Disclaimer: These notes have not been subjected to the usual scrutiny reserved for formal publications.

Recap

Definition 1. A polyhedron \( P \) is integral if \( P = P_I \) where \( P_I : = \text{conv-hull}(P \cap \mathbb{Z}^n) \).

Definition 2. A matrix \( A \) is totally unimodular (TU) if every square submatrix of \( A \) has determinant 0 or 1 or \(-1\).

Theorem 3 (Hoffman-Kruskal). If the constraint matrix \( A \) is TU and the RHS vector \( b \) is integral then the polyhedron \( P = \{ x : Ax \leq b \} \) is integral.

Theorem 4. Let \( A \in \{0, 1, -1\}^{m \times n} \) be a matrix such that

(i) Each column of \( A \) contains at most two non-zeros and

(ii) There exists a partition \( M_1 \cup M_2 = [m] \) of the rows such that every column \( j \) with two non-zero entries satisfies

\[
\sum_{i \in M_1} A_{ij} = \sum_{i \in M_2} A_{ij}.
\]

Then \( A \) is TU.

In the previous lecture, we saw two applications of TU matrices. In this lecture we will see a third application. Before seeing the third application, let us see another sufficient condition to guarantee the TU property.

Corollary 4.1. Let \( A \in \{0, 1, -1\}^{m \times n} \). If each column of \( A \) contains at most one \(+1\) and at most one \(-1\), then \( A \) is TU.

Proof. By Theorem 4 using the partition \( M_1 = [m] \) and \( M_2 = \emptyset \) we have that the matrix \( A \) is TU.

11.1 Application: Max \( s \to t \) flow in directed graphs

Given a digraph \( D = (V, E) \), nodes \( s, t \in V \) and non-negative arc capacities \( h_{ij} \in \mathbb{R}_{\geq 0} \ \forall ij \in E \), the goal is to find a max flow from \( s \) to \( t \). We recall the definition of a flow and its value below.

Definition 5. A flow is a function \( f : E \rightarrow \mathbb{R} \) such that

1. the incoming flow at \( v \) is equal to the outgoing flow at \( v \) for every vertex \( v \in V \setminus \{s, t\} \) and

2. the flow on each arc does not exceed the capacity of that arc.
The value of a flow $f$ is $\sum_{e \in \delta_{\text{out}}(s)} f_e$.

**Formulating the max $s \to t$ flow:** We may assume that there is no incoming arc into $s$ and no outgoing arc from $t$ (even if such arcs exist, we may remove them without changing the optimum since the optimum will never use these arcs). Now, we add an extra arc from $t$ to $s$ to the graph with arc capacity infinity (see Figure 11.1).

![Figure 11.1: Flow problem formulation.](image)

Further, we also define (see Figure 11.2)

\[ V^{\text{in}}(i) := \{ k : \overrightarrow{ki} \in E \} \]
\[ V^{\text{out}}(i) := \{ k : \overrightarrow{ik} \in E \}. \]

Note that $V^{\text{in}}(i) \cap V^{\text{out}}(i)$ could be non-empty.

![Figure 11.2: Incoming and Outgoing neighbors from a node $i$.](image)

We can now formulate the max $s \to t$ flow as a linear program:

\[
\begin{align*}
\text{max} & \quad x_{ls} \\
\sum_{k \in V^{\text{out}}(i)} x_{ik} - \sum_{k \in V^{\text{in}}(i)} x_{ki} &= 0 \quad \forall i \in V & \text{(Flow conservation constraints)} \\
x_{ij} &\leq h_{ij} \quad \forall ij \in E & \text{(Capacity constraints)} \\
x_{ij} &\geq 0 \quad \forall ij \in E & \text{(Non-negativity constraints)}
\end{align*}
\]

**Proposition 6.** The constraint matrix $A$ in Formulation (11.1) is TU.

**Proof.** The constraint matrix $A$ is of the form

\[
\begin{bmatrix}
C \\
-I \\
I
\end{bmatrix}
\]

where $C$ is the constraint matrix for flow conservation. By properties of TU matrices, it is sufficient to show that $C$ is TU. The rows of $C$
are indexed by nodes and the columns of $C$ are indexed by arcs with

$$C[i, e] = \begin{cases} 1 & \text{if } e = ik \text{ for } k \in V^{out}(i), \\ -1 & \text{if } e = ki \text{ for } k \in V^{in}(i), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $C \in \{0, 1, -1\}^{V \times E}$ and every column of $C$ has exactly one $+1$ and one $-1$. Therefore, by Theorem 4.1, the matrix $C$ is TU.

We have the following corollary using Hoffman-Kruskal.

**Corollary 6.1.** If all arc capacities are integral, then the LP has an integral optimum solution.

In particular, if we are interested in the max $s \rightarrow t$ integral flow problem (here, the input is the same as above, but the goal is to find a max integer-valued $s \rightarrow t$ flow, i.e., flow on every arc should be integer-valued), then we can solve it using the LP if the arc capacities are all integral. Let us focus on the strong dual problem to max $s \rightarrow t$ integral flow problem. We will show that the strong dual problem is indeed the min $s \rightarrow t$ cut problem and thereby derive the max flow-min cut theorem.

