number of vertex-disjoint pairs of vertices from 5 is at most 2. Moreover, it is at least 2 as exhibited by the squiggly edges.

**Definition 12.** A vertex cover is a set \(S\) of vertices with each edge adjacent to at least one vertex in \(S\).

Every graph has a vertex cover: for example, consider the set \(S = V\). A natural optimization problem associated with these structures is the problem of finding a minimum cardinality vertex cover:

\[(P_2) \text{ MIN cardinality vertex cover : } \min\{|S| : S \subseteq V \text{ is a VC}\} .\]

**Proposition 13.** \(P_1\) and \(P_2\) form a weak dual pair.

*Proof.* Let \(M = \{u_1v_1, u_2v_2, \ldots, u_nv_n\}\). Any vertex cover \(S\) must contain at least one vertex from each pair \(\{u_iv_i\}, \forall i = 1, \ldots, k\). Therefore \(|R| \geq |M|\).

**Exercise.** Give a graph to show that \(P_1\) and \(P_2\) do not form a strong dual pair.

We will later see that they will form a strong dual pair in certain families of graphs.