**Definition 7.** (i) A $s \rightarrow t$ cut in a directed graph $D$ is a set $U$ such that $s \in U \subseteq V - \{t\}$.

(ii) The capacity of a cut $U$ is $c(U) := \sum_{ij \in \delta^{out}(U)} h_{ij}$.

**Example:** In Figure 11.3 suppose all $h_{ij}$s are equal to one. Then $c(U) = 2$.

![Figure 11.3: A $s \rightarrow t$ cut](image)

Cuts are interesting from the perspective of a network attacker. The following problem gives a quantitative measure of the weakness in connectivity from node $s$ to node $t$.

**Min $s \rightarrow t$ cut problem:** Given a digraph $D = (V, E)$, nodes $s, t \in V$, and non-negative capacities $h_{ij} \in \mathbb{R}_{\geq 0} \forall ij \in E$, the goal is to find a $s \rightarrow t$ cut of minimum capacity.

We will now show that the LP-dual to Formulation [11.1] is in fact formulating the min $s \rightarrow t$ cut problem. The dual to LP [11.1] is

$$\min \sum_{ij \in A} h_{ij}w_{ij}$$

s.t. $u_i - u_j + w_{ij} \geq 0 \forall ij \in E$

$u_t - u_s \geq 1$

$w_{ij} \geq 0 \forall ij \in E$
The constraint matrix of the dual is the transpose of the constraint matrix of the primal. We already saw the the constraint matrix of the primal is TU. Hence, the constraint matrix of the dual is also TU. Therefore, the dual has an integral optimum solution (for all \( h_{ij} \)s which are not necessarily integral). Let \((u, w)\) be a dual optimum solution that is integral. We may assume that \( u_s = 0 \) (otherwise, setting \( u'_i = u_i - u_s \forall i \in V \) gives a dual feasible solution \((u, w)\) with the same obj value). Let \( S := \{ j \in V : u_j \leq 0 \}, \bar{S} := V \setminus S. \)

![Figure 11.4: A cut and the related variables](image)

Observe that \( \bar{S} = \{ j \in V : u_j \geq 1 \} \) since \( u \) is integral. Consider

\[
\bar{u}_j := \begin{cases} 
0 & \text{if } j \in S \\
1 & \text{if } j \in \bar{S}
\end{cases}
\quad \text{and} \quad
\bar{w}_{ij} := \begin{cases} 
1 & \text{if } i \in S, j \in \bar{S}, ij \in E \\
0 & \text{otherwise}
\end{cases}.
\]

Observe that

1. \((\bar{u}, \bar{w})\) is a dual feasible solution. This is verified by verifying the dual constraint for the arcs in all possible positions (see the red arcs in Figure 11.4).

2. Moreover,

\[
\sum_{ij \in E} h_{ij} w_{ij} \geq \sum_{\substack{ij \in E \\
i \in S \\
j \in \bar{S}}} h_{ij} \bar{w}_{ij} \geq \sum_{\substack{ij \in E \\
i \in S \\
j \in \bar{S}}} h_{ij} \bar{w}_{ij} = \sum_{ij \in E} h_{ij} \bar{w}_{ij}.
\]

The second inequality holds since \( w_{ij} \geq u_j - u_i \geq 1 \) for every \( ij \in E \) with \( i \in S, j \in \bar{S} \). The last equation is by definition of \( \bar{w} \). Thus, the objective value of \((\bar{u}, \bar{w})\) is at most the objective value of the dual optimum solution \((u, w)\).

Therefore, \((\bar{u}, \bar{w})\) is a dual optimum solution. We observe that \( s \in S, t \in \bar{S}, \) and \( \delta^{out}(S) = \{ ij : \bar{w}_{ij} = 1 \} \). Therefore, the set \( S \) is a \( s \to t \) cut with capacity equal to the dual optimum value which is equal to the maximum \( s \to t \) flow value in \((D, h)\). Thus, we have shown the following theorem.

**Theorem 8 (max flow-min cut).** Let \( D \) be a digraph with nodes \( s \) and \( t \) and non-negative arc capacities. Then the maximum \( s \to t \) flow value is equal to the minimum \( s \to t \) cut capacity.

### 11.2 Total Dual Integrality

We recall the properties of efficiently solvable IPs defined over integral polyhedra.
Recap

If $P = \{Ax \leq b\}$ is integral, we have

1. explicit inequality description of $P_I$,
2. efficient separation for $P_I$, and
3. a strong dual problem to the IP $\max \{c^T x : Ax \leq b, x \geq 0, x \in \mathbb{Z}^n\}$, namely, the dual to the LP $\max \{c^T x : Ax \leq b, x \geq 0\}$ which is given by $\min \{y^T b : y^T A \leq c^T, y \geq 0\}$.

If $P = \{x : Ax \leq b\}$ is integral, then the primal $\max \{c^T x : Ax \leq b\}$ always has an integral optimum solution along every direction $c$ when the objective value is finite. How about the dual? In particular, we saw that if $A$ is TU and the objective vector $c$ is integral, then the dual also has an integral optimum solution. If $c$ is not necessarily integral, then even if we have a dual optimum solution that is integral, scaling $c$ by $\frac{1}{k}$ for large $k$ leads to a dual optimum solution which is not integral. For several discrete optimization problems which are solvable efficiently, we have integral dual solution if $c$ is integral. In order to unify and describe discrete optimization problems with such strong dual problems (i.e., dual problems which have an integral optimum solution), Edmonds and Giles proposed the following definition.

**Definition 9** (Edmonds-Giles). A rational linear system $Ax \leq b$ is totally dual integral (TDI) if for all integral $c$ with $z^{LP} := \max \{c^T x : Ax \leq b\}$ finite, the dual $\min \{y^T b : y^T A = c^T, y \geq 0\}$ has an integral optimum solution.

Edmonds and Giles developed the theory of TDI also as a tool towards proving integrality of the primal. Let us see how TDI can be used as a tool to show integrality of the primal.

**Theorem 10** (Edmonds-Giles). If $Ax \leq b$ is TDI and $b$ is integral then $P = \{x : Ax \leq b\}$ is integral.

**Proof.** Recall that $P = \{x : Ax \leq b\}$ is integral iff $z^{LP} := \max \{c^T x : Ax \leq b\}$ is integer-valued for all $c \in \mathbb{Z}^n$ when it is finite.

Suppose $z^{LP} = \max \{c^T x : Ax \leq b\}$ is finite. Then by duality,

$$z^{LP} = \min \{y^T b : y^T A = c^T, y \geq 0\}.$$  

The right side has an integer optimum solution $y^*$ by the TDI property of the system $Ax \leq b$. Since $b$ is integral $y^*^T b \in \mathbb{Z}$ and hence $z^{LP}$ is integer-valued. Therefore, for all $c \in \mathbb{Z}^n$, if $Z^{LP}$ is finite then $Z^{LP}$ is integer-valued. This implies that $P = P_I$. \hfill $\square$

We emphasize that TDI is a property of the system $Ax \leq b$ and not a property of the polyhedron $P = \{x : Ax \leq b\}$ that is defined by that system. We illustrate this through the example below.

**Example:** Consider the following two systems.

System 1: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  System 2: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Both define the same polyhedron but we will see that system 1 is not TDI while system 2 is TDI. We write the two primal problems:

Primal 1: \(\max\{c_1x_1 + c_2x_2 : x_1 + x_2 \leq 0, x_1 - x_2 \leq 0\}\) and
Primal 2: \(\max\{c_1x_1 + c_2x_2 : x_1 + x_2 \leq 0, x_1 - x_2 \leq 0, x_1 \leq 0\}\).

The corresponding dual problems are

Dual 1: \(\min\{0 : y_1 + y_2 = c_1, y_1 - y_2 = c_2, y_1, y_2 \geq 0\}\) and
Dual 2: \(\min\{0 : y_1 + y_2 + y_3 = c_1, y_1 - y_2 = c_2, y_1, y_2, y_3 \geq 0\}\)

For instance, suppose \(c_1 = 2, c_2 = 1\). Then

Dual 1: \(y_1 + y_2 = 2, y_1 - y_2 = 1 \implies y_1 = \frac{3}{2}, y_2 = \frac{1}{2}\), i.e., it has a unique dual optimum solution.

Dual 2: \(y_1 + y_2 + y_3 = 2, y_1 - y_2 = 1, y_1, y_2, y_3 \geq 0 \implies y_1 = \frac{3 - y_3}{2}, y_2 = \frac{1 - y_3}{2}\). For instance, \(y_3 = 1, y_1 = 1, y_2 = 0\) is an integral dual optimum solution.

Indeed \(y_1 + y_2 + y_3 = c_1, y_1 - y_2 = c_2, y_1, y_2, y_3 \geq 0\) has integral feasible solution for every \(c_1, c_2 \in \mathbb{Z}\).

We also emphasize that the RHS vector \(b\) being integral is crucial to apply the theorem. The system \(Ax \leq b\) being TDI does not necessarily imply that \(P = \{x : Ax \leq b\}\) is integral.

The above theorem tells us that a TDI system with integral RHS describes an integral polyhedron. The following theorem gives a converse:

**Theorem 11** (Giles-Pulleyblank). Let \(P\) be a rational polyhedron. Then

1. There exists a unique minimal TDI system \(Ax \leq b\) with \(A\) integral such that \(P = \{x : Ax \leq b\}\).
2. \(P = P_I\) iff \(b\) can be chosen to be integral.

The first part of the theorem says that every rational polyhedron has a TDI description. However, the RHS vector \(b\) in this TDI description may not be integral. The second part of the theorem says that the RHS vector \(b\) is integral in this TDI description iff \(P\) is an integral polyhedron.

In the next couple of lectures, we will focus on matroids, and more generally submodular functions, and show that an inequality system associated with these structures is TDI. This will in turn allow us to solve numerous discrete optimization problems by simply solving an LP-relaxation of a suitable IP.